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# Some Mutiplicative and Convolution Products Between $\delta^{(n+2j-1)}(r)$ and $\triangle^j \delta(x)$.

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The purpose of this paper is to obtain a relation between the distribution $\delta^{(n+2j-1)}(r)$ and the operator $\triangle^j \delta(x)$ and to give a sense to some convolutional distributional products.

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## I.1. Introduction

Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the $n$-dimensional Euclidean space $R^n$. We denote by $\varphi(x)$ the $c^\infty$-functions with a compact support defined from $R^n$ to $R$. Let

$$r^2 = x_1^2 + x_2^2 + \ldots + x_n^2 \tag{I.1.1}$$

and consider the functional $r^\lambda$ defined by ([3], p.71):

$$(r^\lambda, \varphi) = \int_{R^n} r^\lambda \varphi(x) dx, \tag{I.1.2}$$

For $\Re(\lambda) > -n$, this integral converges and is an analytic function of $\lambda$. Analytic continuation to $\Re(\lambda) \leq -n$ can be used to extend the definition of $(r^\lambda, \varphi)$.

Denoting by $\Omega_n$ the hypersurface area of the unit sphere imbedded in the $n$-Euclidean space, from [3], p.71, we have that

$$(r^\lambda, \varphi) = \Omega_n \int_0^\infty r^{\lambda+n-1} S_\varphi(r) dr, \tag{I.1.3}$$

where

$$S_\varphi(r) = \frac{1}{\Omega_n}\int_\Omega \varphi dw \tag{I.1.4}$$

and $dw$ is the hypersurface element of the unit sphere.

$S_\varphi(R)$ is the mean value of $\varphi(x)$ on the sphere of radius $r$ (cf.[3], p.71).

The functional $r^\lambda$ (cf.[3], pp.72,73) has a simple pole at

$$\lambda = -n - 2j, \quad j = 0, 1, 2, \ldots \tag{I,1,5}$$

and from [3], p.99, the Laurent series expansion of $r^\lambda$ in a neighbourhood of $\lambda = -n - 2j, \quad j = 0, 1, 2, \ldots$ is

(I.1.6)

$$r^\lambda = \frac{\Omega_n}{(2j)!}\delta^{(2j)}(r)\frac{1}{\lambda + n + 2j} + \Omega_n r^{-2j-n} + \Omega_n(\lambda + n + 2j)r^{-2j-n}\ln(r) + \ldots$$

In (I.1.6) $r^{-2j-n}$ is not the value of the functional $r^\lambda$ at $\lambda = -n - 2j$ (in fact it has a pole at his point), but the value of the regular part of the Laurent expansion of $r^\lambda$ at this point.

From [4], p.133, we know that the neutrix product $r^{-k} \circ \Delta\delta(x)$ exists,

$$r^{-k} \circ \Delta\delta(x) = \frac{\Delta^{k+1}\delta(x)}{2^k(k+1)!(m+2)\ldots(m+2k)} \tag{I.1.7}$$

for $k = 1, 2, \ldots, [\frac{(m-1)}{2}]$ and

$$r^{1-2k} \circ \Delta\delta(x) = 0 \tag{I.1.8}$$

for $k = 1, 2, \ldots, [\frac{m}{2}]$, where $m$ is the dimension of the space and $\Delta$ is the Laplacian operator.

The purpose of this paper is to obtain a relation between the distribution $\delta^{n+2j-1}(r)$ and the operator $\Delta^j\delta(x)$ which is established Section I.2, and to give a sense to the convolution distributional product of the form

$$\delta^{(n+2j-1)}(r) * \delta^{(n+2l-1)}(r), \qquad \frac{r^\lambda}{\Gamma(\frac{\lambda+n}{2})} * \delta^{(n+2j-1)}(r)$$

and to the multiplicative distributional products of $r^{-k}.\Delta^j\delta(x)$, which are showed at the Section II.1, II.2, II.3, respectively. Here

$$\Delta^j = \left\{\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\right\}^j \tag{I.1.9}$$

and by $\triangle^j u$ we understand the $j$-th interation of the Laplacian.

The relation obtained here (see formula (I.2.10)) can not be deduced from (I.1.6) since as Gelfand said ([3], p.99): "In this equation the left-hand side operaters on $\varphi(x)$ and the right-hand side on $S_\varphi(r)$ ...".

Our final formulae (II.3.1) and (II.3.10) give $(Q_c)$ generalization of the equations (I.1.7) and (I.1.8), respectively, due to Li Chen Cuan and B. Fisher (cf. [4], p.133, Th. 2).

