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Joint Root Function Expansion of the Nonlocal Partial Integro-Differential Operators of First and Second Order [1]

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The paper deals with the convolutional structure of a system of the nonlocal partial integro-differential operators of first and second order, whose integral part is of Volterra type, as well as with the convolutional structure and uniqueness theorem for their expansion on joint root functions (joint eigen and associated functions). Sufficient conditions for completeness of the joint root functions of these operators in the spaces $L^p(\Omega), 1 \leq p < \infty, \Omega \subset \mathbf{R}^n$ are considered as well.

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0. Preliminaries

Let $\Omega = \Omega_1 \times \cdots \times \Omega_n$ be a Cartesian product of intervals $\Omega_j, j = 1, \ldots, n$. Then we denote by $L^1_{\text{loc}}(\Omega)$ the space of the Lebesgue locally integrable functions on Ω. Let $AC_j(\Omega)$ denote the space of these functions $f \in L^1_{\text{loc}}(\Omega)$, which are represented in the form

$$f(x_1, \ldots, x_n) = h + \int_0^{x_j} g(x_1, \ldots, \tau_j, \ldots, x_n)\, d\tau_j,$$

where $g, h \in L^1_{\text{loc}}(\Omega)$ and h does not depend on the variable x_j. Analogously, let $AC^1_j(\Omega)$ be the space of these functions $f \in L^1_{\text{loc}}(\Omega_j)$, which are represented in the same form, but with $g \in AC_j(\Omega)$, where h does not depend on the variable x_j.

Many nonlocal boundary value problems, which naturally arize (see [1]-[8]) are related to partial differential operators of first and second order, which

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are special cases of the operators $D_j, j = 1, \dots n$, defined by the partial integro-differential expressions

$$(1) \quad L_j y = A_j(x_j)\frac{\partial y(x_1, \dots, x_n)}{\partial x_j} + B_j(x_j)y(x_1, \dots, x_n) + \int_0^{x_j} Y_j(x_j, \tau_j)y(x_1, \dots, \tau_j, \dots, x_n)\, d\tau_j,$$

$$(2) \quad L_j y = a_j(x_j)\frac{\partial^2 y(x_1, \dots, x_n)}{\partial {x_j}^2} + b_j(x_j)\frac{\partial y(x_1, \dots, x_n)}{\partial x_j} + c_j(x_j)y(x_1, \dots, x_n) + \int_0^{x_j} Y_j(x_j, \tau_j)y(x_1, \dots, \tau_j, \dots, x_n)\, d\tau_j,$$

for $j = 1, \dots, n$, $y = y(x_1, \dots, x_n)$, considered in Ω with the following nonlocal boundary value conditions

$$(1') \qquad N_{j,x_j}[y(x_1, \dots, x_j, \dots, x_n)] = 0$$

for the operators of the form (1) , and

$$(2') \qquad \left[\beta_j \frac{\partial}{\partial x_j} y(x_1, \dots, x_n) + \alpha_j y(x_1, \dots, x_n)\right]\Big|_{x_j = x_j^0} = 0,$$

$$F_{j,x_j}\left[\frac{\partial}{\partial x_j} y(x_1, \dots, x_n)\right] + G_{j,x_j}\left[y(x_1, \dots, x_n)\right] = 0$$

for the operators of the form (2), where N_j, F_j, G_j are continuous linear functionals in $C(\Omega_j)$, and x_j^0 is an end point of the interval Ω_j and $x_j^0 \in \Omega_j$ for these j for which the expresions of the form (2) are considered. (Hereafter, the subscript x_j in these formulas shows that the functionals act on the variable x_j.) It is clear that by a change of the variables the operators of the above form can be reduced to the operators defined by the integro-differential expressions of the form

$$(3) \qquad l_j y = \frac{\partial y(t_1, \dots, t_n)}{\partial t_j} + \int_0^{t_j} V_j(t_j, \tau_j)y(t_1, \dots, \tau_j, \dots, t_n)\, d\tau_j,$$

$$(4) \qquad l_j y = \frac{\partial^2 y(t_1, \dots, t_n)}{\partial {t_j}^2} - q_j(t_j)y(t_1, \dots, t_n) + \int_0^{t_j} V_j(t_j, \tau_j)y(t_1, \dots, \tau_j, \dots, t_n)\, d\tau_j$$

for $j = 1, \dots, n$, $y = y(t_1, \dots, t_n)$, considered in with new nonlocal boundary value conditions

$$(3') \qquad N_{j,t_j}[y(t_1, \dots, t_j, \dots, t_n)] = 0$$

for the operators of the form (1), and

$$(4') \qquad \left[\beta_j \frac{\partial}{\partial t_j} y(t_1, \dots, t_n) + \alpha_j y(t_1, \dots, t_n)\right]\Bigg|_{t_j = t_j^0} = 0,$$

$$F_{j,t_j}\left[\frac{\partial}{\partial t_j} y(t_1, \dots, t_n)\right] + G_{j,t_j}[y(t_1, \dots, t_n)] = 0$$

for the operators of the form (4), with new $\beta_j, \alpha_j \in \mathbf{C}$ and new functionals N_j, F_j, G_j of $C(\Omega_j)^*$, where Ω_j are new intervals and $0 \in \Omega_j$ is a left end point of the interval Ω_j for these j for which the expresions of the form (4) are considered.

The aim of the paper is to study the convolutional structure of a system of partial integro-differential operators generated in the space $L^1_{\mathrm{loc}}(\Omega)$ by expressions of the form (3), (4) and nonlocal boundary value conditions of the form (3′), (4′) as well as the convolutional structure of their multidimensional expansion on joint root functions (eigen and accociated functions), when such an expansion exists. We use a general method for construction of multidimensional convolutions due to Bozhinov in [9].

