Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

---

Mathematica Balkanica

New Series Vol. 12, 1998, Fasc. 1-2

Cnoidal Solution in the Equation FKDV

Ognyan Kamenov

Presented by V. Kiryakova

In the present paper various classes of analytical biperiodic solutions of the nonlinear evolutionary equation FKVD are found based on the Jacobi's elliptic functions. A more general procedure than that described in [7] is used, making possible not only various irregular biperiodic solutions to be found, but also regular ones that are analyzed in another paper of the author.

AMS Subj. Classification: 35Q53; 35B10, 33E05

Key Words: evolutionary partial differential equations, Korteveg-De Vriz equations, biperiodic solutions, Jacobi's elliptic functions

I. Introduction

The nonlinear evolutionary partial differential equation

$$u_t + uu_x = u_{xxxxx} \tag{1}$$

known as the Korteveg-De Vriz equation or FKDV, is a physical model in a proper parametrized region for the propagation of nonlinear magnetoacoustical waves or gravitational capillary waves. The interest shown by a number of authors ([3], [7], [9]) to this equation is due to favourable test opportunities it gives for comparing different analytical, asympthotical and numerical methods. In spite of external similarity between the FKDV-equation and the classical equation KDV:

$$u_t + 6uu_x + u_{xxx} = 0$$

the presence of fifth derivative in the right side of (1) causes debalance between the nonlinear and the convectional terms, that is the reason for the solution bifurcation in the vicinity of the mobile critical points as it is shown in [9]. This is not P-type equation which means that the spectral analysis method is not applicable, i.e. the equation does not have any solution of a solitone type. This fact explains the interest in the searching bounded periodical solutions. Kano and Nakayama [7] have found a specific class of such periodical and irregular solutions. Using the classical procedure of Stokes [13] and assuming small wave amplitudes, Boyd [3] finds that the solution of (1) is a finite Fourier series consisting of cosines of the multiple angles which is the example of a 2-periodical solution. In the present paper an "elliptic" procedure, more general than the one described in [7], is used for determining irregular cnoidal waves in the equation FKDV. All such possible biperiodical waves are considered. Here, we call irregular cnoidal waves all those, for which the invariant $G_3 = 0$ and $G_2 \neq 0$ when

$$cn^2(z,m) = 1 - (e_1 - e_3)[\mathcal{P}(z, G_2, G_3) - e_3)]^{-1}$$

and regular ones if $G_2 \neq 0$ and $G_3 \neq 0$ respectively. Because of the specific character of the irregular cnoidal solutions, they are considered in details in another author's paper.

2. Irregular biperiodical solutions

We search for the solution of equation (1) in the form of stationary waves with parameters $k \neq 0$ and phase velocity v

(2) $$u(t,x) = \zeta(z),$$

where

(3) $$z = k(x - vt).$$

Without loss of generality we could substitute in (3) $k = 1$ because if $\zeta(x - vt)$ is a solution of (1), $k^4\zeta[k(x - vt)]$, where $k^4 = v$ is also a solution of (1), but with a changed phase velocity $- v^{5/4}$. Further on, we shall work as it is given in (3).

Let us now suppose that $\varphi(z)$ is a nontrivial solution of the ordinary differential equation:

(4) $$(\varphi')^2 = 2\alpha_0\varphi^3(z) + \alpha_1\varphi^2(z) + 2\alpha_2\varphi^3(z) + \alpha_3$$

where α_j, $j = 1, 2, 3, 4$ are unknown parameters, eventually depending on the wave number k and also we suppose that $z \in \mathbb{C}$. After raplacing (2) and (3) in (1) and integrating only once, we obtain the following reduced equation:

(5) $$\zeta^2/2 - v\zeta + B = k^4\zeta_{zzzz},$$

where B is the integration constant, which in the case of periodical solutions, being discussed here, is different from zero although it has no dynamic characteristics. We show below when an appropriate choice is made for the parameters

α_j, $j = 1, \ldots, 4$, we can present a particular solution of (5) in the form:

$$\zeta(z) = 3\alpha_0\varphi^2(z) + \alpha_1\varphi(z) + \alpha_2. \tag{6}$$

Let us replace $\zeta(z)$ from (6) into (5) taking into consideration (4). On both sides of (5) we obtain fourth-order polynomes of φ. A sufficient condition for them to be identical (which guarantees that $\zeta(z)$ is a solution of (5)) is the coefficients in front of the corresponding degrees of φ to be equal. So we get to the system (See Annex A):

