Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol. 12, 1998, Fasc. 3-4

Extensions of a Symmetric Second-Order Differential Operator

Ahmet Canoğlu [1], *B. P. Allahverdiev* [2]

Presented by Bl. Sendov

Let L_0 be the closure of the expression of the differential operator

$$l(y) = -y''(x) + \frac{\nu^2 - \frac{1}{4}}{x^2}\, y(x) + q(x)\, y(x), \quad 0 < x < \infty .$$

where $\nu \geq 1$ and $q(x)$ is a real continuous function in $[0, \infty)$. In this case, the defect index of the operator L_0 is $(0,0)$ or $(1,1)$. For the case of defect index $(1,1)$, the space of boundary values of the operator L_0 and all of its maximal dissipative, maximal accretive and selfadjoint extensions are obtained.

AMS Subj. Classification: Primary 34L40, 47A20, 47A45, 47B44; Secondary 34L25

Key Words: symmetric second-order differential operator, defect index

1. Introduction

Let us consider the following differential operator

$$l(y) = -y''(x) + \frac{\nu^2 - \frac{1}{4}}{x^2}\, y(x) + q(x)\, y(x), \quad 0 < x < \infty, \tag{1}$$

where $\nu \geq 1$ and $q(x)$ is a real continuous function in $[0, \infty)$.

Let us denote by L_0 the closure of the minimal symmetric operator ([1]) generated by (1). We denote by D the set of all the functions $y(x)$ from $L_2(0, \infty)$ whose first derivatives are locally absolutely continuous in $(0, \infty)$ and $l(y) \in L_2(0, \infty)$; D is the domain of the maximal operator L, and $L = L_0^*$, [1]. The symmetric operator L_0 has defect index $(0,0)$ or $(1,1)$, [1]. For defect index $(0,0)$ the operator L_0 is selfadjoint, that is, $L_0^* = L_0 = L$.

Previously, for $\nu = \frac{1}{2}$, the space of the boundary values of operator L_0, maximal dissipative and selfadjoint extensions of the operator are studied in [2].

In this work, we assume that the symmetric operator L_0 has defect index $(1,1)$.

Let $v_1(x)$ and $v_2(x)$ denote the solutions of $l(y)=0$ satisfying the inital conditions

$$v_1(0)=1\ ,\ v_1'(0)=1\ ,\ v_2(0)=1\ ;\ v_2'(0)=1\ .$$

Clearly, $v_1(x)$ and $v_2(x)$ are linearly independent and their Wronskians are equal to one:

$$W[v_1,v_2]_x = W[v_1,v_2]_1 = 1\ , \qquad (1\le x\le\infty).$$

We recall that a triple $(\mathcal{H},\Gamma_1,\Gamma_2)$, where $\mathcal{H}$ is a Hilbert space and Γ_1 and Γ_2 are linear maps of $D(A^*)$ into $\mathcal{H}$, is called the space of boundary values (SBV) of closed symmetric operator A in the Hilbert space $\mathcal{H}$ with equal defect indices, if the following two conditions hold:

1) For every $f,g\in D(A^*)$

$$(A^*f,g)_H-(f,A^*g)_H=(\Gamma_1 f,\Gamma_2 g)_{\mathcal{H}}-(\Gamma_2 f,\Gamma_1 g)_{\mathcal{H}}\ ,$$

2) For every $g_1,g_2\in\mathcal{H}$ there is a vector $f\in D(A^*)$.

We consider the following linear maps of D into $\mathbb{C}$:

$$\Gamma_1 f=W[f,v_1]_\infty\ ,\quad \Gamma_2 f=W[f,v_2]_\infty\ ,\quad f\in D\ . \tag{2}$$

2. Results

Theorem 1. *The triple $(\mathbb{C},\Gamma_1,\Gamma_2)$ defined by (2) is the space of boundary values of the operator L_0.*

Proof. In order to check the first condition of (SBV), we first prove the following lemma.