Precisely, let $\Omega = \Omega_1 \times \cdots \times \Omega_n$ be a Cartesian product of intervals $\Omega_j, j = 1, \dots, n$, which contain the origin 0, and let $D_1, \dots, D_n$ be a system of operators generated by expressions of the form (3), (4) and nonlocal boundary value conditions of the form (3′), (4′). In the sake of simplicity here we consider the compact case that $\Omega_j = [-a_j, a_j]$ for these j for which D_j is generated by expression of the form (3) and that $\Omega_j = [0, a_j]$ for these j for which D_j is generated by expression of the form (4). So, now $L^1_{\mathrm{loc}}(\Omega) = L^1(\Omega)$. However, the results of Section 1 hold in the noncompact case too, but in the propositions the space $L^1(\Omega)$ should be replaced by $L^1_{\mathrm{loc}}(\Omega)$. We use the usual short denotation $t = (t_1, \dots, t_n)$. Then the domain X_{D_j} of the operator $D_j, j = 1, \dots, n$ in $L^1_{\mathrm{loc}}(\Omega)$ is defined in the following way:

1. For these j for which D_j is defined by expression of the form (3), let $V_j(t_j, \tau_j)$ be a continuous function in the set $\{(t_j, u_j) : -a_j \le t_j \le u_j \le 0\} \cup \{(t_j, u_j) : 0 \le u_j \le t_j \le a_j\}$ and let $\Phi_j \in C(\Omega_j)^*$ be a fixed nonzero functional. Then we consider

$$X_{D_j} \stackrel{\text{def}}{=} \{f \in AC_j(\Omega) : \Phi_{j,t_j}[f(t)] = 0$$

for all $t_p \in \Omega_p, p \neq j,\ p = 1, \ldots, n\}$.

2. For these j for which D_j is defined by expression of the form (4), let $q_j(t_j) \in L^1(\Omega_j)$ and $V_j(t_j, \tau_j)$ be a continuous function in the set $\{(t_j, u_j) : 0 \leq u_j \leq t_j \leq a_j\}$. Let $\alpha_j, \beta_j \in \mathbf{C}$ and $F_j, G_j \in C(\Omega_j)^*$ be such that $\chi_j^0(f) \stackrel{\text{def}}{=} \beta_j f'(0) + \alpha_j f(0)$ and $\Phi_j(f) \stackrel{\text{def}}{=} F_j(f') + G_j(f),\ f \in C^1(\Omega_j)$ be linearly independent continuous linear functionals in $C^1(\Omega_j)$. Then we consider

$$X_{D_j} \stackrel{\text{def}}{=} \{f \in AC_j^1(\Omega) : \chi^0_{j,t_j}[f(t)] = 0, \Phi_{j,t_j}[f(t)] = 0$$

$$\text{for all } t_p \in \Omega_p, p \neq j\ p = 1, \ldots, n\}.$$

It is clear that now the space $L^1_{\text{loc}}(\Omega_j) = L^1(\Omega_j)$ and the operator D_j be nonlocal integro-differential operator of first or of second order considered by the authors in [10]-[19]. So, according to the results of these papers the operator $D_j,\ j = 1, \ldots, n$ has a convolution $*_j$ in $L^1(\Omega_j)$, which represents its resolvent $R_{\lambda_j,j}$ in the form

$$(5) \qquad R_{\lambda_j,j} f = \left\{ -\frac{y_j(\lambda_j, t_j)}{E_j(\lambda_j)} \right\} *_j f, \quad f \in L^1(\Omega_j) \text{ with } \lambda_j \in \mathbf{C},\ E_j(\lambda_j) \neq 0,$$

where $E_j(\lambda_j) \stackrel{\text{def}}{=} \Phi_{j,t_j}[y_j(\lambda_j, t_j)]$, and $y_j(\lambda_j, t_j)$ is a suitable solution of the equation $D_j y_j = \lambda_j y_j$ in $L^1(\Omega_j)$. Let denote also by $R^0_{\lambda_j,j}$ the "initial resolvent" of the "initial operator" D^0_j, i.e. of the operator generated by the same expression (3) or (4) in $L^1(\Omega_j)$ but by initial conditions (see [10]-[16]). There we used that

$$(6) \qquad R^0_{\lambda_j,j} f = \{y_j(\lambda_j, t_j)\} \overset{0}{*}_j f, \quad f \in L^1(\Omega_j),$$

where $\overset{0}{*}_j$ is a convolution for the initial operator D^0_j in $L^1(\Omega_j)$.

It is clear that both resolvents $R_{\lambda_j,j}$, $R^0_{\lambda_j,j}$, $j = 1, \ldots, n$ can be extended by natural way in $L^1(\Omega)$, if we consider these operators acting only on the variable t_j of an arbitrary function $f \in L^1(\Omega)$. It can be proved easily that $X_{D_j} = R_{\lambda_j,j}(L^1(\Omega))$. That is the reason for $j = 1, \ldots, n$ we call the operators $R_{\lambda_j,j}$, $R^0_{\lambda_j,j}$ *partial resolvent* or *partial initial resolvent* of the operator D_j in $L^1(\Omega)$ respectively.

1. Convolutional structure of the nonlocal partial integro-differential operators of first and second order

Using the method proposed by Bozhinov in [9] we can construct a convolution $*$ in $L^1(\Omega)$ for all nonlocal partial integro-differential operators $D_1, \ldots, D_n$.

This convolution is in essence the tensorial product $* \stackrel{\text{def}}{=} *_1 \otimes \cdots \otimes *_n$ of the bilinear operations $*_1, \dots *_n$, which is defined in the tensorial product $L^1(\Omega_1) \otimes \cdots \otimes L^1(\Omega_n)$ of the spaces $L^1(\Omega_1), \dots, L^1(\Omega_n)$ and is extended by continuity in the space $L^1(\Omega)$. In [9] Bozhinov shows how it is possible to constuct an explicit expression for this extension for all possible combinations of the operators D_1, D_2 of the form (3), (4), when $V_j \equiv 0$ and $n = 2$. The case of arbitrary V_j can be reduced to the case $V_j \equiv 0$ using transmutation operators considered in [20]-[22], which transform the integro-differential operators of first or second order in the operators $\dfrac{\partial}{\partial t_j}$ or $\dfrac{\partial^2}{\partial t_j{}^2}$ respectively. The case of arbitrary n is an evident generalisation of the case $n = 2$, but now the expressions are more complicated. According to the construction proposed in [9] it is clear that the convolution $*$ in $L^1(\Omega)$ has the important "splitting property"

$$(7) \qquad f * g = \prod_{j=1}^{n} f_j(t_j) *_j g_j(t_j) \quad \text{for} \quad f, g \in L^1(\Omega_1) \otimes \cdots \otimes L^1(\Omega_n),$$

i.e. for $f, g \in L^1(\Omega)$ of the form $f(t) = f_1(t_1) \dots f_n(t_n)$, $g(t) = g_1(t_1) \dots g_n(t_n)$. Using this splitting property many propositions for the convolution $*$ can be proved. Let us denote $\overset{(0)}{*} = \overset{0}{*}_1 \otimes \cdots \otimes \overset{0}{*}_n$, which is defined in $L^1(\Omega_1) \otimes \cdots \otimes L^1(\Omega_n)$, and is extended by continuity in $L^1(\Omega)$. (Its existence follows by the general construction in [9] as well.) Also, let $y(\lambda, t) \stackrel{\text{def}}{=} y(\lambda_1, t_1) \dots y(\lambda_n, t_n)$ and $E(\lambda) \stackrel{\text{def}}{=} E_1(\lambda_1) \dots E_n(\lambda_n)$.