$$\alpha_0(1 - 140\alpha_0 k^4) = 0, \tag{7}$$
$$\alpha_0\alpha_1(1 - 140\alpha_0 k^4) = 0, \tag{8}$$
$$v\alpha_0 = \alpha_1^2(1/6 - 21\alpha_0 k^4) + \alpha_0\alpha_2(1 - 168\alpha_0 k^4), \tag{9}$$
$$v\alpha_1 = \alpha_1\alpha_2 - \alpha_1 k^4(\alpha_1^2 + 108\alpha_0\alpha_2) - 180k^4\alpha_0^2\alpha_3, \tag{10}$$
$$B = k^4\alpha_2(\alpha_1^2 + 18\alpha_0\alpha_2) + 30k^4\alpha_0\alpha_1\alpha_3 + \alpha_2(v - \alpha_2/2). \tag{11}$$

Naturally the nontrivial solutions of (7)-(11) are of interest, i.e. those for which $|\alpha_0| + |\alpha_1| + |\alpha_2| + |\alpha_3| \neq 0$ and at the same time $\alpha_j \in \mathbb{R}$. We shall consider the following possible, real and nontrivial solutions:

(a) $\alpha_0 = 0$: The system has the next solution

$$\alpha_0 = \alpha_1 = \alpha_3 = 0, \qquad \alpha_2 = \mu = \text{const}, \qquad B = 2\mu(v - \mu), \tag{12}$$

and then, according to (6) the solution of (1) is

$$u(t, x) = \mu \tag{13}$$

that is obviously a periodical solution, but it is not of physical interest.

(b) $\alpha_0 \neq 0, \quad \alpha_1 = 0$: In this case we obtain

$$\alpha_0 = 1/2\epsilon, \quad \alpha_1 = 0, \quad \alpha_2 = -\mu, \quad \alpha_3 = 0 \quad \text{where } \epsilon = 70k^4 \tag{14}$$

$$v = \mu/5, \qquad B = -4\mu^2/7. \tag{15}$$

The differential equation (4) reduces to

$$\frac{z + C}{2\sqrt{\epsilon}} = \int \frac{dy}{\sqrt{4y^3 - 8\mu\epsilon y}}, \qquad C_0 = \text{const}, \tag{16}$$

which solution is the elliptic function of Weierstrass with invariants $G_2 = 8\mu\epsilon$, $G_3 = 0$ i.e.

$$u(x,t) = \frac{3}{2\epsilon}\mathcal{P}^2\left(\frac{z+C_0}{2\sqrt{\epsilon}}, 8\mu\epsilon, 0\right) - \mu \tag{17}$$

and if we use the homogeneous property for $\mathcal{P}(z, G_2, G_3)$

$$\mathcal{P}(\lambda z, \lambda^{-4}G_2, \lambda^{-6}G_3) = \lambda^{-2}\mathcal{P}(z, G_2, G_3) \tag{18}$$

and replace $k = (280)^{-1/2}$, $\mu = 5\sigma$, $G_2 = \sigma/28$, $B = -100\sigma^2/7$ we get just the solution:

$$u(t,x) = 1680\mathcal{P}^2((x - \sigma t + C_0), G_2, 0) - 140G_2, \tag{19}$$

obtained by Kano and Nakayama in [7]. In spite of the fact that it gives a bounded biperiodical solution in the form:

$$u(t,x) = (15\sigma/2)cn^4\left[\sqrt[4]{(\sigma/28)}(x - \sigma t) + C_0\right] - 5\sigma/2, \tag{20}$$

we obviously have to relate it to the irregular cases, because the second invariant G_3 in the Weierstrass function (19) is equal to zero. The procedure described in [7] does not allow for determining other bounded periodic solutions. We show in the next item (c) that in the case of a suitable solution of the system (7)-(11), we can achieve it.

(c) $\alpha_3 = 0$: It is understood that $\alpha_j \neq 0$, $j = 0, 1, 2$. We obtain easily the solutions of (6)-(11) in the form

$$\alpha_0 = 1/(2\epsilon), \qquad \alpha_1 = 3(\mu/\epsilon)^{1/2}, \qquad \alpha_2 = \mu, \qquad \alpha_3 = 0. \tag{21}$$

$$v = \mu/10, \qquad B = -\mu^2/7. \tag{22}$$

The above indicated solutions determine the following type of the differential equation (4):

$$(\varphi')^2 = (\varphi^3 + 3\sqrt{\mu\epsilon}\varphi^2 + 2\mu\epsilon\varphi)/\epsilon$$

the solution of which consists in "converting" the integral

$$\frac{z + C_0}{\epsilon} = \int \frac{dy}{\sqrt{y^3 + 3(\epsilon\mu)^{1/2}y^2 + 2\mu\epsilon y}}. \tag{23}$$