To obtain our results we need the following formulae:

$$\text{(I.1.10)}\quad (\delta^{(k)}(r-c),\varphi) = (-1)^k\Omega_n\left[\frac{\partial^k}{\partial r^k}(r^{n-1}S_\varphi(r))\right]_{r=c} \quad ([1], \text{p. } 58, (\text{II}.2.5)),$$

where

(I.1.11)

$$(\delta^{(k)}(r-c),\varphi) = \int \delta^{(k)}(r-c)\varphi dx = \frac{(-1)^k}{c^{n-1}}\int_{O_c}\frac{\partial^k}{\partial r^k}(\varphi r^{n-1})dO_c \quad ([3], \text{p. } 231, (10),$$

$O_c$ is the sphere $r-c=0$ and $dO_c$ is the Euclidean element of area of it;

(I.1.12)

$$\text{Res}_{\lambda=-n-2j}(r^\lambda,\varphi) = \frac{\Omega_n}{2^j j! n(n+2)\dots(n+2j-2)}(\triangle^j\delta,\varphi(x)) \quad ([3], \text{p.}72, 73),$$

where

$$\text{(I.1.13)}\qquad \Omega_n = 2\pi^{n/2}/\Gamma(\pi/2),$$

$$\text{(I.1.14)}\qquad \Gamma(z+k) = z(2+1)\dots(z+k-1)\Gamma(z), \quad ([2], \text{p.}3, (2))$$

$$\text{(I.1.15)}\qquad \Gamma(z)\Gamma(1-z) = \pi c^3 c(\pi z), \quad ([2], \text{p.}3, (6))$$

$$\text{(I.1.16)}\qquad \Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+\frac{1}{2}), \quad ([2], \text{p.}5, (15))$$

and

$$\text{(I.1.17)}\qquad \text{Res}_{\mu=-k,k=1,2,\dots}(x_+^\mu,\varphi) = \frac{\varphi^{(k-1)}(0)}{(k-1)!} \quad ([3], \text{p.}49),$$

where $x_+^\mu$ is the functional defined by

$$\text{(I.1.18)}\qquad (x_+^\mu,\varphi) = \int_0^\infty x^\mu\varphi(x)dx \quad ([3], \text{p.}48)$$

which is analytic for $Re(\mu) > -1$ and can be analytically continued to the entire $\mu$-plane except for the points $\mu = -1, -2, \ldots$, where it has a simple pole.

## I.2. The relation between the distributoin $\delta^{(n+2j-1)}(r)$ and the operator $\Delta^j\delta(x)$

In this paragraph we obtain a formula relating the distribution $\delta^{(n+2j-1)}(r)$ with the operator $\Delta^j\delta(x)$.

From (I.1.12) and (I.1.13), the residue of $(r^\lambda, \varphi)$ at $\lambda = -n - 2j$ for non-negative integer $j$ is given by

$$\text{(I.2.1)} \qquad \text{Res}_{\lambda=-n-2j}(r^\lambda, \varphi) = \frac{\Omega_n \Gamma(\frac{n}{2})}{2^{2j} j! \Gamma(j + n/2)} (\Delta^j \delta, \varphi(x)),$$

where $\Delta^j$ is defined by the equation (I.1.7), $\Omega)n$ by (I.1.11), $n$ is the dimension of the space and $j = 1, 2, \ldots$

From [3], p.72, $S_\varphi(r)$ is an even functon of the simple variable $r$ in $K$, where $K$ is the space of infinitely differentiable functions with bounded support. Then the integral (I.1.3) represents the application of $\Omega_n x_+^\mu$ (with $\mu = -\lambda + n - 1$) to $S_\varphi(r)$, where $x_+^\mu$ is defined by the equation (I.1.16).

Therefore according to the residue of $(x_+^\mu, \varphi)$ at $\mu = -k$ for $k$ positive integer (see formula (I.1.15)) from (I.1.3) we have,

$$\text{(I.2.2)} \qquad \text{Res}_{\lambda=-n-2j}(r^\lambda, \varphi) = \text{Res}_{\lambda=-n-2j}\Omega_n(r^{\lambda+n-1}, S_\varphi(r))$$

$$= \frac{\Omega_n}{(n+2j-1)!}\left[\frac{\partial^{n+2j-1}}{\partial r^{n+2j-1}}(r^{n-1} S_\varphi(r))\right]_{r=0},$$

where $S_\varphi(r)$ is defined by (I.1.4) and $j = 0, 1, 2, \ldots$

From (I.2.2) and considering (I.1.10) we have,

$$\text{(I.2.3)} \qquad \text{Res}_{\lambda=-n-2j}(r^\lambda, \varphi) = \frac{(-1)^{n+2j-1}}{(n+2j-1)!}, (\delta^{(n+2j-1)}(r), \varphi) \quad \text{for } j = 1, 2, \ldots$$