Theorem 1. a) *The operation $* : L^1(\Omega) \times L^1(\Omega) \to L^1(\Omega)$ is a continuous, bilinear, commutative and associative operation in $L^1(\Omega)$ for the operators $D_1, \dots, D_n$ and for their partial resolvents $R_{\lambda_1,1}, \dots, R_{\lambda_n,n}$, i.e. their domains $X_{D_1}, \dots, X_{D_n}$ are ideals of the convolution algebra $(L^1(\Omega), *)$ and the equalities*

$$(8) \qquad D_j(f * g) = (D_j f) * g, \quad f \in X_{D_j},\ g \in L^1(\Omega),$$

$$(9) \qquad R_{\lambda_j,j}(f * g) = (R_{\lambda_j,j}) * g, \quad f, g \in L^1(\Omega)$$

hold for arbitrary $\lambda \in \mathbf{C}^n$ with $E(\lambda) \neq 0$ and $j = 1, \dots, n$.

b) *For each fixed $\lambda \in \mathbf{C}^n$ with $E(\lambda) \neq 0$ the equality*

$$(10) \qquad R_\lambda f \stackrel{\text{def}}{=} R_{\lambda_1,1} \dots R_{\lambda_n,n} f = \left\{ -\frac{y(\lambda, t)}{E(\lambda)} \right\} * f \quad \textit{for } f \in L^1(\Omega).$$

holds. The element $r_\lambda \stackrel{\text{def}}{=} \{-y(\lambda, t)/E(\lambda)\}$ is a nondivisor of zero of the convolution algebra $(L^1(\Omega), *)$.

c) *The operation* $\overset{(0)}{*}$ *is a coninuous convolution in* $L^1(\Omega)$ *of the partial initial resolvents* $R^0_{\lambda_1,1},\dots,R^0_{\lambda_n,n}$ *and the equality*

$$(10') \qquad R^{(0)}_\lambda f \overset{\text{def}}{=} R^0_{\lambda_1,1}\dots R^0_{\lambda_n,n} f = \{y(\lambda,t)\}\overset{(0)}{*} f \quad \text{for } f\in L^1(\Omega),\ \lambda\in \mathbf{C}^n$$

holds.

Proof. The continuity of the operation $\overset{(0)}{*}$ as well as the continuity of $*$ follows from the construction of these operations as an extension by continuity on the closure $L^1(\Omega)$ of the tensorial product $L^1(\Omega_1)\otimes\dots\otimes L^1(\Omega_n)$. We note that the operation $\overset{(0)}{*} = \overset{0}{*}_1\otimes\dots\otimes\overset{0}{*}_n$ has a "splitting property" analogous to (7), but with respect to the operations $\overset{0}{*}_1,\dots,\overset{0}{*}_n$. Then, using this property and the relations (5), (6) as well as the "splitting property" (7) we can prove easily that the equalities (9), (10) and (10′) hold for "splitting" functions of the form $f(t) = f_1(t_1)\dots f_n(t_n)$, $g(t) = g_1(t_1)\dots g_n(t_n)$. Hence the bilinearity of both sides of these relations imply that they are true for linear combinations of "splitting functions", which are dense in $L^1(\Omega)$ since the the polynomials are special kind of such linear combinations. Then using this density and the $L^1(\Omega)\to L^1(\Omega)$ continuity of right and left sides of the equalities (9), (10), (10′) we obtain that these equalities are true in $L^1(\Omega)$ as well. The equality (8) follows from (9). ■

2. Convolutional structure of the expansion on joint root functions of the nonlocal partial integro-differential operators of first and second order

The results in the present section are true by the assumption that Ω_j, $j = 1,\dots,n$ are compact intervals in contrast to previous section, where this assumption was not necessary. Precisely, let $\Omega_j = [-a_j, a_j]$ for these j for which D_j is generated by expression of the form (3) and that $\Omega_j = [0, a_j]$ for these j for which D_j is generated by expression of the form (4).

Let $\Lambda_j = \{\lambda^j_{k_j}\}^\infty_{k_j=0}$ be the set of the zeros of the entire function $E_j(\lambda_j)$ with corresponding multiplicities $\{\varkappa^j_{k_j}\}^\infty_{k_j=0}$ and let for each fixed $j = 1,\dots,n$ let $f_j(t_j)\sim\sum^\infty_{k_j=0} P^j_{k_j} f_j$ for $f\in L^1(\Omega_j)$ be the one-dimensional root function expansion of the operator D_j in $L^1(\Omega_j)$ considered in [18], [19] for ordinary integro-differential operators of the form (3) or (4) respectively.

A necessary and sufficient condition for validity of a uniqueness theorem for this one-dimensional expansion, i.e. for the totality of the projection system

$\{P^j_{k_j}\}_{k_j=0}^{\infty}$ in $L^1(\Omega_j)$, is the compactness of the interval Ω_j and the condition $-a_j, a_j \in \operatorname{supp} \Phi_j$ or the condition $a_j \in \operatorname{supp} \Phi_j$ when the operator D_j is of the form (3) or (4) respectively. According to the results in [18], [19] for $j = 1, \dots, n$ the one-dimensional root projections

$$(11) \qquad P^j_{k_j} f_j = \frac{1}{2\pi i} \int\limits_{\Gamma^j_{k_j}} \frac{\Phi_j(R^0_{\lambda_j, j} f_j) y(\lambda_j, t_j)}{E_j(\lambda_j)} d\lambda_j, \quad f_j \in L^1(\Omega_j)$$

have the convolutional representation

$$(11') \quad P^j_{k_j} f_j = f_j *_j \varphi^j_{k_j}, \quad f_j \in L^1(\Omega_j) \quad \text{with} \quad \varphi^j_{k_j} = \frac{1}{2\pi i} \int\limits_{\Gamma^j_{k_j}} \frac{y(\lambda_j, t_j)}{E_j(\lambda_j)} d\lambda_j,$$

where $\Gamma^j_{k_j}$ are contours enclosing $\lambda^j_{k_j}$. It is clear that

$$\mathcal{V}^j_{k_j} = \left\{ \frac{1}{s_j!} \frac{\partial^{s_j}}{\partial \lambda_j^{s_j}} y_j(\lambda^j_{k_j}, t_j) : 0 \le s_j \le \varkappa^j_{k_j} \right\}$$

is a chain of root functions of the operator D_j in the root subspace $\operatorname{Ker}(D_j - \lambda^j_{k_j})^{\varkappa^j_{k_j}} \subset L^1(\Omega_j)$. (A system $\mathcal{W}^j_{k_j} = \{w_0^{k_j}(t_j), \dots, w^{k_j}_{\varkappa^j_{k_j}-1}(t_j)\} \subset \operatorname{Ker}(D_j - \lambda^j_{k_j})^{\varkappa^j_{k_j}}$ is said to be a *chain* of root functions (eigen and associated functions) of the operator D_j corresponding to the eigenvalue $\lambda^j_{k_j}$, iff $D_j w_0^{k_j} = \lambda^j_{k_j} w_0^{k_j}$, $D_j w_s^{k_j} = \lambda^j_{k_j} w_s^{k_j} + w^{k_j}_{s-1}$, $1 \le s \le \varkappa^j_{k_j} - 1$ for these j for which $\varkappa^j_{k_j} > 1$.)

