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A Property of Certain Analytic Functions Involving Ruscheweyh Derivatives, II

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In this paper we obtain some inequalities for certain analytic functions involving Ruscheweyh derivative defined in the unit disc $U = \{z : |z| < 1\}$.

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1. Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. For functions $f_j(z)$ $(j = 1, 2)$ defined by

$$f_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k, \tag{1.2}$$

we define the *convolution* $f_1 * f_2(z)$ of functions $f_1(z)$ and $f_2(z)$ by

$$f_1 * f_2(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k. \tag{1.3}$$

In terms of (1.3), we define

$$D^{n+p-1} f(z) = \left(\frac{z^p}{(1-z)^{n+p}} \right) * f(z) \qquad (f(z) \in A(p)), \tag{1.4}$$

where n is any integer greater than $-p$. We note that

$$D^{n+p-1}f(z) = \frac{z^p(z^{n-1}f(z))^{(n+p-1)}}{(n+p-1)!}. \tag{1.5}$$

The symbol D^{n+p-1}, when $p = 1$, was introduced by Ruscheweyh [6], and the symbol D^{n+p-1} was introduced by Goel and Sohi [4]. Therefore, the symbol $D^{n+p-1}f(z)$ is usually called the $(n+p-1)$–th order *Ruscheweyh derivative* of $f(z)$. It follows from (1.5) that

$$z(D^{n+p-1}f(z))'' = (n+p)(D^{n+p}f(z))' - (n+1)(D^{n+p-1}f(z))'. \tag{1.6}$$

Recently, Chen and Lan ([1], [2]) and Chen and Owa [3] have proved some interesting results of certain analytic functions involving Ruscheweyh derivatives.

In [3] Chen and Owa have proved the following:

Theorem A. *If a function $f(z) \in A(p)$ satisfies*

$$\Re\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\right\} > \alpha \qquad (z \in U) \tag{1.7}$$

for some $\alpha(0 \le \alpha < 1)$, then

$$\Re\left\{\frac{D^{n+p-1}f(z)}{z^p}\right\}^{\beta} > \gamma \qquad (z \in U), \tag{1.8}$$

where

$$0 < \beta \le \frac{1}{2(n+p)(1-\alpha)} \tag{1.9}$$

and

$$\gamma = \frac{1}{2\beta(n+p)(1-\alpha)+1}. \tag{1.10}$$

Corollary A. *If $f(z) \in A(1)$ satisfies*

$$\Re\left\{\frac{zf''(z)}{f'(z)}\right\} > 2\alpha - 1 \qquad (z \in U), \tag{1.11}$$

then

$$\Re\{f'(z)\}^{\beta} > \frac{1}{4\beta(1-\alpha)+1} \qquad (z \in U), \tag{1.12}$$

where $0 < \beta \le (1-\alpha)/4$.

2. Main result

In order to prove our main result, we need the following lemma due to Miller and Mocanu [5].

Lemma 1. *Let $\varphi(u,v)$ be a complex-valued function,*

$$\varphi : D \longrightarrow \mathbb{C}, \quad D \subset \mathbb{C}\times\mathbb{C} \qquad (\mathbb{C} \textit{ is the complex plane}),$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\varphi(u,v)$ satisfies

(i) *$\varphi(u,v)$ is continuous in D;*

(ii) *$(1,0) \in D$ and $\Re\{\varphi(1,0)\} > 0$;*

(iii) *for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\dfrac{(1+u_2^2)}{2}$, $\Re\{\varphi(iu_2,v_1)\} \leq 0$.*

Let $q(z) = 1 + q_1 z + q_2 z^2 + \ldots$ be regular in the unit disc U such that $(q(z), zq'(z)) \in D$ for all $z \in U$, if

$$\Re\{\varphi(q(z), zq'(z))\} > 0 \qquad (z \in U),$$

then $\Re\{q(z)\} > 0 \qquad (z \in U)$.

Applying the same technique as in proof of Theorem A (used by Chen and Owa [3]) and using (1.6) or putting $\dfrac{zf'(z)}{p}$ in Theorem 1 instead of $f(z)$, we get the following

Theorem 1. *If a function $f(z) \in A(p)$ satisfies*

$$\Re\left\{\frac{(D^{n+p}f(z))'}{(D^{n+p-1}f(z))'}\right\} > \alpha \qquad (z \in U) \tag{2.1}$$

for some $\alpha (0 \leq \alpha < 1)$, then

$$\Re\left\{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}\right\}^{\beta} > \gamma \qquad (z \in U), \tag{2.2}$$

where β and γ are given by (1.9) and (1.10), respectively.

Taking $\beta = 1/2$ in Theorem 1, we have

Corollary 1. *If $f(z) \in A(p)$ satisfies the condition (2.1), then*

$$\Re\left\{\sqrt{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}}\right\} > \frac{1}{(n+p)(1-\alpha)+1} \qquad (z \in U), \tag{2.3}$$

where $1-\frac{1}{(n+p)}\leq\alpha<1$.

Letting $\beta=\frac{1}{2(n+p)(1-\alpha)}$ in Theorem 1, we have

Corollary 2. *If* $f(z)\in A(p)$ *satisfies*

$$\Re\left\{\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}\right\}^{\frac{1}{2(n+p)(1-\alpha)}}>\frac{1}{2}\qquad(z\in U). \tag{2.4}$$

Putting $p=n=1$ in Theorem 1, we have

Corollary 3. *If* $f(z)\in A(1)$ *satisfies*

$$\Re\left\{1+\frac{z(zf'(z))''}{(zf'(z))'}\right\}>2\alpha-1\qquad(z\in U), \tag{2.5}$$

then

$$\Re\{(zf'(z))'\}^{\beta}>\frac{1}{4\beta(1-\alpha)+1}\qquad(z\in U), \tag{2.6}$$

where $0<\beta\leq\frac{1}{4(1-\alpha)}$.

Remark. On our opinion, Corollary A of Theorem A is incorrect and should read as follows:

Corollary B. *If* $f(z)\in A(1)$ *satisfies*

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>2\alpha-1\qquad(z\in U), \tag{2.7}$$

then

$$\Re\{f'(z)\}^{\beta}>\frac{1}{4\beta(1-\alpha)+1}\qquad(z\in U), \tag{2.8}$$

where $0<\beta\leq\frac{1}{4(1-\alpha)}$.

This can be obtained by putting $n=p=1$ in Theorem A.

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