Comparing (I.2.3) with (I.2.1) we see that

$$\text{(I.2.4)} \qquad \delta^{(n+2j-1}(r) = a_{j,n} \Delta^j \delta(x) \quad \text{for } j = 1, 2, \ldots,$$

where

$$\text{(I.2.5)} \qquad a_{j,n} = \frac{(n+2j-1)!(-1)^{n+2j-1}\Omega_n \Gamma(\frac{n}{2})}{2^{2j} j! \Gamma(\frac{n}{2} + j)}.$$

Considering (I.1.13) and having in mind the formula (I.1.16), we rewrite the formula (I.2.5):

$$
\begin{aligned}
a_{j,n} &= \frac{2^{n+2j-1}\pi^{-1/2}\gamma(j+n/2)\Gamma(j+(n+1)/2)2\pi^{n/2}(-1)^{n+2j-1}}{2^{2j}j!\Gamma(j+n/2)} \\
&= \frac{2^n\pi^{n-1/2}\Gamma(j+(n+1)/2)(-1)^{n+2j-1}}{j!}.
\end{aligned} \tag{I.2.6}
$$

For the case $j=0$, from [3], pp. 72-73, we have,

$$
\text{Res}_{\lambda=-n}(r^\lambda,\varphi)=\Omega_n S_\varphi(0)=\Omega_n\varphi(0)=\Omega_n(\delta(x),\varphi(x)). \tag{I.2.7}
$$

That means that the generalized function $r^\lambda$ at $\lambda=-n$ has a simple pole whose residue is $\Omega_n.\delta(x)$.

From (I.2.3) we have,

$$
\text{Res}_{\lambda=-n}(r^\lambda,\varphi)=\frac{(-1)^{n-1}}{(n-1)!}(\delta^{(n-1)}(r),\varphi). \tag{I.2.8}
$$

Comparing (I.2.8) with (I.2.7) we see that

$$
\delta^{(n-1)}(r)=(n-1)!(-1)^{(n-1)}\Omega_n\delta(x). \tag{I.2.9}
$$

From (I.2.4),(I.2.6) and (I.2.9) we obtain the following formula:

$$
\delta^{(n+2j-1)}(r)=\begin{cases} a_{j,n}\Delta^j\delta(x) & \text{if } j=1,2,\ldots \\ (-1)^{n-1}(n-1)!\Omega_n\delta(x) & \text{if } j=0, \end{cases} \tag{I.2.10}
$$

where $a_{j,n}$ is defined by (I.2.6).

Actually, taking into account (I.2.8) we note that the formula:

$$
\frac{2}{\Omega_n}\frac{r^\lambda}{\Gamma(\frac{\lambda+n}{2})}\Big|_{\lambda=-n}=\delta(x_1,x_2,\ldots,x_n) \quad ([3],\ \text{p. } 74,76)
$$

is deduced from (I.2.9).

## II. Applications of the basic formula

In this paragraph we give a sense to the convolution distributional products of the form:

$$
\delta^{(n+2j-1)}(r)*\delta^{(n+2j-1)}(r) \quad \text{and} \quad \frac{r^\lambda}{\Gamma((\lambda+n)/2)}*\delta^{(n+2j-1)}(r)
$$

and to multiplicative distributional products $r^{-k} \cdot \Delta\delta(x)$.

**II.1. Convolution distributional product of the form $\delta^{(n+2j-1)}(r) * \delta^{(n+2l-1)}(r)$.**

Along this paragraph, by * we denote the convolution.

To give a sense to the convolution products of $\delta^{(n+2j-1)}(r) * \delta^{(n+2l-1)}(r)$ we use the convolutional theorem for the Fourier transformation with respect to $\delta^{(n+2j-1)}(r) * \delta^{(n+2l-1)}(r)$ and also use the formula

$$\text{(II.1.1)} \qquad \{\Delta^j \delta\}^\wedge = (-1)^j \rho^{2j} \quad ([3],\ \text{p. } 201),$$

$$\text{(II.1.2)} \qquad \rho^2 = y_1^2 + \ldots + y_n^2,$$

where $\Delta^j$ is defined by (I.1.9) and $\wedge$ denotes the Fourier transform:

$$\widehat{f} = \int_{R^n} f(x) e^{-i<x,y>} dx.$$

Let us observe that $\delta^{(n+2j-1)}(r)$, by virtue of (I.2.10) is a finite linear combination of $\delta$ and its derivatives; in consequence we conclude, that $\delta^{(n+2j-1)}(r)$ is a convolutor of $\mathcal{D}'$ (space of distributions), that is $\delta^{(n+2j-1)}(r)$ is a distribution of the class $Q_c$, where $Q_c$ ([5], p.244) is the space of rapidly decreasing distributions. Therefore, considering the classical theorem of Schwartz ([5], p. 268, (IV.8.5)), the following formula is valid:

$$\text{(II.1.3)} \qquad \{\delta^{(n+2j-1)}(r) * \delta^{(n+2l-1)}(r)\}^\wedge = \{\delta^{(n+2j-1)}(r)\}^\wedge . \{\delta^{(n+2l-1)}(r)\}^\wedge.$$

From (II.1.3) and considering (I.2.3), (II.1.2) we have,

$$\{\delta^{(n+2j-1)}(r) * \delta^{(n+2l-1)}(r)\}^\wedge = a_{j,n} . a_{l,n} \{\Delta^j \delta(x)\}^\wedge \cdot \{\Delta^l \delta(x)\}^\wedge$$

$$\text{(II.1.4)} \qquad = a_{j,n} a_{l,n} (-1)^{j+l} \rho^{2j} \cdot \rho^{2l} = a_{j,n} a_{l,n} (-1)^{j+l} \rho^{2(j+l)}$$

$$= a_{j,n} a_{l,n} \{\Delta^{j+l} \delta\}^\wedge = \frac{a_{i,n} a_{l,n}}{a_{j+l,n}} \{\delta^{(n+2(j+l)-1)}(r)\}^\wedge$$

for $j = 1, 2, \ldots$ and $l = 1, 2, \ldots$ and

$$\text{(II.1.5)} \qquad \{\delta^{(n-1)}(r) * \delta^{(n-1)}(r)\}^\wedge = (-1)^{n-1}(n-1)!\Omega_n \cdot (-1)^{n-1}(n-1)!\Omega_n$$

$$= (-1)^{n-1}(n-1)!\Omega_n \cdot ((-1)^{n-1}(n-1)!\Omega_n \delta(x))^\wedge = (-1)^{n-1}(n-1)!\Omega_n \cdot \{\delta^{(n-1)}(r)\}^\wedge.$$

Using the uniqueness theorem for the Fourier transform, from (II.1.3) and (II.1.4) we conclude that

$$\delta^{(n+2j-1)}(r) * \delta^{(n+2l-1)}(r) = \frac{a_{i,n}a_{l,n}}{a_{j+l,n}}\delta^{(n+2(j+l)-1)}(r), \tag{II.1.6}$$

$j = 1, 2, \dots \quad l = 1, 2, \dots$ and

$$\delta^{(n-1)}(r) * \delta^{(}n-1)(r) = (-1)^{(n-1)}\Omega_n(n-1)!\delta^{(n-1)}(r), \tag{II.1.7}$$

where using (I.2.6) we have

$$\frac{a_{i,n}a_{l,n}}{a_{j+l,n}} = \frac{2^n\pi^{\frac{n-1}{2}}\Gamma(\frac{n}{2}+j+\frac{1}{2})}{j!}\cdot\frac{2^n\pi^{\frac{n-1}{2}}\Gamma(\frac{n}{2}+l+\frac{1}{2})}{l!}.$$

(II.1.8)

$$\times\frac{(-1)^{n+2l-1}(j+l)!}{2^n\pi^{\frac{n-1}{2}}\Gamma(\frac{n}{2}+l+j+\frac{1}{2})} = \frac{(-1)^{n-1}\pi^{\frac{n-1}{2}}2^n(j+l)!\Gamma(\frac{n}{2}+j+\frac{1}{2})\Gamma(\frac{n}{2}+l+\frac{1}{2})}{j!l!\Gamma(\frac{n}{2}+l+j+\frac{1}{2})}.$$

Therefore from (II.1.6),(II.1.7) and considering (II.1.8) we obtain the following formula
(II.1.9)

$$\delta^{(n+2j-1)}(r) * \delta^{(n+2l-1)}(r) = \begin{cases} b_{j,l}\delta^{(n+2(l+j))}(r), & \text{for } j = 1, 2, \dots \\ (-1)^{n-1}(n-1)!\Omega_n\delta^{(n-1)}(r), & \text{for } j = 0, l = 0, \end{cases}$$

where

$$b_{j,l} = (-1)^{n-1}\pi^{\frac{n-1}{2}}2^n\frac{(j+l)!}{j!l!}\frac{\Gamma(\frac{n}{2}+j+\frac{1}{2})\Gamma(\frac{n}{2}+l+\frac{1}{2})}{\Gamma(\frac{n}{2}+l+j+\frac{1}{2})}. \tag{II.1.10}$$