Here as in (16) integration constant C_0 represents the phase shift. Now if we substitute in the integral (23) we obtain:

$$\frac{z + C_0}{2\epsilon} = \int \frac{dy}{\sqrt{4y^3 - 4[\epsilon\mu(4 - 3\sqrt{\epsilon\mu})]y}} \tag{24}$$

which means that the solution $\varphi(z)$ is:

$$\varphi(z) = 4\epsilon^2 \mathcal{P}\left(z + C_0, \mu(4 - 3\sqrt{\epsilon\mu})/(4\epsilon^3), 0\right). \tag{25}$$

Then according to (6) the final result in the case of (c) will be:

$$u(t, x) = 24\epsilon^3 \mathcal{P}^2(z + C_0, G_2, 0) + 12\epsilon\sqrt{\epsilon\mu}\mathcal{P}(z + C_0, G_2, 0) + \mu, \tag{26}$$

where $G_2 = \mu(4 - 3\sqrt{\epsilon\mu})/(4\epsilon^3)$.

It is obvious that this solution also have to be related to the irregular cases because here $G_3 = 0$ too but the solution (26) does not follow from (17). Since the invariants of the function $\mathcal{P}(z+C_0, G_2, 0)$ are real, its poles are also real (moreover they are double). In fact those are the z points for which $z = -C_0$. For these "movable" poles to be avoided in the solution it is sufficient to accept:

$$C_0 = C(e_1 - e_3)^{-1/2} + \omega_2/2, \tag{27}$$

where ω_2 is a purely imaginary number characterizing the one of both periods of $\mathcal{P}(z + C_0, G_2, 0)$, being supposed to have $Im(\omega_2/\omega_1) > 0$ and C is an arbitrary real constant. Taking into consideration the relations, described in Annex B in details, we get:

$$\mathcal{P}(z + C_0, G_2, 0) = e_3 + (e_2 - e_3)sn^2(z\sqrt{e_1 - e_3}, m),$$

where m is the modulas of the elliptic function of Jacobi

$$0 \leq m^2 = (e_2 - e_3)/(e_1 - e_3) \leq 1$$

and we suppose that e_j, $j = 1, 2, 3$ are different real roots of the cubic algebraic equation

$$E^3 - [\mu(4 - 3\sqrt{\epsilon\mu})/(4\epsilon^3)]E = 0. \tag{28}$$

Such real and different roots exist only when

$$0 < \mu < 8/(315k^4), \tag{29}$$

and they are (arrenged in ascedent order), i.e. $e_1 > e_2 > e_3$

$$e_1 = \frac{1}{2\epsilon}\sqrt{\frac{\mu}{\epsilon}}\left(4 - 3\sqrt{\frac{\mu}{\epsilon}}\right), \qquad e_2 = 0, \qquad e_3 = -e_1. \tag{30}$$

So we get the necessary for us identity:

$$\mathcal{P}(z + C_0, G_2, 0) = e_3 cn^2(z\sqrt{2e_1} + C, 1/\sqrt{2}) \tag{31}$$

by the help of which and of (6) we obtain a bounded cnoidal solution of the equation (1):

$$\begin{aligned} u(t,x) &= \mu + 6\mu\sqrt{4 - 3\sqrt{\mu/\epsilon}}\left[cn^2(z\sqrt{2e_1} + C, 1/\sqrt{2})\sqrt{4 - 3\sqrt{\mu/\epsilon}} + 1\right] \\ &\times cn^2(z\sqrt{2e_1} + C, 1/\sqrt{2}), \end{aligned} \tag{32}$$

where $z = k(x - \mu t/10)$, $C \in \mathbb{R}$ and the constant μ satisfies (29). This solution is biperiodical in x with periods: $2\pi(2e_1k^2)^{-1/2}\ \theta_3^2(0|\tau)$ and $\pi(1 + \tau)(2e_1k^2)^{-1/2}\ \theta_3^2(0|\tau)$, where $\tau = \omega_2/\omega_1$ and $\theta_3(0|\tau)$ is the Jacobi's function:

$$\theta_3(0|\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2 \tau}$$

(d) $\alpha_j \neq 0, \quad j = 0, 1, 2, 3,$

So far in the considered cases the solution of the system (7)-(11) have been presented as a function of the wave number k that we suppose here positive. If $\alpha_j \neq 0$, $j = 0, 1, 2, 3$, the solution of the system as a function of k is:

$$\alpha_0 = 1/2\epsilon, \qquad \alpha_1 = \eta, \qquad \alpha_2 = \xi, \qquad \alpha_3 = 2\epsilon\eta(9\xi - \epsilon\eta^2)/27, \tag{33}$$

$$v = (\epsilon\eta^2 - 6\xi)/30, \tag{34}$$

$$B = \epsilon\eta^2(12\xi - \eta^2)/63 - 4\xi^2/7, \tag{35}$$

where as in the case of (c) $\epsilon = 1/(70k^4)$ and the nonzero constants η and ξ are chosen so that

$$\xi \neq \epsilon\eta^2/9, \qquad \eta \neq 0, \qquad \xi \neq \epsilon\eta^2/6 \tag{36}$$

for $\alpha_3 \neq 0$ and $v \neq 0$ to be provided. The stationary solution of (5) will be of the form:

$$\zeta(z) = (3/2\epsilon)\varphi^2 + \eta\varphi + \xi, \tag{37}$$

but $\varphi(z)$ is already a solution of the differential equation:

$$\epsilon(\varphi')^2 = \varphi^3 + \eta\epsilon\varphi^2 + 2\epsilon\xi\varphi + 2\epsilon^2\eta(9\xi - \epsilon\eta^2)/27, \tag{38}$$

that is connected with the "conversion" of the elliptic integral:

$$\frac{z + C_1}{\epsilon} = \int \frac{dy}{\sqrt{y^3 + \eta\epsilon y^2 + 2\epsilon\xi y + 2\epsilon^2\eta(9\xi - \epsilon\eta^2)/27}} \tag{39}$$

where C_1 is an arbitrary integration constant. If we make the substitution $y = Y - \eta\epsilon/3$ in the integral, we get the solution in the form:

$$\varphi(z) = 4\epsilon^2\mathcal{P}(z + C_1, G_2, 0) \tag{40}$$

that is presented again by means of a Weirstrass's irregular function $\mathcal{P}(w, G_2, 0)$ because $G_3 = 0$, but this time:

$$G_2 = (\epsilon\eta^3/3 - 2\xi)/(4\epsilon^3). \tag{41}$$

The solution of (1) becomes:

$$u(t, x) = 24\epsilon^3\mathcal{P}^2(z + C_1, G_2, 0) + 4\eta\epsilon^2\mathcal{P}(z + C_1, G_2, 0) + \xi, \tag{42}$$

and if we use here the procedure of translation of the real poles half a period, we obtain again a biperiodical cnoidal solution of the form:

$$\begin{aligned} u(t, x) &= \sqrt{\epsilon\eta^3 - 6\xi}cn^2\left(z\sqrt{2e_1'} + C, 1/\sqrt{2}\right)\left[cn^2\left(z\sqrt{2e_1'} + C, 1/\sqrt{2}\right)\right. \\ &\left.\sqrt{\epsilon\eta^3 - 6\xi} - 2\eta\sqrt{\xi}\right] + \xi, \end{aligned} \tag{43}$$

where $e_1' = (\eta^3 - 6\xi/\epsilon)^{1/2}/(2\epsilon)$, $e_2' = 0$, $e_3' = -e_1'$ and the modulus of the function cn as in the case of (c) is $M_1 = 1/\sqrt{2}$, $C \in \mathbb{R}$. As in the previous case (c) it is necessary the following condition to be fulfilled

$$\xi < \epsilon\eta^3/6 \tag{44}$$

for the numbers e_j', $j = 1, 2, 3$ to be real and different.

3. Analysis of results and concluding remarks

A common feature of bounded beperiodical solutions obtained in (19), (32) and (43) is that they originate from an irregular function of Weierstrass, i.e. when the second invariant is equal to zero. But while (19) depends only on an independant constant (characterizing the phase shift), the solution (32) depends on two constants – μ, C and (43) depends on three – μ, η, C, when, of course, the conditions (29) and (44) are satisfied. This makes the solution in each particular case more common than previous ones, which means that the solution (43) can satisfy more initial and boundary conditions adequate to the periodicity requirement. Together with this advantage of (32) and (43) over (19) they have the disadvantage that in the boundary case $G_2 \to 0$ (a strongly irregular case) the solutions (32) and (43) coincide with the solution (13) while the (19) becomes $u(x,t) = u(x) = 1680/(x+C)^4$ when $G_2 \to 0$ i.e. when $\mu \to 0$, that is in fact the stationary solution of (1). In the general case the solutions (19), (32) and (43) (excluding the trivial solution (13)) have diffenent periods depending on the cnoidal irregular waves from (20) and (26). They have one-way propagation in positive or nagative direction of the abscissa depending on the sign (), while the cnoidal waves of (43) could have two-way propagation according to (34).

