Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica Balkanica

New Series Vol. 12, 1998, Fasc. 3-4

On Neighborhood Structures in Intuitionistic Topological Spaces

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The purpose of this paper is to introduce the neighborhood structures of intuitionistic topological spaces and give some introductory information about interiors in these spaces.

AMS Subj. Classification: Primary 54A99, Secondary 04A99

Key Words: Intuitionistic set, intuitionistic point, vanishing intuitionistic point, intuitionistic topology, intuitionistic topological space, neighborhood

1. Introduction

After the introduction of intuitionistic fuzzy sets by K. Atanassov [1,2,3], as a generalization of fuzzy sets developped by L. Zadeh [7], intuitionistic sets and intuitionistic points were defined by D. Çoker in [4]. Later, intuitionistic fuzzy topological spaces were introduced in [5], and then intuitionistic topological spaces were defined as a classical version of this concept [6].

2. Preliminaries

For the sake of completeness, we shall give some preliminary information about intuitionistic sets, intuitionistic points and intuitionistic topological spaces given in [4,6].

Definition 2.1. ([4]) Let X be a nonempty fixed set. An intuitionistic set (IS for short) A is an object having the form

$$A = < X, A_1, A_2 >,$$

where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \emptyset$. The set A_1 is called the set of members of A, while A_2 is called the set of nonmembers of A.

Obviously, every set A on a nonempty set X is an IS having the form $< X, A, A^c >$. One can define several relations and operations between IS's as follows:

Definition 2.2. ([4]) Let X be a nonempty set and the IS's A and B be in the form $A =< X, A_1, A_2 >$, $B =< X, B_1, B_2 >$, respectively and let $\{A_i : i \in J\}$ be an arbitrary family of IS's in X, where $A_i =< X, A_i^{(1)}, A_i^{(2)} >$. Then

(a) $A \subseteq B$ iff $A_1 \subseteq B_1$ and $A_2 \supseteq B_2$;
(b) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;
(c) $\bar{A} =< X, A_2, A_1 >$;
(d) $\underset{\sim}{\emptyset} =< X, \emptyset, X >$ and $\underset{\sim}{X} =< X, X, \emptyset >$;
(e) $\cup A_i =< X, \cup A_i^{(1)}, \cap A_i^{(2)} >$;
(f) $\cap A_i =< X, \cap A_i^{(1)}, \cup A_i^{(2)} >$.

Definition 2.3. ([4]) Let X be a nonempty set and $p \in X$ a fixed element in X. The IS $\underset{\sim}{p} =< X, \{p\}, \{p\}^c >$ is called an intuitionistic point (IP, for short) in X, and the IS $\underset{\approx}{p} =< X, \emptyset, \{p\}^c >$ is called a vanishing intuitionistic point (VIP, for short) in X.

Now we shall present some types of inclusion of an IP or a VIP to an IS:

Definition 2.4. ([4]) Let $p \in X$ and $A =< X, A_1, A_2 >$ be an IS in X.

(a) $\underset{\sim}{p}$ is said to be contained in A ($\underset{\sim}{p} \in A$ for short) iff $p \in A_1$.

(b) $\underset{\approx}{p}$ is said to be contained in A ($\underset{\approx}{p} \in A$ for short) iff $p \notin A_2$.

Proposition 2.5. ([4]) *Let $\{A_i : i \in J\}$ be a family of IS's in X. Then*
(a1) *$\underset{\sim}{p} \in \cap_{i \in J} A_i$ iff $\underset{\sim}{p} \in A_i$ for each $i \in J$.*
(a2) *$\underset{\approx}{p} \in \cap_{i \in J} A_i$ iff $\underset{\approx}{p} \in A_i$ for each $i \in J$.*
(b1) *$\underset{\sim}{p} \in \cup_{i \in J} A_i$ iff $\exists i \in J$ such that $\underset{\sim}{p} \in A_i$.*
(b2) *$\underset{\approx}{p} \in \cup_{i \in J} A_i$ iff $\exists i \in J$ such that $\underset{\approx}{p} \in A_i$.*