## II.2. Convolution distributional product of $\frac{r^\lambda}{\Gamma(\frac{\lambda+n}{2})} * \delta^{(n+2j-1)}(r)$.

To give a sence to the convolution product $\frac{r^\lambda}{\Gamma(\frac{\lambda+n}{2})} * \delta^{(n+2j-1)}(r)$, we observe that the gamma-function $\Gamma((\lambda+n)/2)$ as well as $r^\lambda$ have simple poles at $\lambda = -n-2j$, $j = 0, 1, 2, \dots$ then $\frac{r^\lambda}{\Gamma((\lambda+n)/2)}$ is entire distribution in $\lambda$. Since $\delta^{(n+2j-1)}(r) \in Q_c$ and taking into account the theorem of Schwartz ([5], p.268, (IV.8.5)), the following formula follows:

$$\{\frac{r^\lambda}{\Gamma((\lambda+n)/2)} * \delta^{(n+2j-1)}(r)\}^\wedge = \{\frac{r^\lambda}{\Gamma((\lambda+n)/2)}\}\cdot\{\delta^{(n+2j-1)}(r)\}^\wedge. \tag{II.2.1}$$

From [3], p.194 the Fourier transform of $r^\lambda$ is

$$\{r^\lambda\}^\wedge = c_{\lambda,n}\Gamma((\lambda+n)/2)\,\rho^{-\lambda-n} \tag{II.2.2}$$

for all values of $\lambda$ except for $\lambda = -n-2j,\ j = 1,2,\ldots$, where $\rho^2$ is defined by (II.1.2), and

$$c_{\lambda,n} = \frac{2^{\lambda_n}\pi^{n/2}}{\Gamma(-\lambda/2)}. \tag{II.2.3}$$

From (II.2.1) and considering (I.2.10),(II.1.1),(II.2.2) we have,

(II.2.4)

$$\begin{aligned}\left\{\frac{r^\lambda}{\Gamma((\lambda+n)/2)} * \delta^{(n+2j-1)}(r)\right\}^\wedge &= c_{\lambda,n}a_{j,n}(-1)^j\rho^{-(\lambda-2j)-n}\\ &= \frac{c_{\lambda,n}a_{j,n}}{c_{\lambda-2j,n}}(-1)^j\left\{\frac{r^{\lambda-2j}}{\Gamma((\lambda-2j+n)/2)}\right\}^\wedge,\end{aligned}$$

if $j = 1,2\ldots$ and

(II.2.5)

$$\begin{aligned}\left\{\frac{r^\lambda}{\Gamma((\lambda+n)/2)} * \delta^{(n+2j-1)}(r)\right\}^\wedge &= c_{\lambda,n}(-1)^{n-1}(n-1)!\,\Omega_n\rho^{-\lambda-n}\\ &= (-1)^{n-1}(n-1)!\,\Omega_n\left\{\frac{r^\lambda}{\Gamma((\lambda+n)/2)}\right\}^\wedge,\end{aligned}$$

if $j = 0$, where $c_{\gamma,n}$ is defined by (II.2.3) and $a_{j,n}$ by (I.2.6).

Using the uniqueness for the Fourier transform, from (II.2.4) and (II.2.5) we obtain

$$\frac{r^\lambda}{\Gamma(\frac{\lambda+n}{2})} = \begin{cases} c_{\lambda,n}a_{j,n}(-1)^j\dfrac{r^{\lambda-2j}}{\Gamma((\lambda-2j+n)/2)}, & \text{if } j = 1,2,\ldots\\[2ex] (-1)^{n-1}(n-1)!\,\Omega_n\left\{\dfrac{r^\lambda}{\Gamma((\lambda+n)/2)}\right\}, & \text{if } j = 0\end{cases} \tag{II.2.6}$$

where from (I.2.6) and (II.2.3) we have

(II.2.7)

$$\begin{aligned}\frac{c_{\lambda,n}a_{j,n}}{c_{\lambda-2j,n}} &= \frac{2^{\lambda+n}\pi^{n/2}}{\Gamma(-\lambda/2)}\frac{\Gamma(-(\lambda-2j)/2)}{2^{\lambda-2j+n}\pi^{n/2}}\frac{2^n\pi^{(n-1)/2}\Gamma(j+(n+1)/2)(-1)^{n}2^{j-1}}{j!}\\ &= \frac{2^{2j+n}\pi^{(n-1)/2}(-1)^{n-1}\Gamma(j-\lambda/2)\Gamma(j+(n+1)/2)}{j!\Gamma(-\lambda/2)}.\end{aligned}$$

