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# On Generating Functions for Certain Special Functions by Weisner's Group Theoretic Method [1]

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Weisner's group theoretic method of obtaining generating functions with the single interpretation to the index $n$ has been applied in the study of a modification of extended Jacobi polynomials. In Section 2, a set of infinitesimal operators (known as raising and lowering operators), their commutators (or alternants or Lie products) and the groups corresponding to the infinitesimal operators have been introduced and on showing that they generate a three dimensional Lie algebra we have obtained in Section 3, a novel generating relation of the polynomial under consideration which, as special cases, yields a good number of results (both new and known) involving various special funtions. Besides, it has been shown that those generating functions which could have been obtained by double interpretation either to $(n, \alpha)$ or to $(n, \beta)$ in Weisner's method, can also be derived from our results. At the end, we have mentioned some important observations in the application of Weisner's method to the study of generating functions for the classical orthogonal polynomials.

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## 1. Introduction

The unified presentation for the classical orthogonal polynomials was originally introduced by I. Fujiwara [1] which was subsequently designated by N.K. Thakare [2] as extended Jacobi polynomials, $F_n(\alpha, \beta, x)$ and was given by

$$F_n(\alpha,\beta,x) = \frac{(-1)^n}{n!}(x-a)^{-\alpha}(b-x)^{-\beta}\left(\frac{\lambda}{b-a}\right)^n$$

$$\times D^n\left[(x-a)^{n+\alpha}(b-x)^{n+\beta}\right] \quad , \quad D = \frac{d}{dx}\,.$$

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In this paper we consider $F_n(\alpha - n, \beta, x)$, a modified form of $F_n(\alpha, \beta, x)$ which satisfies the following ordinary differential equation :

$$(1.1)\quad [(x-a)(b-x)D^2+\{(\alpha-n+1)(b-x)-(\beta+1)(x-a)\}D+n(1+\alpha+\beta)]y = 0\,.$$

Very recently some attempts in connection with the application of Weisner's group theoretic method [3-5] in deriving generating functions of extended Jacobi polynomials and its different modifications, have been made by some authors. For previous works, see [6-10].

The aim of writing this article is to present a unified form of generating functions for some special functions in terms of modified extended Jacobi polynomials $F_n(\alpha - n, \beta, x)$ by the application of Weisner's group theoretic method, which is lucidly presented in the monograph "Obtaining Generating Functions" written by E.B. McBride [10], and then to show that those results which could have been obtained by double interpretations either to $(n, \alpha)$ or to $(n, \beta)$ simultaneously in Weisner's method can also be derived from our results. The main results of this article are given in Section 3.

## 2. Group theoretic discussion

Replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, $n$ by $y\frac{\partial}{\partial y}$ and $y$ by $u(x, y)$ in (1.1), we get

$$(2.1)\quad \Big[(x-a)(b-x)\frac{\partial^2}{\partial x^2} + \{(\alpha+1)(b-x) - (\beta+1)(x-a)\}\frac{\partial}{\partial x} - (b-x)y\frac{\partial^2}{\partial x \partial y} + (1+\alpha+\beta)y\frac{\partial}{\partial y}\Big]u = 0\,.$$

Thus $u_1(x, y) = F_n(\alpha - n, \beta, x)y^n$ is a solution of (2.1) since $F_n(\alpha - n, \beta, x)$ is a solution of (1.1).

We now look for two first order linear partial differential operators $A_2$ and $A_3$ such that

$$(2.2)\quad A_2(F_n(\alpha - n, \beta, x)y^n) = a_2(n)F_{n+1}(\alpha - n - 1, \beta, x)y^{n+1}$$

and

$$(2.3)\quad A_3(F_n(\alpha - n, \beta, x)y^n) = a_3(n)F_{n-1}(\alpha - n + 1, \beta, x)y^{n-1}\,.$$

Using (2.2) and the relation

$$(2.4)\quad (x-a)(b-x)\, DF_n(\alpha - n, \beta, x) = [(x-b)(\alpha - n)$$

$$+ (n+\beta+1)(x-a)]F_n(\alpha-n,\beta,x) - \frac{(n+1)(b-a)}{\lambda}F_{n+1}(\alpha-n-1,\beta,x) ,$$

we easily get

$$A_2 = \frac{\lambda}{b-a}(x-a)(b-x)y\frac{\partial}{\partial x} - \lambda y^2\frac{\partial}{\partial y} + [(b-x)\alpha - (\beta+1)(x-a)]\frac{\lambda y}{b-a} ,$$

so that

$$(2.5) \qquad A_2(F_n(\alpha-n,\beta,x)y^n) = -(n+1)F_{n+1}(\alpha-n-1,\beta,x)y^{n+1} .$$