For simplicity sake, hereafter we use some multiindex denotations proposed in [20], 2.4.3.

The multiindices $(p) = (p_1, \dots, p_n)$, $p_j \ge 0, j = 1, \dots, n$ are considered as nonnegative integer n-dimensional vectors, provided with the summation operation and a partial order relation $(p) \le (q) \iff p_j \le q_j$, $j = 1, \le n$. If $(p) \le (q)$ and $(p) \ne (q)$ we use the denotation $(p) < (q)$. Also, let $(0) \overset{\text{def}}{=} (0, \dots, 0)$, $(1) \overset{\text{def}}{=} (1, \dots, 1)$, $(\infty) \overset{\text{def}}{=} (\infty, \dots, \infty)$, $\lambda^{(k)} \overset{\text{def}}{=} \lambda_1^{k_1} \dots \lambda_n^{k_n}$, $(k)! \overset{\text{def}}{=} = k_1! \dots k_n!$ The sign $\sum\limits_{(s)=(p)}^{(q)}$ denotes summation over all multiindices (s) with $(p) \le (s) \le (q)$, i.e. $\sum\limits_{(s)=(p)}^{(q)} = \sum\limits_{s_1=p_1}^{q_1} \dots \sum\limits_{s_n=p_n}^{q_n}$. Let $\Lambda = \{\lambda_{(k)}\}_{(k)=(0)}^{(\infty)}$ with $\lambda_{(k)} \overset{\text{def}}{=} (\lambda^1_{k_1}, \dots, \lambda^n_{k_n})$ be the set of the joint multieigenvalues $\lambda_{(k)}$ of the operators $D_1, \dots, D_n$ of their joint spectrum. Let $\Gamma_{(k)} = \Gamma^1_{k_1} \times \dots \times \Gamma^n_{k_n}$.

Also, let $D \stackrel{\text{def}}{=} D_1, \ldots, D_n$,

$$\int_{\zeta}^{z} f(\tau)\,d\tau \stackrel{\text{def}}{=} \int_{\zeta_1}^{z_1} \cdots \int_{\zeta_n}^{z_n} f(\tau_1, \ldots, \tau_n)\,d\tau_1 \ldots d\tau_n, \quad z, \zeta \in \mathbf{C}^n,$$

$$\partial^{(k)} \stackrel{\text{def}}{=} \frac{\partial^{(k)}}{\partial\lambda^{(k)}} \stackrel{\text{def}}{=} \frac{\partial^{k_1}}{\partial{\lambda_1}^{k_1}} \cdots \frac{\partial^{k_n}}{\partial{\lambda_n}^{k_n}},$$

$$D^{(k)} \stackrel{\text{def}}{=} D_1^{k_1} \ldots D_n^{k_n}, \quad (D-\lambda)^{(k)} \stackrel{\text{def}}{=} (D_1-\lambda_1)^{k_1} \ldots (D_n-\lambda_n)^{k_n}.$$

It can be easily proved that for each multiindex (k) the tensorial product

$$V_{(k)} = V_{k_1}^1 \otimes \cdots \otimes V_{k_n}^n = \left\{ \frac{1}{(s)!} \frac{\partial^{(s)}}{\partial\lambda^{(s)}} y(\lambda_{(k)}, t) : \quad (0) \le (s) \le \varkappa(k) \right\}$$

is a chain basis system in the joint root subspace

$$H_{(k)} \stackrel{\text{def}}{=} \operatorname{Ker}(D_1 - \lambda_{k_1}^1)^{\varkappa_{k_1}^1} \cap \ldots \cap \operatorname{Ker}(D_n - \lambda_{k_n}^n)^{\varkappa_{k_n}^n} \subset L^1(\Omega)$$

corresponding to $\lambda_{(k)}$.

Let $R_\lambda^{(0)} \stackrel{\text{def}}{=} R_{\lambda_1,1}^0 \ldots R_{\lambda_n,n}^0$. Also, let us consider the functional

$$\Phi f \stackrel{\text{def}}{=} \Phi_t[f(t)] = \Phi_{1,t_1} \ldots \Phi_{n,t_n}[f(t_1, \ldots t_n)],$$

which is defined for these $f \in C(\Omega)$ for which the right part of these equalities exists (we recall that $\Phi_j \in C^1(\Omega)^*$). We do not give here the exact domain of the functional Φ, but we note that the functional $\Phi \circ R_\lambda^{(0)}$ is well-defined continuous linear functional in $L^1(\Omega)$, since $R_{\lambda_j}^j$ maps $L^1(\Omega_j)$ to the space $AC(\Omega_j)$ or to the space $AC^1(\Omega_j)$ depending on the fact whether D_j is of the form (3) or (4) respectively. Indeed, for each $f \in L^1(\Omega)$ we have

$$\begin{aligned} \Phi(R_\lambda^{(0)} f) &= \Phi_{1,t_1} \ldots \Phi_{n,t_n}[(R_{\lambda_1,1}^0 \ldots R_{\lambda_n,n}^0 f)(t_1, \ldots t_n)] \\ &= (\Phi_1 \circ R_{\lambda_1,1}^0)_{t_1} \ldots (\Phi_n \circ R_{\lambda_n,n}^0)_{t_n}[f(t_1, \ldots t_n)]. \end{aligned}$$

Also, let $\varkappa_{(k)} \stackrel{\text{def}}{=} (\varkappa_{k_1}^1, \ldots, \varkappa_{k_n}^n)$, $\nu_{(k)} \stackrel{\text{def}}{=} \varkappa_{(k)} - (1) = (\varkappa_{k_1}^1 - 1, \ldots, \varkappa_{k_n}^n - 1)$. By these denotations the next proposition make clear the convolutional structure of the joint root subspace $H_{\lambda_{(k)}}$:

Theorem 2. a) *For each multiindex $(k) \ge (0)$ the operator*

$$P_{(k)} f \stackrel{\text{def}}{=} f * \varphi_{(k)}, \quad f \in L^1(\Omega), \tag{12}$$

where

$$\varphi_{(k)}(t) \overset{\text{def}}{=} \prod_{j=1}^{n} \varphi^{j}_{k_j}(t_j) = \frac{1}{(2\pi i)^n} \int_{\Gamma_{(k)}} \frac{y(\lambda,t)}{E(\lambda)}\, d\lambda$$

$$= \frac{1}{\nu_{(k)}!} \frac{\partial^{\nu_{(k)}}}{\partial \lambda^{\nu_{(k)}}} \left[\frac{(\lambda - \lambda_{(k)})^{\varkappa_{(k)}} y(\lambda,t)}{E(\lambda)} \right] \Bigg|_{\lambda=\lambda_{(k)}} \in H_{(k)},$$

is a convolutional multiplier projection, mapping $L^1(\Omega)$ onto the joint root subspace $H_{(k)}$ of the operator system $D_1, \dots, D_n$. The projection $P_{(k)}$ commutes with all $D_1, \dots, D_n$ and it is represented in the form

$$(13) \qquad P_{(k)} f = \frac{1}{(2\pi i)^n} \int_{\Gamma_{(k)}} \frac{\Phi(R_\lambda^{(0)} f) y(\lambda,t)}{E(\lambda)}\, d\lambda$$

$$= \frac{1}{\nu_{(k)}!} \frac{\partial^{\nu_{(k)}}}{\partial \lambda^{\nu_{(k)}}} \left[\frac{(\lambda - \lambda_{(k)})^{\varkappa_{(k)}} \Phi(R_\lambda^{(0)} f) y(\lambda,t)}{E(\lambda)} \right] \Bigg|_{\lambda=\lambda_{(k)}}$$

The projection $P_{(k)}$ is the unique continuous projection mapping $L^1(\Omega)$ onto $H_{(k)}$ and commuting with $D_1, \dots, D_n$. It is the unique nontrivial continuous projection mapping $L^1(\Omega)$ in $H_{(k)}$ and commuting with $D_1, \dots, D_n$.

b) *The function $\varphi_{(k)} \in H_{(k)}$ is an idempotent of the convolution algebra $(L^1(\Omega), *)$, i.e. $\varphi_{(k)} * \varphi_{(k)} = \varphi_{(k)}$ for each $(k) \geq (0)$. If $(k) \neq (p)$, then $\varphi_{(k)} * \varphi_{(p)} = 0$. The functions*

$$(14) \qquad \overset{(k)}{u}_{(s)} = \frac{1}{(2\pi i)^n} \int_{\Gamma_{(k)}} \frac{(\lambda - \lambda_{(k)})^{\nu_{(k)} - (s)} y(\lambda,t)}{E(\lambda)}\, d\lambda$$

$$= \frac{1}{(s)!} \frac{\partial^{(s)}}{\partial \lambda^{(s)}} \left[\frac{(\lambda - \lambda_{(k)})^{\varkappa_{(k)}} y(\lambda,t)}{E(\lambda)} \right] \Bigg|_{\lambda=\lambda_{(k)}}, \quad (0) \leq (s) \leq \nu_{(k)}$$

form a "good" chain basis in $H_{(k)}$ with respect to $*$,i.e.

$$(15) \qquad \overset{(k)}{u}_{(p)} * \overset{(k)}{u}_{(q)} = \begin{cases} \overset{(k)}{u}_{(p)+(q)-\nu_{(k)}}, & \text{if } (p)+(q) \geq \nu_{(k)} \\ 0\,, & \text{if } (p)+(q) \text{ is not } \geq \nu_{(k)}. \end{cases}$$

Moreover, $\varphi_{(k)} = \overset{(k)}{u}_{\nu_{(k)}}$, $\overset{(k)}{u}_{(s)} = (D - \lambda_{(k)})^{\nu_{(k)} - (s)} \overset{(k)}{u}_{\nu_{(k)}}$, $(0) \leq (s) \leq \nu_{(k)}$.

c) *Let* $E(\lambda)=(\lambda-\lambda_{(k)})^{\varkappa_{(k)}}\sum\limits_{(l)=(0)}^{(\infty)}\overset{(k)}{\alpha}_{(l)}(\lambda-\lambda_{(k)})^{(l)}$, $(0)\le(s)\le\nu_{(k)}$ *with* $\overset{(k)}{\alpha}_{(0)}\neq 0$ *is the Taylor expansion of* $E(\lambda)$ *around* $\lambda_{(k)}$. *Then*

$$\frac{1}{(s)!}\frac{\partial^{(s)}}{\partial\lambda^{(s)}}y(\lambda_{(k)},t)=\sum_{(l)=(0)}^{(s)}\overset{(k)}{\alpha}_{(s)-(l)}\overset{(k)}{u}_{(l)}(t), \tag{16}$$

$$\overset{(k)}{u}_{(s)}(t)=\sum_{(l)=(0)}^{(s)}\overset{(k)}{\beta}_{(s)-(l)}\frac{1}{(l)!}\frac{\partial^{(l)}}{\partial\lambda^{(l)}}y(\lambda_{(k)},t),$$

where $\overset{(k)}{\beta}_{(s)}$, $(0)\le(s)\le\nu_{(k)}$ *are the coefficients in the Taylor expansion of the function* $(\lambda-\lambda_{(k)})^{\varkappa_{(k)}}/E(\lambda)$ *around* $\lambda_{(k)}$ *and the equalities*

$$\overset{(k)}{\alpha}_{(0)}\overset{(k)}{\beta}_{(0)}=1,\qquad \sum_{(p)=(0)}^{(s)}\overset{(k)}{\alpha}_{(p)}\overset{(k)}{\beta}_{(s)-(p)}\quad for\quad (0)<(s)\le\nu_{(k)},$$

hold, when (k) *is such that* $\nu_{(k)}>(0)$, *i.e.* $\varkappa_{(k)}>(1)$. *For* $(0)\le(s)\le\nu_{(k)}$ *and* $(l)\overset{\mathrm{def}}{=}(p)+(q)-\nu_{(k)}$ *the following equalities hold:*