From (II.2.6) and considering (II.2.7) we conclude
(II.2.8)

$$\frac{r^\lambda}{\Gamma((\lambda+n)/2)}*\delta^{(n+2j-1)}(r)=\begin{cases} d_{\lambda,n,j}\dfrac{r^{\lambda-2j}}{\Gamma((\lambda-2j+n)/2)}, & \text{if } j=1,2,\ldots \\ (-1)^{n-1}(n-1)!\,\Omega_n\left\{\dfrac{r^\lambda}{\Gamma((\lambda+n)/2)}\right\}, & \text{if } j=0, \end{cases}$$

where

(II.2.9) $$d_{\lambda,n,j}=\frac{(-1)^j(-1)^{n-1}2^{2j+n}\pi^{(n-1)/2}\Gamma(j-\lambda/2)\Gamma(j+(n+1)/2)}{j!\Gamma(-\lambda/2)}.$$

N o t e: We note that putting $\lambda=-n-2l$, $l=0,1,2,\ldots$ in (II.2.8) and considering the equation (II.2.3), we obtain the formula (II.1.9).

In fact, from (II.2.8) and (II.2.9) we have
(II.2.10)

$$\lim_{\lambda\to-n-2l}\left(\frac{r^\lambda}{\Gamma((\lambda+n)/2)}*\delta^{(n+2j-1)}(r)\right)=d_{l,n,j}\lim_{\lambda\to-n-2l}\frac{r^{\lambda-2j}}{\Gamma((\lambda-2j+n)/2)}$$

if $l=1,2,\ldots$ and $j=1,2,\ldots$, and
(II.2.11)

$$\lim_{\lambda\to-n-2l}\frac{r^\lambda}{\Gamma((\lambda+n)/2)}*\delta^{(n+2j-1)}(r)=(-1)^{n-1}(n-1)!\,\Omega_n\lim_{\lambda\to-n-2l}\frac{r^\lambda}{\Gamma((\lambda+n)/2)},$$

if $j=0$ and $l=0$, where

(II.2.12) $$d_{l,n}=\frac{(-1)^j(-1)^{n-1}2^{2jn}\pi^{(n-1)/2}\Gamma(j+(n+1)/2)}{j!\Gamma(l+n/2)}.$$

On the other hand, the gamma-function $\Gamma((\lambda+n)/2)$ as well as $r^\lambda$ have simple poles at $\lambda=-n-2l$, $l=0,1,2\ldots$ and cnsidering the formulae (I.2.3), (I.1.14) and

(II.2.13) $$\operatorname{Res}_{z=-s}\Gamma(z)=\frac{(-1)^s}{s!}\quad \text{for } s=0,1,2,\ldots,$$

we have

(II.2.14) $$\lim_{\lambda\to-n-2l}\frac{r^\lambda}{\Gamma((\lambda+n)/2)}=\frac{(-1)^{n-1}l!}{2(-1)^l(n+2l-1)!}\delta^{(n+2l-1)}(r)$$

$$=\frac{(-1)^{n-1}l!\delta^{(n+2l-1)}(r)}{2(-1)^l2^{n+2l-1}\pi^{1/2}\Gamma(l+n/2)\Gamma(l+(n+1)/2)}$$

and

$$\lim_{\lambda\to-n-2l}\frac{r^{\lambda-2j}}{\Gamma((\lambda-2j+n)/2)}=\frac{(-1)^{n+2(l+j)-1}(l+j)!}{(n+2(l+j)-1)!2(-1)^{l+j}}\delta^{(n+2(l+j)-1)}(r)$$

(II.2.15) $$=\frac{(-1)^{n-1}(l+j)!\delta^{(n+2(l+j)-1)}}{2(-1)^{l+j}2^{n+2(l+j)-1}\pi^{1/2}\Gamma(l+j+n/2)\Gamma(l+j+(n+1)/2)}.$$

From (II.2.10), (II.2.11) and considering the formulae (II.2.12),(II.2.14) and (II.2.15), one easily verifies that

(II.2.16) $$\delta^{(n+2l-1)}(r)*\delta^{(n+2j-1)}(r)=$$

$$=\frac{2^n\pi^{(n-1)/2}(-1)^{n-2}(l+j)!\Gamma(l+(n+1)/2)\Gamma(j+(n+1)/2)}{j!\Gamma(l+j+(n+1)/2)}\delta^{(n+2(l+j)-1)}(r),$$

if $j=1,2,\dots$ and $l=1,2,\dots$.