Similarly, using (2.3) and the relation

$$(2.6) \quad (x-b)DF_n(\alpha-n,\beta,x) = nF_n(\alpha-n,\beta,x) - \lambda(\beta+n)F_{n-1}(\alpha-n+1,\beta,x) ,$$

we get

$$A_3 = (x-b)y^{-1}\lambda^{-1}\frac{\partial}{\partial x} - \lambda^{-1}\frac{\partial}{\partial y} ,$$

so that

$$(2.7) \qquad A_3(F_n(\alpha-n,\beta,x)y^n) = (-\beta-n)F_{n-1}(\alpha-n+1,\beta,x)y^{n-1} .$$

To find the group of operators let us write $A_1 = y\frac{\partial}{\partial y}$, then we have the following commutator relations:

$$[A_1, A_j] = (-1)^j A_j \quad (j = 2,3) ,$$

$$[A_2, A_3] = -(2A_1 + \beta + 1) ,$$

where

$$[A, B]u = (AB - BA)u .$$

From the above commutator relations, we arrive to the following theorem.

**Theorem.** *The set of operators* $\{1, A_i(i = 1,2,3)\}$, *where* 1 *stands for the identity operator, generates a Lie algebra* $\mathcal{L}$.

It can be shown that the partial differential operator

$$L = (x-a)(b-x)\frac{\partial^2}{\partial x^2} + \{(\alpha+1)(b-x) - (\beta+1)(x-a)\}\frac{\partial}{\partial x}$$

$$- (b-x)y\frac{\partial^2}{\partial x \partial y} + (1+\alpha+\beta)y\frac{\partial}{\partial y} ,$$

which can be expressed as follows

$$(x-b)L=(b-a)[A_2A_3-(A_1+\beta)A_1], \tag{2.8}$$

commutes with each of the operators $A_i$ $(i=1,2,3)$ , i.e. ,

$$[(x-b)L, A_i]=0 \quad , \quad i=1,2,3 \, . \tag{2.9}$$

The extended form of the groups generated by $A_i$ $(i=1,2,3)$ are as follows

$$\mathrm{e}^{a_1A_1} f(x,y)=f(x,\mathrm{e}^{a_1}y) \, , \tag{2.10}$$

$$\mathrm{e}^{a_2A_2} f(x,y)=(1+a_2\lambda y)^{\alpha}\left\{1+a_2\lambda\frac{x-a}{b-a}\,y\right\}^{-\alpha-\beta-1} \times f\left(\frac{x+a_2\lambda b\dfrac{x-a}{b-a}y}{1+a_2\lambda\dfrac{x-a}{b-a}y}\,,\;\frac{y}{1+a_2\lambda y}\right), \tag{2.11}$$

$$\mathrm{e}^{a_3A_3} f(x,y)=f\left(\frac{x-\dfrac{a_3b}{\lambda y}}{1-\dfrac{a_3}{\lambda y}}, y\left(1-\frac{a_3}{\lambda y}\right)\right) \, . \tag{2.12}$$

Thus we have

$$\begin{aligned} &\mathrm{e}^{a_3A_3}\mathrm{e}^{a_2A_2}\mathrm{e}^{a_1A_1} f(x,y) \\ &= \left\{1+a_2\lambda\left(y-\frac{a_3}{\lambda}\right)\right\}^{\alpha}\left\{1+a_2\lambda\left(\frac{x-a}{b-a}y-\frac{a_3}{\lambda}\right)\right\}^{-\alpha-\beta-1} \\ &\times f\left(\frac{xy-\dfrac{a_3b}{\lambda}+a_2\lambda b\left(y-\dfrac{a_3}{\lambda}\right)\left(\dfrac{x-a}{b-a}y-\dfrac{a_3}{\lambda}\right)}{\left(y-\dfrac{a_3}{\lambda}\right)\left\{1+a_2\lambda\left(\dfrac{x-a}{b-a}y-\dfrac{a_3}{\lambda}\right)\right\}}, \mathrm{e}^{a_1}\frac{y-\dfrac{a_3}{\lambda}}{1+a_2\lambda\left(y-\dfrac{a_3}{\lambda}\right)}\right) . \end{aligned} \tag{2.13}$$

## 3. Generating functions

From the above discussion it follows that $u(x,y)=F_n(\alpha-n,\beta,x)y^n$ is a solution of the system

$$\begin{cases} Lu=0 \\ (A_1-n)u=0 \, . \end{cases}$$

From (2.9), it is obvious that

$$S(x-b)L(F_n(\alpha-n,\beta,x)y^n)=(x-b)LS(F_n(\alpha-n,\beta,x)y^n)=0\ ,$$

where

$$S=e^{a_3A_3}e^{a_2A_2}e^{a_1A_1}\ .$$

Therefore the transformation $(SF_n(\alpha-n,\beta,x)y^n)$ is annihilated by $(x-b)L$.