Lemma 1. *For arbitrary functions $y(x),z(x)\in D$ we have*

$$W[y,\overline{z}]_x=W[y,v_1]_x\,.\,W[\overline{z},v_2]_x-W[y,v_2]_x\,.\,W[\overline{z},v_1]_x\ ,\quad 0\le x\le\infty. \tag{3}$$

Proof. Observing that $v_1(x)$ and $v_2(x)$ are real functions, we have,

$$\begin{aligned}&W[y,v_1]_x\,.\,W[\overline{z},v_2]_x-W[y,v_2]_x\,.\,W[\overline{z},v_1]_x\\&=(\,y(x)\,v_1'(x)-y'(x)\,v_1(x)\,)\,(\,\overline{z}(x)\,v_2'(x)-\overline{z}'(x)\,v_2(x))\\&-(\,y(x)\,v_2'(x)-y'(x)\,v_2(x)\,)\,(\,\overline{z}\,v_1'(x)-\overline{z}'(x)\,v_1(x)\,)\\&=y(x)\,\overline{z}'(x)-y'(x)\,\overline{z}(x)=W[y,\overline{z}]_x\ ,\quad 0\le x\le\infty.\end{aligned}$$

Thus, Lemma 1 is proved. ■

For $\nu \geq 1$ and operator L_0 having defect index $(1,1)$, and for every $y(x), z(x) \in D$ satisfying $W[y,\bar{z}]_0 = 0$, the following Lagrange formula is satisfied:

$$(4) \qquad (Ly,z)_{L_2(0,\infty)} - (y,Lz)_{L_2(0,\infty)} = W[y,z]_\infty$$

Using Lemma 1, from (4) the following relation is obtained

$$(\Gamma_1 y, \Gamma_2 z)_{\mathbb{C}} - (\Gamma_2 y, \Gamma_1 z)_{\mathbb{C}} = W[y,v_1]_\infty \cdot W[\bar{z},v_2]_\infty - W[y,v_2]_\infty \cdot W[\bar{z},v_1]_\infty = W[y,\bar{z}]_\infty .$$

Therefore, the first requirement of the SBV is fulfilled. The second requirement is proved by the use of the following lemma.

Lemma 2. *For any complex numbers β_0, β_1, there is a function $y \in D$ satisfying*

$$(5) \qquad W[y,v_1]_\infty = \beta_0 , \quad W[y,v_2]_\infty = \beta_1 .$$

P r o o f. Let us denote by L_1 the closure of the minimal symmetric operator generated by $l(y)$ in $1 \leq x < \infty$. For any complex numbers $\gamma_0, \gamma_1, \beta_0$ and β_1, there is a function $y_1(x) \in D(L_1^*)$ which satisfies the following conditions

$$(6) \qquad y(1) = \gamma_0 , \quad y'(1) = \gamma_1 , \quad W[y,v_1]_\infty = \beta_0 , \quad W[y,v_2]_\infty = \beta_1 .$$

Now, let us prove these relations. We consider a function $f(x) \in L_2(1,\infty)$ satisfying

$$(7) \qquad (f,v_1)_{L_2(1,\infty)} = \beta_0 + \gamma_1 , \quad (f,v_2)_{L_1(1,\infty)} = \beta_1 - \gamma_0 .$$

Let $y_1(x)$ denote the solution of equation $l(y) = f(x)$, $(1 \leq x < \infty)$ satisfying the initial conditions

$$y(1) = \gamma_0 , \quad y'(1) = \gamma_1 .$$

This solution can be written as

$$y_1(x) = \gamma_0 \, v_1(x) + \gamma_1 \, v_2(x) + \int_1^x \{v_1(x)\, v_2(\xi) - v_1(\xi)\, v_2(x)\}\, f(\xi)\, d\xi .$$

This expression shows that $y_1(x) \in D(L_1{}^*)$. Let us apply Lagrange's formula to the functions $y_1(x)$ and $v_j(x)$ $(j = 1,2)$