Similarly for the cases $j=0$ and $l=0$ from (II.2.8) we have,

(II.2.17)

$$\lim_{lambda\to-n}\frac{r^{\lambda}}{\Gamma((\lambda+n)/2)}*\delta^{(n-1)}(r)=(-1)^{n-1}(n-1)!\Omega_n\lim_{\lambda\to-n}\frac{r^{\lambda}}{\Gamma((\lambda+n)/2)}$$

and considering the equation (I.2.3) for $j=0$ we have,

(II.2.18) $$\frac{(-1)^{n-1}}{(n-1)!}\frac{1}{2}*\delta^{(n-1)}(r)=\frac{(-1)^{n-1}(n-1)!\Omega_n(-1)^{n-1}}{2(n-1)!}\delta^{(n-1)}(r).$$

Finally, from (II.2.18) we obtain

(II.2.19) $$\delta^{(n-1)}(r)*\delta^{(n-1)}(r)=(-1)^{n-1}(n-1)!\Omega_n\delta^{(n-1)}(r).$$

The formulae (II.2.16) and (II.2.19) are $Q_c$-coinciding with the formula (II.1.9).

### II.3. The multiplicative distributional products of $r^{-k}\cdot\Delta^j\delta(x)$

We know ([3], p.72) that $r^{-k}$ is locally summable in $R^n$ for $k>n/2$.

To give a sense to the multiplicative distributional products of $r^{-k}\cdot\Delta^j\delta(x)$ we are to study the cases $r^{-2k}\cdot\Delta^j\delta(x)$ and $r^{1-2k}\cdot\Delta^j\delta(x)$ and to use essentially formula (I.2.10).

**Theorem 1.** *Let $k$ be a positive integer such that $k<n/2$ and $j$ be a non-negative integer, then*

(II.3.1) $$r^{-2k}\cdot\Delta^j\delta(x)=h_{j,n,k}\,\Delta^{j+k}\delta(x),$$

*where*

(II.3.2) $$h_{j,n,k} = \frac{j!\Gamma(j+n/2)}{2^{2k}(j+k)!\Gamma(j+k+n/2)}.$$

Proof. Taking into account the condition (I.1.5) for $\lambda = -2k$, $k = 1,2,\ldots$, we have

(II.3.3) $$k \neq \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, \ldots$$

From (I.2.10) and considering (II.3.3) we have,

(II.3.4) $$r^{-2k}\cdot \Delta^j\delta(x) = (a_{j,n})^{-1}r^{-2k}\cdot \delta^{(n+2j-1)}(r)$$

for $j - 1,2,\ldots$ and $k \neq \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2,\ldots$, where $a_{j,n}$ is defined by (I.2.6).

It follows from (I.2.5) that

(II.3.5) $$\delta^{(n+2j-1)}(r) = \frac{(n+2j-1)!}{(-1)^{n-1}} \lim_{lambda\to -n-2j} (\lambda+n+2j)r^{\lambda}$$

$$= \frac{(n+2j-1)!}{(-1)^{n-1}} \lim_{\alpha\to 0} \alpha r^{\alpha-n-2j}.$$

Denote by $A$ the left-hand side of (II.3.4) and considering (II.3.5) we have,

$$A = (a_{j,n})^{-1}r^{-2k}\frac{(n+2j-1)!}{(-1)^{n-1}} \lim_{\alpha\to 0} \alpha r^{\alpha-n-2j}$$

$$= (a_{j,n})^{-1}\frac{(n+2j-1)!}{(-1)^{n-1}} \lim_{\alpha\to 0} \alpha r^{\alpha-n-2(j+k)}$$

(II.3.6) $$= (a_{j,n})^{-1}\frac{(n+2j-1)!}{(-1)^{n-1}(n+2(j+k)-1)!}\delta^{(n+2(j+k)-1)}(r)$$

$$= (a_{j,n})^{-1}\frac{(n+2j-1)!}{n+2(j+k)-1)!}a_{j+k,n}\,\Delta^{j+k}\delta(x)$$

$$= \frac{j!}{2^{2k}(j+k)!}\,\frac{\Gamma(j+n/2)}{\Gamma(j+k+n/2)}\Delta^{j+k}\delta(x).$$