(17)

$$\overset{(k)}{\alpha}_{(s)}=\frac{1}{(2\pi i)^n}\int\limits_{\Gamma_{(k)}}\frac{E(\lambda)}{(\lambda-\lambda_{(k)})^{\varkappa_{(k)}+(s)+(1)}}\,d\lambda=\frac{1}{(s)!}\frac{\partial^{(s)}}{\partial\lambda^{(s)}}\left[\frac{E(\lambda)}{(\lambda-\lambda_{(k)})^{\varkappa_{(k)}}}\right]\Bigg|_{\lambda=\lambda_{(k)}},$$

$$\overset{(k)}{\beta}_{(s)}=\frac{1}{(2\pi i)^n}\int\limits_{\Gamma_{(k)}}\frac{(\lambda-\lambda_{(k)})^{\nu_{(k)}-(s)}}{E(\lambda)}\,d\lambda=\frac{1}{(s)!}\frac{\partial^{(s)}}{\partial\lambda^{(s)}}\left[\frac{(\lambda-\lambda_{(k)})^{\varkappa_{(k)}}}{E(\lambda)}\right]\Bigg|_{\lambda=\lambda_{(k)}},$$

$$\frac{1}{(p)!}\frac{\partial^{(p)}}{\partial\lambda^{(p)}}y(\lambda_{(k)},t)*\frac{1}{(q)!}\frac{\partial^{(q)}}{\partial\lambda^{(q)}}y(\lambda_{(k)},t)= \tag{18}$$

$$=\begin{cases}0 & \text{for these } (p),(q) \text{ for which } (p)+(q) \text{ is not} \ge\nu_{(k)}\\ \dfrac{1}{(l)!}\dfrac{\partial^{(l)}}{\partial\lambda^{(l)}}\left[\dfrac{E(\lambda)}{(\lambda-\lambda_{(k)})^{\varkappa_{(k)}}y(\lambda,t)}\right]\Bigg|_{\lambda=\lambda_{(k)}} & \text{if } (p)+(q)\ge\nu_{(k)},\end{cases}$$

$$=\begin{cases}0 & \text{for these } (p),(q) \text{ for which } (p)+(q) \text{ is not} \ge\nu_{(k)}\\ \sum\limits_{(s)=(0)}^{(l)}\overset{(k)}{\alpha}_{(l)-(s)}\dfrac{1}{(s)!}\dfrac{\partial^{(s)}}{\partial\lambda^{(s)}}y(\lambda_{(k)},t), & \text{if } (p)+(q)\ge\nu_{(k)}.\end{cases}$$

The projection representation

$$(19)\quad P_{(k)}f = \sum_{(l)=(0)}^{\nu(k)} \overset{(k)}{C}_{\nu_{(k)}-(l)}(f)\overset{(k)}{u}_{(l)}(t) = \sum_{(l)=(0)}^{\nu(k)} \overset{(k)}{A}_{\nu_{(k)}-(l)}(f)\frac{1}{(l)!}\frac{\partial^{(l)}}{\partial\lambda^{(l)}}y(\lambda_{(k)},t)$$

holds for $f \in L^1(\Omega)$ *with respect to the "good" chain basis* $\mathcal{U}_{(k)} = \{\overset{(k)}{u}_{(s)} : (0) \leq (s) \leq \nu_{(k)}\}$ *and the "traditional" chain basis* $\mathcal{V}_{(k)}$ *in* $H_{(k)}$. *The coefficient functionals* $\overset{(k)}{C}_{(s)}(f)$, $\overset{(k)}{A}_{(s)}(f)$ *in the last formula are represented in the form*

$$(20)\quad \overset{(k)}{C}_{(s)}(f) = \frac{1}{(2\pi i)^n}\int_{\Gamma_{(k)}} \frac{\Phi(R_\lambda^{(0)}f)}{(\lambda-\lambda_{(k)})^{(s)+(1)}}\,d\lambda = \frac{1}{(s)!}\frac{\partial^{(s)}}{\partial\lambda^{(s)}}\Phi(R_\lambda^{(0)}f)\Big|_{\lambda=\lambda_{(k)}},$$

$$= \Phi\Big\{f \overset{(0)}{*} \frac{1}{(s)!}\frac{\partial^{(s)}}{\partial\lambda^{(s)}}y(\lambda_{(k)},t)\Big\},$$

$$(20')\quad \overset{(k)}{A}_{(s)}(f) = \frac{1}{(2\pi i)^n}\int_{\Gamma_{(k)}} \frac{(\lambda-\lambda_{(k)})^{\nu_{(k)}-(s)}\Phi(R_\lambda^{(0)}f)}{E(\lambda)}\,d\lambda$$

$$= \frac{1}{(s)!}\frac{\partial^{(s)}}{\partial\lambda^{(s)}}\left[\frac{(\lambda-\lambda_{(k)})^{\varkappa(k)}\Phi(R_\lambda^{(0)}f)}{E(\lambda)}\right]\Bigg|_{\lambda=\lambda_{(k)}} = \Phi\{f \overset{(0)}{*} \overset{(k)}{u}_{(s)}\}.$$

For $(0) \leq (s) \leq \nu_{(k)}$ *these coefficient functionals are connected with the equalities*

$$(21)\qquad \overset{(k)}{C}_{(s)} = \sum_{(l)=(0)}^{(s)} \overset{(k)}{\alpha}_{(s)-(l)}\overset{(k)}{A}_{(s)},\quad \overset{(k)}{A}_{(s)} = \sum_{(l)=(0)}^{(s)} \overset{(k)}{\beta}_{(s)-(l)}\overset{(k)}{C}_{(s)}$$

and have the following multiplicative convolutional propeties

$$(22)\qquad \overset{(k)}{C}_{(s)}(f*g) = \sum_{(l)=(0)}^{(s)} \overset{(k)}{C}_{(s)-(l)}(f)\overset{(k)}{C}_{(s)}(f),$$

$$\overset{(k)}{A}_{(s)}(f*g) = \sum_{(l)=(0)}^{(s)} \overset{(k)}{\alpha}_{(s)-(l)} \sum_{(j)=(0)}^{(l)} A_{(l)-(j)}(f)\overset{(k)}{A}_{(j)}(f)$$

for all $f, g \in L^1(\Omega)$.

P r o o f. In the one-dimensional case $n = 1$ the operator D_1 is generated by ordinary integro-differential expression of the form

$$ly = y'(t_1) + \int_0^{t_1} V(t_1, u_1)y(u_1)du_1, \quad t_1 \in [-a_1, a_1]$$

or of the form

$$ly = y''(t_1) - q(t_1)y(t_1) + \int_0^{t_1} V(t_1, u_1)y(u_1)du_1, \quad t_1 \in [0, a_1].$$

The cases of these operators were considered by the authors in [16] and [12] respectively, where theorem 2 were proved in the one-dimensional case $n = 1$.