Proposition 2.6. ([4]) *Let A and B be two IS's in X. Then*

(a) *$A \subseteq B$ iff for each $\underset{\sim}{p}$ we have $\underset{\sim}{p} \in A \Rightarrow \underset{\sim}{p} \in B$ and for each $\underset{\approx}{p}$ we have $\underset{\approx}{p} \in A \Rightarrow \underset{\approx}{p} \in B$.*

(b) *$A = B$ iff for each $\underset{\sim}{p}$ we have $\underset{\sim}{p} \in A \Leftrightarrow \underset{\sim}{p} \in B$ and for each $\underset{\approx}{p}$ we have $\underset{\approx}{p} \in A \Leftrightarrow \underset{\approx}{p} \in B$.*

Proposition 2.7. [4] *Let A be any IS in X. Then*

$$A = (\cup\{\underset{\sim}{p} : \underset{\sim}{p} \in A\}) \cup (\cup\{\underset{\approx}{p} : \underset{\approx}{p} \in A\}).$$

Proposition 2.7 states that any IS in X can be written in the form $A = \underset{\sim}{A} \cup \underset{\approx}{A}$, where $\underset{\sim}{A} = \cup\{\underset{\sim}{p} : \underset{\sim}{p} \in A\}$ and $\underset{\approx}{A} = \cup\{\underset{\approx}{p} : \underset{\approx}{p} \in A\}$. Furhermore it is easy to show that, if $A =< X, A_1, A_2 >$, then $\underset{\sim}{A} =< X, A_1, A_1^c >$ and $\underset{\approx}{A} =< X, \emptyset, A_2 >$.

Definition 2.8. ([6]) An intuitionistic topology (IT, for short) on a nonempty set X is a family τ of IS's in X containing $\underset{\sim}{\emptyset}$, $\underset{\sim}{X}$ and closed under arbitrary suprema and finite infima. In this case the pair (X, τ) is called an intuitionistic topological space (ITS for short) and any IS in τ is known as an intuitionistic open set (IOS, for short) in X. The complement $\bar{A}$ of an IOS A is an ITS (X, τ) is called an intuitionistic closed set (ICS, for short) in X.

Obviously, any topological space (X, τ_0) is an ITS in the form $\tau = \{A' : A \in \tau_0\}$ whenever we identify a subset A in X with its counterpart $A' =< X, A, A^c >$.

Definition 2.9. ([6]) Let (X, τ) be an ITS and $A =< X, A_1, A_2 >$ be an IS in X. Then the interior and closure of A are defined by

$$int(A) = \cup\{G : G \text{ is an IOS in } X, G \subseteq A\},$$

$$cl(A) = \cap\{K : K \text{ is an ICS in } X, A \subseteq K\}.$$

3. Neigborhood structures in intutionistic topological spaces

Definition 3.1.

(a) Let $\underset{\sim}{p}$ be an IP in X. A subset N of X said to be a neighborhood of $\underset{\sim}{p}$ in X, if an IOS $G \in \tau$ exists such that $\underset{\sim}{p} \in G \subseteq N$.

(b) Let $\underset{\approx}{p}$ be an VIP in X. A subset N of X said to be a neighborhood of $\underset{\approx}{p}$ in X, if an IOS $G \in \tau$ exists such that $\underset{\approx}{p} \in G \subseteq N$.

We shall denote the set of all neighborhoods of $\underset{\sim}{p}$ by $N(\underset{\sim}{p})$, and the set of all neighborhoods of $\underset{\approx}{p}$ by $N(\underset{\approx}{p})$.

The systems $N(\underset{\sim}{p})$ and $N(\underset{\approx}{p})$ of neighborhoods satisfy the following properties:

Proposition 3.2. *The neighborhood system $N(\underset{\sim}{p})$ in the ITS (X, τ) satisfies the following properties:*

(N1) *If $N \in N(\underset{\sim}{p})$, then $\underset{\sim}{p} \in N$.*

(N2) *If $N \in N(\underset{\sim}{p})$ and $N \subseteq M$, then $M \in N(\underset{\sim}{p})$.*

(N3) *If $N_1, N_2 \in N(\underset{\sim}{p})$, then $N_1 \cap N_2 \in N(\underset{\sim}{p})$.*

(N4) *If $N \in N(\underset{\sim}{p})$, then there exists $M \in N(\underset{\sim}{p})$ such that $N \in N(\underset{\sim}{q})$ for each $\underset{\sim}{q} \in M$.*

Proof. (N1), (N2) and (N4) are easy to prove.