4. Concluding remarks

The assumption made in (4) and the dependence on the particular solution of (5) have determined a procedure that is connected to the construction of identical polynomes using the unknown function $\varphi(z)$. Obviously such an approach could be applied to nonlinear differential equations of more general structure, for example to the equations:

$$u_t + \Phi(u, m+1)_x = u_{(2n+1)x},$$

where $\Phi(u, m+1)$ is the polynom of $m+1$ degree and $m, n \in \mathbb{N}$. So far it is not known if the procedure described above could be effective when $\Phi(u, m+1)$ contains only odd powers of u or if it has both even and odd powers, of course, on different assumptions for the type of the reduced equation. In the concrete equation FKDV (1) is obtained when $m = 1$ and $n = 2$. The two classes of periodical solutions derived, are bounded and two periods depending on the space variable x. Generally speaking these solutions are complex, but it was pointed out during the analysis that their physical adequacy it is necessary a proper choice of the parameters to be made for the solutions to be real. That was shown in (29). A nature generalization of the solution, given in (26) and (42), should be the one for which the invariants G_2 and G_3 of the Weierstrass function are nonzero, i.e. the regular case. Such solutions are considered in another author's paper.

Annex A

$$\zeta^2 = 9\alpha_0^2\varphi^4(z) + 6\alpha_0\alpha_1\varphi^3(z) + (\alpha_1^2 + 6\alpha_0\alpha_2)\varphi^2(z) + 2\alpha_1\alpha_2\varphi(z) + \alpha_2^2;$$

$$\zeta_{zzzz} = 630\alpha_0^3\varphi^4(z) + 420\alpha_0^2\alpha_1\varphi^3(z) + (504\alpha_0^2\alpha_2 + 63\alpha_0\alpha_1)\varphi^2(z)$$
$$+(\alpha_1^3 + 108\alpha_1\alpha_2 + 180\alpha_0^2\alpha_3)\varphi(z) + (\alpha_1^2\alpha_2 + 18\alpha_0\alpha_2^2 + 30\alpha_0\alpha_1\alpha_2).$$

Annex B

$$\mathcal{P}(z + C_0, G_2, G_3) = e_3 + (e_1 - e_3)sn^{-2}[(z + C_0)\sqrt{e_1 - e_3}, M]$$
$$sn^{-2}(z\sqrt{e_1 - e_3} + C_0\sqrt{e_1 - e_3}, M) = sn^{-2}[z\sqrt{e_1 - e_3} + \frac{\omega_2}{2}\sqrt{e_1 - e_3}, M]$$
$$= M^2 sn^2(z\sqrt{e_1 - e_3} + C, M) = (e_2 - e_3)/(e_1 - e_3)sn^2(z\sqrt{e_1 - e_3} + C, M) \therefore$$
$$\mathcal{P}(z + C_0, G_2, G_3) = e_3 + (e_2 - e_3)sn^2[z\sqrt{e_1 - e_3} + C, M).$$

References

[1] M. A b r a m o w i t z, I. S t e g u n. *Handbook of Mathematical Functions*, 1964.
[2] N. A h i e z e r. *Elements of the Theory of Elliptic Functions*, Moscow, 1970 (In Russian), p. 303.
[3] J. B o y d. *Physica D*, **21**, p.227.
[4] J. B o y d. *Adv. Appl. Mech.*, **27**, No 1, 1990.
[5] M. I m a d a. *J. Phys. Soc. Japan*, **52**, 1983, p.1946.
[6] T. K a w a h a r a. *J. Phys. Soc. Japan*, **33**, 1972, p.260.
[7] K. K a n o, T. N a k a y a m a. *J. Phys. Soc. Japan*, **50**, 1981, p.361.
[8] O. K a m e n o v. *Math. Balkanica*, **6**, 1992, p.367.
[9] O. K a m e n o v. *Math. Balkanica*, **7**, 1993, p. 99.
[10] A. N a k a m u r a, Y. M a t s u n o. *J Phys. Soc. Japan*, 48, 1980, p.653.
[11] A. P a r k e r. *J. Phys. Math. Gen.*, **25**, 1992, p. 2005.
[12] G. K o r n, T. K o r n. *Math. Handbook*, New York, 1968.
[13] G. S t o k e s. *Camb. Trans.*, **8**, 1947, 441-473.
[14] G. W h i t h m a n. *Linear and Nonlinear Waves*, New York, 1974.

Institute for Appl. Math. and Computer Science
Technical University – Sofia, P.O. Box 384
1000 Sofia, BULGARIA

Received: 28.03.1996