From (I.2.5) and (I.1.11) we have

(II.3.7) $$\frac{a_{j+k,n}}{a_{j,n}} = \frac{\Gamma(n+2(j+k))(-1)^{n-1}2\pi^{n/2}2^{2j}j!\Gamma(j+n/2)}{2^{2(j+k)}(j+k)!\Gamma(j+k+n/2)\Gamma(n+2j)(-1)^{n-1}2\pi^{n/2}}$$

and it is easily seen that

(II.3.8) $$\frac{(n+2j-1)!}{(n+2(j+k)-1)!}\frac{a_{j+k,n}}{a_{j,n}}=\frac{j!}{2^{2k}(j+k)!}\frac{\Gamma(j+n/2)}{\Gamma(j+k+n/2)}.$$

The theorem follows from (II.3.6) and (II.3.8). ∎

Theorem 1, formula (II.3.1), generalizes the neutrix product $r^{-2k}\circ\Delta\delta(x)$ given by Li Chen Kuan and Brian Fisher ([4], p.133, Theorem 2).

In fact, putting $j=1$ on (II.3.1) and considering (II.3.2) and (I.1.14) we have,

(II.3.9)
$$r^{-2k}.\Delta\delta(x) = b_{1,n,k}\Delta^{k+1}\delta(x)=\frac{\Gamma(1+n/2)}{2^{2k}(k+1)!\Gamma(k+1+n/2)}\Delta^{k+1}\delta(x)$$

$$=\frac{1}{2^k(k+1)!(n+2)(n+4)\dots(n+2k)}\Delta^{k+1}\delta(x)\quad \text{for } k<n/2.$$

The formula (II.3.9) coincides with the formula (I.1.7).

**Theorem 2.** *Let $k$ be a positive integer such that $k\le n/2$ and $j$ be a non-negative integer, then*

(II.3.10) $$r^{1-2k}.\Delta^j\delta(x)=0.$$

**Proof.** From (I.1.5) for $\lambda=1-2k$, $k=1,2,\dots$ we have

(II.3.11) $$k\neq\frac{n+1}{2},\frac{n+3}{2},\frac{n+5}{2},\dots$$

From(I.3.12) and considering (II.3.11) we have,

(II.3.12) $$r^{1-2k}.\Delta^j\delta(x)=(a_{j,n})^{-1}r^{1-2k}.\delta^{(n+2j-1)}(r)$$

for $j=1,2,\dots$

It follows from (II.3.12) and (I.2.3) that

$$r^{1-2k}.\Delta^j\delta(x)=(a_{j,n})^{-1}r^{1-2k}.\mathrm{Res}_{\lambda=-n-2j}r^\lambda\frac{(n+2j-1)!}{(-1)^{n+2j-1}}$$

(II.3.13) $$=(a_{j,n})^{-1}\frac{(n+2j-1)!}{(-1)^{n+2j-1}}\lim_{\lambda\to-n-2j}[(\lambda+n+2j)r^\lambda].$$

Denote by $B$ the left-hand side of (II.3.13) and considering the formula (II.2.13) we have

$$B = (a_{j,n})^{-1} r^{1-2k} \frac{2(-1)^j}{j!} \frac{(n+2j-1)!}{(-1)^{n+2j-1}} \lim_{\lambda \to -n-2j} \left\{ \frac{r^\lambda}{\Gamma((\lambda+n)/2)} \right\}. \tag{II.3.14}$$

From (II.3.14), using (I.2.5) and taking into account the condition (II.3.11) we have,

$$B = \frac{2^{2j}(-1)^j \Gamma(j+n/2)}{\pi^{n/2}} \lim_{\lambda \to -n-2j} \frac{r^{\lambda-2k+1}}{\Gamma((\lambda+n)/2)} \tag{II.3.15}$$

for $k \neq \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \dots$.

According to (I.1.5), $r^\lambda$ has simple pole at $\lambda = -n, -n-2, -n-4, \dots$ Therefore, the functional $r^{-n-[(2k+2j)-1]}$ exists and from (I.1.15) the following formula holds

$$\lim_{\lambda \to -n-2j} \frac{r^{\lambda-2k+1}}{\Gamma((\lambda+n)/2)}. \tag{II.3.16}$$

Theorem 2 follows from (II.3.13), (II.3.15) and (II.3.16). ■

Formula (II.3.10) generalizes the neutrix product $r^{1-2k} \circ \Delta\delta(x)$ given by Li Chen Kuan and Brian Fisher ([4], p.133, Th. 2). In fact, putting $j-1$ in (II.3.10) and taking into accaunt that the condition $k \neq \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \dots$ is equivalent to $k \leq n/2$, we have

$$r^{1-2k}.\Delta\delta(x) = 0 \quad \text{for } k = 1, 2, \dots n/2. \tag{II.3.17}$$

Formula (II.3.17) concides with formula (I.1.8).

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