(N3) Let $N_1, N_2 \in N(\underset{\sim}{p})$. Then there exist the IOS's G_1 and G_2 such that $\underset{\sim}{p} \in G_i \subseteq N_i$ $(i = 1, 2)$. For the IOS $G := G_1 \cap G_2$, we have also $\underset{\sim}{p} \in G \subseteq N_1 \cap N_2$, and so $N_1 \cap N_2 \in N(\underset{\sim}{p})$. ■

Proposition 3.3. *The neighborhood system $N(\underset{\approx}{p})$ in the ITS (X, τ) satisfies the following properties:*

(N1) *If $N \in N(\underset{\approx}{p})$, then $\underset{\approx}{p} \in N$.*

(N2) *If $N \in N(\underset{\approx}{p})$ and $N \subseteq M$, then $M \in N(\underset{\approx}{p})$.*

(N3) *If $N_1, N_2 \in N(\underset{\approx}{p})$, then $N_1 \cap N_2 \in N(\underset{\approx}{p})$.*

(N4) *If $N \in N(\underset{\approx}{p})$, then there exists $M \in N(\underset{\approx}{p})$ such that $N \in N(\underset{\approx}{q})$ for each $\underset{\approx}{q} \in M$.*

Proof. Similar to the proof of the previous proposition. ■

Now let us define the families

$$\underset{\sim}{\tau} = \{G : G \in N(\underset{\sim}{p}) \text{ for each } \underset{\sim}{p} \in G\}$$

and

$$\underset{\approx}{\tau} = \{G : G \in N(\underset{\approx}{p}) \text{ for each } \underset{\approx}{p} \in G\}.$$

Clearly, $G_1 \cap G_2 \in \underset{\approx}{\tau}$ for each $G_1, G_2 \in \underset{\approx}{\tau}$. Moreover, let $(G_i)_{i \in J}$ be a family of IS's in $\underset{\approx}{\tau}$ and $G := \bigcup_{i \in J} G_i$. Then, for any $\underset{\approx}{p} \in G$, there exists an index $i_0 \in J$

such that $\underset{\approx}{p} \in G_{i_0}$. Hence $G_{i_0} \in N(\underset{\approx}{p})$ follows. Since $G_{i_0} \subseteq G$, we get from (N2) that $G \in N(\underset{\approx}{p})$, i.e $G \in \underset{\approx}{\tau}$. Thus we obtain the following proposition:

Proposition 3.4. *$\underset{\sim}{\tau}$ and $\underset{\approx}{\tau}$ are IT's on X.*

Proposition 3.5. *$\tau \subseteq \underset{\sim}{\tau}$ and $\tau \subseteq \underset{\approx}{\tau}$.*

P r o o f. Let $G \in \tau$. We obviously have $G \in N(\underset{\sim}{p})$ and $G \in N(\underset{\approx}{p})$ for each $\underset{\sim}{p} \in G$ and $\underset{\approx}{p} \in G$, respectively. Hence $G \in \underset{\sim}{\tau}$ and $G \in \underset{\approx}{\tau}$ follow, directly. ∎

E x a m p l e 3.6. Let $X = \{a, b, c, d\}$ and consider the family

$$\tau = \{\underset{\sim}{\emptyset}, \underset{\sim}{X}, A_1, A_2, A_3, A_4\}$$

of IS's, where

$$A_1 =< X, \{a, b\}, \{d\} >, \qquad A_2 =< X, \{c\}, \{b, d\} >,$$

$$A_3 =< X, \emptyset, \{b, d\} >, \qquad A_4 =< X, \{a, b, c\}, \{d\} > .$$

Then (X, τ) is an ITS on X from which we get the IT's $\underset{\sim}{\tau}$ and $\underset{\approx}{\tau}$ as follows:

$$\underset{\sim}{\tau} = \tau \cup \{A_i : i = 5, 6, ..., 23\},$$

where

$$\begin{aligned}
A_5 &= < X, \{c\}, \{b\} >, \qquad A_6 =< X, \{c\}, \{d\} >, \qquad A_7 =< X, \{a, b\}, \emptyset >,\\
A_8 &= < X, \{a, b, c\}, \emptyset >, \qquad A_9 =< X, \{c\}, \emptyset >, \qquad A_{10} =< X, \emptyset, \{a\} >,\\
A_{11} &= < X, \emptyset, \{b\} >, \qquad A_{12} =< X, \emptyset, \{c\} >, \qquad A_{13} =< X, \emptyset, \{d\} >,\\
A_{14} &= < X, \emptyset, \{a, b\} >, \qquad A_{15} =< X, \emptyset, \{a, c\} >, \qquad A_{16} =< X, \emptyset, \{a, d\} >,\\
A_{17} &= < X, \emptyset, \{b, c\} >, \qquad A_{18} =< X, \emptyset, \{c, d\} >, \qquad A_{19} =< X, \emptyset, \{a, b, c\} >,\\
A_{20} &= < X, \emptyset, \{a, b, d\} >, \qquad A_{21} =< X, \emptyset, \{a, c, d\} >, \qquad A_{22} =< X, \emptyset, \{b, c, d\} >,\\
A_{23} &= < X, \emptyset, \emptyset >
\end{aligned}$$

and

$$\underset{\approx}{\tau} = \tau \cup \{A_{24}, A_{25}\},$$

where $A_{24} =< X, \{a, c\}, \{b, d\} >$, $A_{25} =< X, \{a\}, \{b, d\} >$.

Here come the reverse implications of Proposition 3.2 and Proposition 3.3.

Proposition 3.7. *If to each element p of a set X there corresponds a set $N(\underset{\approx}{p})$ of IS's of X such that the properties (N1),(N2),(N3) and (N4) in Proposition 3.3 are satisfied, then there exists an intuitionistic topology on X such that for each $p \in X$, $N(\underset{\approx}{p})$ is the set of all neighborhoods of $\underset{\approx}{p}$ in this intuitionistic topology.*

P r o o f. Let

$$\underset{\approx}{\tau} = \{G : G \in N(\underset{\approx}{p}) \text{ for each } \underset{\approx}{p} \in G\}.$$

It is easy to show that $(X, \underset{\approx}{\tau})$ is an ITS on X. We will show that $N(\underset{\approx}{p})$ is the set of all neighborhoods of $\underset{\approx}{p}$ for $p \in X$. From (N2) it follows that each neighborhood of $\underset{\approx}{p}$ belongs to $N(\underset{\approx}{p})$.

Conversely, let V be an IS belonging to $N(\underset{\approx}{p})$, and let U be the union of all the VIP's $\underset{\approx}{q} \in X$ such that $V \in N(\underset{\approx}{q})$. If we can show that $\underset{\approx}{p} \in U$, $U \subseteq V$ and $U \in \underset{\approx}{\tau}$, then the proof will be complete. We have $\underset{\approx}{p} \in U$, since $V \in N(\underset{\approx}{p})$ and also $U \subseteq V$. Now if $\underset{\approx}{q} \in U$, then by (N4), there exists an IS $W \in N(\underset{\approx}{q})$ such that for each $\underset{\approx}{r} \in W$ we have $V \in N(\underset{\approx}{r})$, which means that $\underset{\approx}{r} \in U$. It follows that $W \subseteq U$, and therefore, by (N2), $U \in N(\underset{\approx}{q})$ for each $\underset{\approx}{q} \in U$. Hence we get $U \in \underset{\approx}{\tau}$ as required. ■

Proposition 3.8. *If to each element p of a set X there corresponds a set $\eta(\underset{\sim}{p})$ of IS's of X such that the properties (N1), (N2), (N3) and (N4) in Proposition 3.2 are satisfied, then there exists an intuitionistic topology on X such that for $p \in X$, $N(\underset{\sim}{p})$ is the set of all neighborhood of $\underset{\sim}{p}$ in this intuitionistic topology.*

P r o o f. Similar to the proof of Proposition 3.7. ■

Now we shall give the relations between these two ITS's:

Proposition 3.9. $\tau = \underset{\sim}{\tau} \cap \underset{\approx}{\tau}$.