(8)

$$(f,v_j)_{L_2(1,\infty)} = (l(y_1),v_j)_{L_2(1,\infty)} = W[y_1,v_j]_\infty - W[y_1,v_j]_1 + (y,l(v_j))_{L_2(1,\infty)} .$$

If we let

$$l(v_j) = 0 \quad , \quad y_1(1) = \gamma_0 \quad , \quad y_1'(1) = \gamma_1$$

in (8), we find

$$W[y_1, v_j] = \begin{cases} -\gamma_1 & \text{for } j = 1 \\ \gamma_0 & \text{for } j = 2 \end{cases}$$

and

$$(f, v_1)_{L_2(1,\infty)} = W[y_1, v_1]_\infty + \gamma_1 \quad ,$$

$$(f, v_2)_{L_2(1,\infty)} = W[y_1, v_2] - \gamma_0 \; .$$

From (7) we obtain

$$W[y_1, v_1]_\infty = \beta_0 \quad , \quad W[y_1, v_2]_\infty = \beta_1$$

Hence, we have proved that there exists a function $y_1 \in D(L_1{}^*)$ which satisfies (6). For any complex numbers γ_0 and γ_1, let

$$y_1(x) = \gamma_0 \, v_1(x) + \gamma_1 \, v_2(x) \quad , \quad (0 < x \leq 1).$$

Then, let us define

$$y(x) = \begin{cases} y_2(x) & 0 < x \leq 1 \\ y_1(x) & 1 \leq x < \infty. \end{cases}$$

It is clear that $y \in D$. With respect to the condition (6) we obtain

$$W[y, v_1]_\infty = W[y_1, v_1]_\infty = \beta_0 \quad , \quad W[y, v_2]_\infty = W[y_1, v_2]_\infty = \beta_1 \; .$$

Hence, Lemma 2 and Theorem 1 are proved. Using Theorem 1 and Theorem 1.6 [3], we can state the following theorem.

Theorem 2. *For every number $h \in \mathbb{C}$, $Im\,h \geq 0$ or $h = \infty$, the restriction of L to the set of functions $y \in D$ satisfying either*

$$W[y, v_1]_\infty - h\,W[y, v_2]_\infty = 0 \; , \tag{9}$$

or

$$W[y, v_1]_\infty + h\,W[y, v_2]_\infty = 0 \; , \tag{10}$$

is respectively the maximal dissipative and accretive extension of the symmetric operator L_0. Conversely, every maximally dissipative (accretive) extention of L_0 is the compression (restriction) of L to the set of functions $y(x) \in D$ satisfying (9),(10). If $Imh = 0$ or $h = \infty$, the conditions (9) and (10) define *selfadjoint extensions of the operator L_0. For $h = \infty$, the conditions (9),(10) form $W[y, v_2]_\infty = 0$.*

References

[1] M. A. N a i m a r k, *Linear Differential Operators*, 2nd Ed., Nauka, Moscow, 1969; English Trans. of 1st Ed., Parts I,II Ungar, N. York, 1967, 1968.

[2] B. P. A l l a h v e r d i e v, On dilation theory and spectral analysis of dissipative Schrödinger operators in Weyl's limit-circle case, *Izv. Akad. Nauk. SSSR* **54**, No 2, 1990; English Transl. in: *Math USSR, Izv.* **36**, No2, 1991, 247-262.

[3] V. I. G o r b a c h u k, M. L. G o r b a c h u k, *Boundary Value Problems for Operator Differential Equations*, Naukova Dumka, Kiev, 1984; English Trans.: Kluwer, Dordrecht, 1991.

[1] *Mathematics Department*
Istanbul Technical University
80626 Maslak, Istanbul, TURKEY

Received: 25.09.1995

[2] *Institute of Mathematics and Mechanics*
Academy of Sciences of Azerbaijan
370 602 Baku, AZERBAIJAN