Let us consider the multi-dimensional case $n > 1$. Now we prove formulae (12), (13) for "splitting functions" using formulae (11), (11′), the "splitting property" (7) of the convolution $*$ and the fact that

$$P_{(k)}f = \prod_{j=1}^{n} P_{k_j,j} f_j,$$

where $P_{k_j,j}$ is of the form (11) and $f(t) = f_1(t_1)\dots f_n(t_n)$ is a "splitting function". Hence the bilinearity of both sides of these relations implies that they are true for linear combinations of "splitting functions", which are dense in $L^1(\Omega)$. Then using this density and the $L^1(\Omega) \to L^1(\Omega)$ continuity of right and left sides of the equalities (12), (13) we obtain that they are true in $L^1(\Omega)$ as well.

Other part of the proof follows analogously from the one-dimensional case by approximation with linear combinations of "splitting functions". ∎

From theorem 2 b) it follows that the projections $P_{(k)}$ form an orthogonal projection system $\{P_{(k)}\}_{(k)=(0)}^{(\infty)}$ in $L^1(\Omega)$, i.e. $P_{(k)}P_{(p)} = 0$, if $(k) \neq (p)$.

The projection system $\{P_{(k)}\}_{(k)=(0)}^{(\infty)}$ is said to be *total* in the space $L^1(\Omega)$, if the equalities $P_{(k)}f = 0, (k) \geq (0)$ for $f \in L^1(\Omega)$ imply $f = 0$ almost everywhere in Ω. The next theorem gives necessary and sufficient conditions for the totality of this system, i.e. necessary and sufficient conditions for validity of a uniqueness theorem for the formal expansion $f \sim \sum_{(k)=(0)}^{(\infty)} P_{(k)}f$.

First we formulate a lemma.

Lemma 1. *Let $g(t_1, \dots, t_n) \in L^1(\Omega)$ and let $j = 1, \dots, n$ be arbitrarily fixed. Let*

$$P_{k_j,j,x_j}\{g(t_1, \dots, x_j, \dots, t_n)\}(t_j) = 0$$

for all $k_j = 1, 2, 3, \dots$, for almost all $t_j \in \Omega_j$ and almost all $t_p \in \Omega_p$, where $1 \leq p \leq n, p \neq j$ with respect to the corresponding Lebesgue measure in Ω_j, Ω_p. Then $g = 0$ almost everywhere in Ω with respect to the Lebesgue measure in Ω.

P r o o f. The statement of the lemma follows from the one-dimensional uniqueness theorem for the corresponding one-dimensional root expansions proved by Bozhinov in [18],[19]. ∎

Theorem 3. *The conditions* $-a_j, a_j \in \operatorname{supp}\Phi_j$ *when* D_j *is of the form* (3) *and* $a_j \in \operatorname{supp}\Phi_j$ *when* D_j *is of the form* (4), $j = 1 \dots n$ *are necessary and sufficient for the projection system* $\{P_{(k)}\}_{(k)=(0)}^{(\infty)}$ *to be total in* $L^1(\Omega)$, *i.e. they are necessary and sufficient conditions for validity of a uniqueness theorem for the formal expansion* $f \sim \sum_{(k)=(0)}^{(\infty)} P_{(k)}f$ *on joint root projections of the operators* $D_1, \dots D_n$.

P r o o f. It is easy to see that the equality $P_{(k)}f = P_{k_1,1}\{P_{k_2,2}[\dots(P_{k_n,n}f)]\}$ holds for each $f \in L^1(\Omega)$ and arbitrary $(k) = (k_1, \dots, k_n) \geq (0)$. It follows by approximation by "splitting functions". Now, let $f \in L^1(\Omega)$ be such that $P_{(k)}f = P_{k_1,1}\{P_{k_2,2}[\dots(P_{k_n,n}f)]\} = 0$ for all $(k) = (k_1, \dots, k_n) \geq (0)$. Since $P_{k_1,1}, \{P_{k_2,2}, \dots, (P_{k_n,n})$ are one-dimensional projections acting on the variables $t_1, t_2, \dots, t_n$ respectively, the proposition follows by successive application of lemma 1. ∎

Analogously to [17], 2.4.3 we introduce the inner Cauchy convolution $*_{\mathcal{X}}$ in the space $\mathcal{X} \overset{\text{def}}{=} \prod_{(k)=(0)}^{(\infty)} \mathbf{C}^{\varkappa(k)}$ with $\mathbf{C}^{\varkappa(k)} = \mathbf{C}^{\varkappa^1_{k_1}} \times \dots \times \mathbf{C}^{\varkappa^n_{k_n}}$ by the equality

$$\xi *_{\mathcal{X}} \eta \overset{\text{def}}{=} \{ \sum_{(l)=(0)}^{(s)} \overset{(k)}{\xi}_{(s)-(l)} \overset{(k)}{\eta}_{(l)} : (0) \leq (s) \leq \nu_{(k)} \}_{(k)=(0)}^{(\infty)},$$

where $\xi = \{ \overset{(k)}{\xi}_{(l)} : (0) \leq (s) \leq \nu_{(k)} \}_{(k)=(0)}^{(\infty)}$, $\eta = \{ \overset{(k)}{\eta}_{(l)} : (0) \leq (s) \leq \nu_{(k)} \}_{(k)=(0)}^{(\infty)}$ are arbitrary elements of $\mathcal{X}$. A system $\mathcal{W}_{(k)} = \{ \overset{(k)}{w}_{(s)} : (0) \leq (s) \leq \nu_{(k)} \} \subset H_{(k)}$ is said to be *chain system* of joint root functions, if

$$\overset{(k)}{w}_{(s)}(t) = (D - \lambda_{(k)})^{\nu(k)-(s)} \overset{(k)}{w}_{\nu(k)}(t), \quad (0) \leq (s) \leq \nu_{(k)}.$$

Let $\mathcal{H}$ be the set of all systems $\mathcal{W} = \{\mathcal{W}_{(k)}\}_{(k)=(0)}^{(\infty)}$, where $\mathcal{W}_{(k)} = \{ \overset{(k)}{w}_{(s)} : (0) \leq (s) \leq \nu_{(k)} \}$ is a chain system in $H_{(k)}$. A system $\mathcal{W}_{(k)} = \{ \overset{(k)}{w}_{(s)} : (0) \leq (s) \leq \nu_{(k)} \}$ in $H_{(k)}$ is said to be *tensorial chain system* of joint root root functions, if it is a tensorial product

$$\mathcal{W}_{(k)} = \mathcal{W}^1_{k_1} \otimes \dots \otimes \mathcal{W}^n_{k_n}$$

of chain root function systems in $\operatorname{Ker}(D_j - \lambda^j_{k_j})^{\varkappa^j_{k_j}} \subset L^1(\Omega_j)$ $j = 1, \dots, n$.