P r o o f. By Proposition 3.5, we clearly have $\tau \subseteq \underset{\sim}{\tau} \cap \underset{\approx}{\tau}$. Conversely, let $G \in \underset{\sim}{\tau} \cap \underset{\approx}{\tau}$. Then $G \in \underset{\sim}{\tau}$ and $G \in \underset{\approx}{\tau}$ follow. Hence G is a neighborhood of each of its IP's $\underset{\sim}{p}$ and its VIP's $\underset{\approx}{p}$, respectively. Therefore there exist IOS's $G_{\underset{\sim}{p}}, G_{\underset{\approx}{p}} \in \tau$ such that $\underset{\sim}{p} \in G_{\underset{\sim}{p}} \subseteq G$ and $\underset{\approx}{p} \in G_{\underset{\approx}{p}} \subseteq G$. Consequently,

$$\underset{\sim}{G} = \bigcup_{\underset{\sim}{p} \in G} \underset{\sim}{p} \subseteq \bigcup_{\underset{\sim}{p} \in G} \underset{\sim}{G_p} \subseteq G \text{ and } \underset{\approx}{G} = \bigcup_{\underset{\approx}{p} \in G} \underset{\approx}{p} \subseteq \bigcup_{\underset{\approx}{p} \in G} \underset{\approx}{G_p} \subseteq G,$$

and so

$$G = \underset{\sim}{G} \cup \underset{\approx}{G} \subseteq (\bigcup_{\underset{\sim}{p} \in \underset{\sim}{G}} p) \cup (\bigcup_{\underset{\approx}{p} \in \underset{\approx}{G}} p) \subseteq G.$$

Since $(\bigcup_{\underset{\sim}{p} \in \underset{\sim}{G}} p) \cup (\bigcup_{\underset{\approx}{p} \in \underset{\approx}{G}} p) \in \tau$, we get $G \in \tau$, as required. ■

Proposition 3.10. *Let A be an IS in ITS (X,τ). Then*

$$int(A) = int_{\underset{\sim}{\tau}}(A) \cap int_{\underset{\approx}{\tau}}(A).$$

Proof. Since $\tau \subseteq \underset{\sim}{\tau}$ and $\tau \subseteq \underset{\approx}{\tau}$ by Proposition 3.5, we easily obtain

$$int(A) \subseteq int_{\underset{\sim}{\tau}}(A), \quad int(A) \subseteq int_{\underset{\approx}{\tau}}(A) \Rightarrow int(A) \subseteq int_{\underset{\sim}{\tau}}(A) \cap int_{\underset{\approx}{\tau}}(A).$$

Now let $\underset{\sim}{p} \in int_{\underset{\sim}{\tau}}(A) \cap int_{\underset{\approx}{\tau}}(A)$. Then, by Proposition 2.5, we see, in particular, that $\underset{\sim}{p} \in int_{\underset{\sim}{\tau}}(A)$ from which we obtain $A \in \underset{\sim}{N}(\underset{\sim}{p})$, i.e. there exists $G \in \tau$ such that $\underset{\sim}{p} \in G \subseteq A$, i.e. $\underset{\sim}{p} \in int(A)$. Similarly, we have

$$\underset{\approx}{p} \in int_{\underset{\sim}{\tau}}(A) \cap int_{\underset{\approx}{\tau}}(A) \Rightarrow \underset{\approx}{p} \in int_{\underset{\approx}{\tau}}(A) \Rightarrow A \in N(\underset{\approx}{p})$$

$$\Rightarrow \text{ there exists } G \in \tau \text{ such that } \underset{\approx}{p} \in G \subseteq A \Rightarrow \underset{\approx}{p} \in int(A).$$

Hence, by Proposition 2.6, $int_{\underset{\sim}{\tau}}(A) \cap int_{\underset{\approx}{\tau}}(A) \subseteq int(A)$ follows. □

In general, we have the inclusions:

$$int(A) \subseteq int_{\underset{\sim}{\tau}}(A) \text{ and } int(A) \subseteq int_{\underset{\approx}{\tau}}(A).$$

But the reverse inclusions do not need to be true:

Example 3.11. Consider the ITS (X,τ) in Example 3.6. If we take $A = < X, \{a,c\}, \{d\} >$, then it is easy to see that

$$int(A) = < X, \{c\}, \{b,d\} >$$

and

$$int_{\underset{\sim}{\tau}}(A) = < X, \{c\}, \{d\} >, \quad int_{\underset{\approx}{\tau}}(A) = < X, \{a,c\}, \{b,d\} > .$$

4. Interiors of intuitionistic sets

Given an ITS (X,τ), one can obtain the interiors of an arbitrary IS A with respect to the IT's $(X,\tau), (X,\underset{\sim}{\tau})$ and $(X,\underset{\approx}{\tau})$. Furthermore, we can construct another type of interior in terms of the following concepts:

Definition 4.1.

(a) Let A be an IS in X, $\underset{\sim}{p} \in A$ an IP and $\underset{\approx}{p} \in A$ an VIP. Then $\underset{\sim}{p}$ is said to be a $\underset{\sim}{\tau}$-interior point of A, if A is a neighborhood of $\underset{\sim}{p}$, and $\underset{\approx}{p}$ is said to be a $\underset{\approx}{\tau}$-interior point of A, if A is a neighborhood of $\underset{\approx}{p}$.

(b) The unions of all $\underset{\sim}{\tau}$-interior and $\underset{\approx}{\tau}$-interior points of A are denoted by $\underset{\sim}{\tau} - int(A)$ and $\underset{\approx}{\tau} - int(A)$, respectively.

Proposition 4.2.

(1) $A \in \underset{\sim}{\tau}$ *iff* $\underset{\sim}{A} = \underset{\sim}{\tau} - int(A)$.

(2) $A \in \underset{\approx}{\tau}$ *iff* $\underset{\approx}{A} = \underset{\approx}{\tau} - int(A)$.

Proof. (1) Let $A \in \underset{\sim}{\tau}$ and $\underset{\sim}{p} \in \underset{\sim}{A}$ be given, i.e. $\underset{\sim}{p} \in A$. Hence A is a neighborhood of $\underset{\sim}{p}$. We have then $\underset{\sim}{p} \in \underset{\sim}{\tau} - int(A)$, i.e. $\underset{\sim}{A} \subseteq \underset{\sim}{\tau} - int(A)$. Conversely, since $\underset{\sim}{\tau} - int(A) = \cup\{\underset{\sim}{p} : A \in \tilde{N}(\underset{\sim}{p})\}$ and if $A \in \tilde{N}(\underset{\sim}{p})$, then $\underset{\sim}{p} \in A$, i.e. $\underset{\sim}{p} \in \underset{\sim}{A}$, and we obtain $p \in \underset{\sim}{\tau} - int(A)$. Thus $\underset{\sim}{\tau} - int(A) \subseteq \underset{\sim}{A}$ follows.

Now let $\underset{\sim}{A} = \underset{\sim}{\tau} - int(A)$. We shall show that $A \in \underset{\sim}{\tau}$. If $\underset{\sim}{p} \in A$ is an arbitrary IP, then $\underset{\sim}{p} \in \underset{\sim}{A}$, and hence A is a neighborhood of $\underset{\sim}{p} \in A$. Hence, by definition, $A \in \underset{\sim}{\tau}$.

(2) It is similar to that of (1). ∎

Lemma 4.3. *Let* $(G_i)_{i\in J}$ *be a family of IS's in* X *and* $G = \cup_{i\in J} G_i$. *Then we have*

(1) $\underset{\sim}{G} = \cup_{i\in J} \underset{\sim}{G}_i$,

(2) $\underset{\approx}{G} = \cup_{i\in J} \underset{\approx}{G}_i$.

Proof. Since the proof of (2) is similar to (1), we shall provide only the proof of (1).