Let $\mathcal{H}_0$ be the set of all systems $\mathcal{W} = \{\mathcal{W}_{(k)}\}_{(k)=(0)}^{(\infty)}$, where $\mathcal{W}_{(k)} = \{\overset{(k)}{w}_{(s)} : (0) \leq (s) \leq \nu_{(k)}\}$ is a tensorial chain system in $H_{(k)}$.

Analogously to [17], 2.4.3 we introduce the transformations $\hat{}$ and $\tilde{}$, defined by the coefficient functionals (20) with respect to the "good" and the "traditional" chain root basis systems

$$\mathcal{U} = \{\overset{(k)}{u}_{(s)} : (0) \leq (s) \leq \nu_{(k)}\}_{(k)=(0)}^{(\infty)},$$
$$\mathcal{V} = \left\{\frac{1}{(s)!}\frac{\partial^{(s)}}{\partial\lambda^{(s)}} y(\lambda_{(k)}, t) : (0) \leq (s) \leq \nu_{(k)}\right\}_{(k)=(0)}^{(\infty)}$$

in $L^1(\Omega)$, i.e.

$$f \in L^1(\Omega) \to \hat{f} \overset{\text{def}}{=} \{\overset{(k)}{C}_{(s)}(f) : (0) \leq (s) \leq \nu_{(k)}\}_{(k)=(0)}^{(\infty)} \in \mathcal{X},$$
$$f \in L^1(\Omega) \to \tilde{f} \overset{\text{def}}{=} \{\overset{(k)}{A}_{(s)}(f) : (0) \leq (s) \leq \nu_{(k)}\}_{(k)=(0)}^{(\infty)} \in \mathcal{X}.$$

By these compact denotations now theorem 2 implies:

Theorem 4. *The equalities*

(23) $$\mathcal{V} = \alpha *_{\mathcal{H}} \mathcal{U}, \quad \mathcal{U} = \beta *_{\mathcal{H}} \mathcal{V},$$
$$\tilde{f} = \beta *_{\mathcal{X}} \hat{f}, \quad \hat{f} = \alpha *_{\mathcal{X}} \tilde{f} \quad \textit{hold for } f \in L^1(\Omega),$$

as well as the equalities

(24) $$(f * g)\hat{} = \hat{f} *_{\mathcal{X}} \hat{g}, \quad (f * g)\tilde{} = \alpha *_{\mathcal{X}} \tilde{f} *_{\mathcal{X}} \tilde{g} \quad \textit{hold for all } f, g \in L^1(\Omega),$$

i.e. $*$ *is a coefficient convolution of the expansion* $f \sim \sum_{(k)=(0)}^{(\infty)} P_{(k)} f$ *and the transformation* $\hat{}$ *, which is "good" with respect to the convolution* $*$ *, is an injective homomorphism of the convolutional algebra* $(L^1(\Omega), *)$ *in the algebra of the cellular "sequences"* $(\mathcal{X}, *_{\mathcal{X}})$.

Both transformations $\hat{}$ *and* $\tilde{}$ *are represented by the dual formulas*

(25) $$\hat{f} = \Phi\{f \overset{(0)}{*} \mathcal{V}\}, \quad \tilde{f} = \Phi\{f \overset{(0)}{*} \mathcal{U}\} \quad \textit{for } f \in L^1(\Omega).$$

In [18],[19] for the one-dimensional case $n = 1$ Bozhinov gives several types sufficient conditions imposed on the nonlocal boundary value conditions, which assure the *completeness* of the root functions in the space $L^p(\Omega)$, $1 \leq p < \infty$, i.e. that the linear combinations of the root functions are dense in this space. In the present multi-dimensional case that means, that for each

$j = 1, \dots, n$ we have a series of sufficient conditions for completenes of the root function system of the operator D_j in the space $L^p(\Omega_j)$, $1 \le p < \infty$. Now we give a theorem, which shows how these one-dimensional results can be used in the multidimensional case $n > 1$ to obtain a series of results for completenes in of the joint root functions in each space $L^p(\Omega)$, $1 \le p < \infty$.

Theorem 5. *For $j = 1, \dots, n$, $n \ge 1$ let the functional Φ_j be such that the root function system*

$$\mathcal{V}^j \stackrel{\text{def}}{=} \bigcup_{k_j=0}^{\infty} \mathcal{V}^j_{k_j} = \left\{ \frac{1}{s_j!} \frac{\partial^{s_j}}{\partial \lambda_j^{s_j}} y_j(\lambda^j_{k_j}, t_j) : 0 \le s_j \le \varkappa^j_{k_j} \right\}_{k_j=0}^{\infty}$$

of the operator D_j be complete in the space $L^p(\Omega_j)$, $1 \le p < \infty$. Then the joint root function system

$$\mathcal{V} = \left\{ \frac{1}{(s)!} \frac{\partial^{(s)}}{\partial \lambda^{(s)}} y(\lambda_{(k)}, t) \ : \ (0) \le (s) \le \nu_{(k)} \right\}_{(k)=(0)}^{(\infty)}$$

of the operator system $D_1, \dots, D_n$ is complete in the space $L^p(\Omega)$, $1 \le p < \infty$.

Proof. This theorem follows immediately from the fact that the space $L^p(\Omega)$, provided with the usual $\|.\|_p$-norm, is a completion of the tensorial product $L^p(\Omega_1) \otimes \dots \otimes L^p(\Omega_n)$ of the spaces $L^p(\Omega_1), \dots, L^p(\Omega_n)$ and that the restriction of this norm on this tensorial product is a cross-norm, i.e. it has a "splitting property" for functions of $f \in L^p(\Omega)$ in the form $f(t) = f_1(t_1) \dots f_n(t_n)$, where $f(t_j) \in L^p(\Omega_j)$ for $j = 1, \dots, n$. Now, if the linear combinations of the system $\mathcal{V}^j$ are dense in $L^p(\Omega_j)$ for all $j = 1, \dots, n$, then it is not difficult to see that the linear combinations of the functions of the system $\mathcal{V} = \mathcal{V}_1 \otimes \dots \otimes \mathcal{V}_n$ are dense in the space $L^p(\Omega)$, since they are products of the functions of $\mathcal{V}_1, \dots, \mathcal{V}_n$. ∎

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