(1) Let $G_i = < X, G_{i1}, G_{i2} >$, $i \in J$. Then we have

$$G = \cup_{i\in J} G_i = < \cup_{i\in J} G_{i1}, \cap_{i\in J} G_{i2} > .$$

Now choose any IP $\underset{\sim}{p} \in G$. Then $p \in \cup_{i\in J} G_{i1}$. Thus there exists G_i such that $p \in G_{i1}$ meaning that $\underset{\sim}{p} \in \underset{\sim}{G}_i$ and so $\underset{\sim}{p} \in \cup_{i\in J} \underset{\sim}{G}_i$.

Conversely let $\underset{\sim}{p} \in \cup_{i\in J} \underset{\sim}{G}_i$, then there exists $i \in J$ with $\underset{\sim}{p} \in \underset{\sim}{G}_i$. Hence $p \in G_{i1}$, and so $p \in \cup_{i\in J} G_{i1}$. Consequently, $\cup_{i\in J} \underset{\sim}{G}_i \subseteq \underset{\sim}{G}$ follows, as required. ∎

Proposition 4.4.
(1) $\underset{\sim}{\tau} - int(A) = \bigcup_{G \subseteq A, G \in \underset{\sim}{\tau}} \underset{\sim}{G}$,
(2) $\underset{\approx}{\tau} - int(A) = \bigcup_{G \subseteq A, G \in \underset{\approx}{\tau}} \underset{\approx}{G}$.

Proof. We shall prove only (1), the assertion (2) is similar to (1).

(1) Let $\underset{\sim}{p} \in \bigcup\{\underset{\sim}{G} : G \subseteq A, G \in \underset{\sim}{\tau}\}$. Then there exists $G \in \underset{\sim}{\tau}$ such that $G \subseteq A$ and $\underset{\sim}{p} \in \underset{\sim}{G}$. Since $G \in \underset{\sim}{\tau}$ and $\underset{\sim}{p} \in G$, we have $G \in \tilde{N}(\underset{\sim}{p})$ and so $A \in N(\underset{\sim}{p})$, which means that $\underset{\sim}{p} \in \underset{\sim}{\tau} - int(A)$.

Conversely, let $\underset{\sim}{p} \in \underset{\sim}{\tau} - int(A)$. Hence there exists $G \in \tau$ such that $\underset{\sim}{p} \in G \subseteq A$. But from $\underset{\sim}{p} \in \underset{\sim}{G}$ and $G \in \underset{\sim}{\tau}$, we get the assertion. ∎

Notice that we always have the inclusions:

$$\underset{\sim}{\tau} - int(A) \subseteq int_{\underset{\sim}{\tau}}(A) \text{ and } \underset{\approx}{\tau} - int(A) \subseteq int_{\underset{\approx}{\tau}}(A).$$

The following counterexample shows that the reverse inclusions do not hold in general:

Example 4.5. Consider the ITS (X, τ), where $X = \{a, b, c, d, e\}$ and

$$\tau = \{\underset{\sim}{\emptyset}, \underset{\sim}{X}, < X, \{a,b,c\}, \{e\} >, < X, \{c,d\}, \{e\} >, < X, \{c\}, \{d,e\} >, \\ < X, \{a,b,c,d\}, \emptyset >\}.$$

Then it is easy to show that

$$\underset{\sim}{\tau} = \tau \cup \{< X, \{a,b,c\}, \emptyset >, < X, \{c,d\}, \emptyset >, < X, \{c\}, \{d\} >, < X, \{c\}, \{e\} >, \\ < X, \{c\}, \emptyset >\} \cup \{< X, \emptyset, S >: S \subseteq X\}$$

and

$$\underset{\approx}{\tau} = \tau \cup \{< X, \{a,b,c,d\}, \{e\} >, < X, \{a,b,c\}, \{d\} >, < X, \{a,c,d\}, \{e\} >, \\ < X, \{b,c,d\}, \{e\} >, < X, \{a,b,c\}, \{d,e\} >, < X, \{a,c\}, \{d,e\} >, \\ < X, \{b,c\}, \{d,e\} >\}.$$

If we let $B =< X, \{b,c\}, \{d\} >$, then $int(B) =< X, \{c\}, \{d,e\} >$ and

$$int_{\underset{\sim}{\tau}}(B) =< X, \{c\}, \{d\} >, \qquad int_{\underset{\approx}{\tau}}(B) =< X, \{b,c\}, \{d,c\} >,$$

$$\underset{\sim}{\tau} - int(B) =< X, \{c\}, \{a,b,d,e\} >, \qquad \underset{\approx}{\tau} - int(B) =< X, \emptyset, \{d,e\} >,$$

follow, i.e. we have the strict inclusions

$$\underset{\sim}{int}_{\tau}(A) \supseteq \underset{\sim}{\tau} - int(A), \qquad \underset{\sim}{int}_{\tau}(A) \neq \underset{\sim}{\tau} - int(A),$$

$$\underset{\approx}{int}_{\tau}(A) \supseteq \underset{\approx}{\tau} - int(A), \qquad \underset{\approx}{int}_{\tau}(A) \neq \underset{\approx}{\tau} - int(A).$$

Lastly, we present the properties of the interior operator $\underset{\sim}{\tau} - int$ and $\underset{\approx}{\tau} - int$.

Proposition 4.6. *Let (X, τ) be an ITS and A, B two IS's in X. Then*

(a) $\underset{\sim}{\tau} - int(A) \subseteq \underset{\sim}{A}$

(a1) $\underset{\approx}{\tau} - int(A) \subseteq \underset{\approx}{A}$

(b) $A \subseteq B \Longrightarrow \underset{\sim}{\tau} - int(A) \subseteq \underset{\sim}{\tau} - int(B)$

(b1) $A \subseteq B \Longrightarrow \underset{\approx}{\tau} - int(A) \subseteq \underset{\approx}{\tau} - int(B)$

(c) $\underset{\sim}{\tau} - int(A \cap B) = \underset{\sim}{\tau} - int(A) \cap \underset{\sim}{\tau} - int(B)$

(c1) $\underset{\approx}{\tau} - int(A \cap B) = \underset{\approx}{\tau} - int(A) \cap \underset{\approx}{\tau} - int(B)$

(d) $\underset{\sim}{\tau} - int(\underset{\sim}{X}) = \underset{\sim}{X}$

(d1) $\underset{\approx}{\tau} - int(\underset{\sim}{X}) = \underset{\sim}{X}$

P r o o f. We shall only give the proof of (c), the others can be verified directly:
(c) First, let $\underset{\sim}{p}$ be an $\underset{\sim}{\tau}$-interior point of $A \cap B$, i.e. $A \cap B \in N(\underset{\sim}{p})$. Hence $A \in N(\underset{\sim}{p})$ and $B \in N(\underset{\sim}{p})$ follow, meaning that $\underset{\sim}{p}$ is a $\underset{\sim}{\tau}$-interior point of both A and B, i.e.

$$\underset{\sim}{p} \in \underset{\sim}{\tau} - int(A) \text{ and } \underset{\sim}{p} \in \underset{\sim}{\tau} - int(B) \Longrightarrow \underset{\sim}{p} \in \underset{\sim}{\tau} - int(A) \cap \underset{\sim}{\tau} - int(B).$$

On the other hand, if $\underset{\sim}{p} \in \underset{\sim}{\tau} - int(A) \cap \underset{\sim}{\tau} - int(B)$, then $A \in N(\underset{\sim}{p})$ and $B \in N(\underset{\sim}{p})$, i.e. $A \cap B \in N(\underset{\sim}{p})$ which means that $\underset{\sim}{p}$ is an $\underset{\sim}{\tau}$-interior point of $A \cap B$. ■

Notice that, in general, the equalities $\underset{\sim}{\tau} - int(\underset{\sim}{\tau} - int(A)) = \underset{\sim}{\tau} - int(A)$, $\underset{\approx}{\tau} - int(\underset{\approx}{\tau} - int(A)) = \underset{\approx}{\tau} - int(A)$ do not hold. For this purpose, consider the ITS (X, τ) in Example 4.5 and take $A = < X, \{b, c\}, \{d\} >$. In this case one can obtain the following:

$$\underset{\sim}{\tau} - int(A) = < X, \{c\}, \{a, b, d, e\} > \text{ and } \underset{\sim}{\tau} - int(\underset{\sim}{\tau} - int(A)) = < X, \emptyset, X > = \underset{\sim}{\emptyset}.$$

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