By putting $a_1 = 0$ in (2.13) and then replacing $f(x,y)$ by $F_n(\alpha-n,\beta,x)y^n$, we get

$$\text{(3.1)}\qquad e^{a_3A_3}e^{a_2A_2}(F_n(\alpha-n,\beta,x)y^n)$$

$$=\left\{1+a_2\lambda\left(y-\frac{a_3}{\lambda}\right)\right\}^{\alpha-n}\left\{1+a_2\lambda\left(\frac{x-a}{b-a}y-\frac{a_3}{\lambda}\right)\right\}^{-\alpha-\beta-1}$$

$$\times F_n\left(\alpha-n,\beta,\frac{xy-\dfrac{a_3b}{\lambda}+a_2\lambda b\left(y-\dfrac{a_3}{\lambda}\right)\left(\dfrac{x-a}{b-a}y-\dfrac{a_3}{\lambda}\right)}{\left(y-\dfrac{a_3}{\lambda}\right)\left\{1+a_2\lambda\left(\dfrac{x-a}{b-a}y-\dfrac{a_3}{\lambda}\right)\right\}}\right)\left(y-\frac{a_3}{\lambda}\right)^n .$$

But

$$\text{(3.2)}\qquad e^{a_3A_3}e^{a_2A_2}(F_n(\alpha-n,\beta,x)y^n)$$

$$=\sum_{k=0}^{\infty}\sum_{p=0}^{n+k}\frac{(a_3/y)^p}{p!}\frac{(-a_2y)^k}{k!}(n+1)_k(-\beta-n-k)_p\cdot F_{n+k-p}(\alpha-n-k+p,\beta,x)y^n .$$

Equating (3.1) and (3.2), we get

$$\text{(3.3)}\quad (1-\frac{a_3}{\lambda y})^n\left\{1+a_2\lambda\left(y-\frac{a_3}{\lambda}\right)\right\}^{\alpha-n}\cdot\left\{1+a_2\lambda y\left(\frac{x-a}{b-a}y-\frac{a_3}{\lambda}\right)\right\}^{-\alpha-\beta-1}$$

$$\times F_n\left(\alpha-n,\beta,\frac{x-\dfrac{a_3b}{\lambda y}+a_2\lambda by\left(1-\dfrac{a_3}{\lambda y}\right)\left(\dfrac{x-a}{b-a}-\dfrac{a_3}{\lambda y}\right)}{\left(1-\dfrac{a_3}{\lambda y}\right)\left\{1+a_2\lambda y\left(\dfrac{x-a}{b-a}-\dfrac{a_3}{\lambda y}\right)\right\}}\right)$$

$$=\sum_{k=0}^{\infty}\sum_{p=0}^{n+k}\frac{(a_3/y)^p}{p!}\frac{(-a_2y)^k}{k!}(n+1)_k(-\beta-n-k)_p\cdot F_{n+k-p}(\alpha-n-k+p,\beta,x)\ ,$$

which seems to be new.

Before discussing particular cases of the result (3.3), we would like to point it out that the operators $A_2, A_3$ being non-commutative, as seen from the commutator relation $[A_2, A_3] = -(2A_1 + \beta + 1)$, the relation (3.3) will change if their order be interchanged in $e^{a_2 A_2} e^{a_3 A_3}$, which is given in Section 4.

We now consider the following particular cases:

Case 1: Putting $a_3 = 0$ and replacing $-a_3 y$ by $t$, we get

$$(3.4) \qquad (1-\lambda t)^{\alpha-n}\left\{1-\lambda t\frac{x-a}{b-a}\right\}^{-\alpha-\beta-1} \cdot F_n\left(\alpha-n,\beta,\frac{x-\lambda t\frac{x-a}{b-a}b}{1-\lambda t\frac{x-a}{b-a}}\right)$$

$$= \sum_{k=0}^{\infty}\frac{(n+1)_k}{k!}F_{n+k}(\alpha-n-k,\beta,x)t^k .$$

Sub-case: If we put $n = 0$, then

$$(3.5) \qquad (1-\lambda t)^{\alpha}\left\{1-\lambda t\frac{x-a}{b-a}\right\}^{-\alpha-\beta-1} = \sum_{k=0}^{\infty}F_k(\alpha-k,\beta,x)t^k .$$

Case 2: Putting $a_2 = 0$ and replacing $a_2/y$ by $t$ in (3.3), we get

$$(3.6) \qquad \left(1-\frac{t}{\lambda}\right)^n F_n\left(\alpha-n,\beta,\frac{x-tb/\lambda}{1-t/\lambda}\right)$$

$$= \sum_{p=0}^{\infty}\frac{(-\beta-n)_p}{p!}F_{n-p}(\alpha-n+p,\beta,x)t^p .$$

Case 3: Taking $a_2 a_3 \neq 0$,without any loss of generality we can choose $a_2 = \omega$ , $a_3 = -1$ and replacing $y$ by $t^{-1}$, we get

$$(3.7) \qquad \left(1+\frac{t}{\lambda}\right)^n\left\{1+\frac{\lambda}{\omega}\left(1+\frac{t}{\lambda}\right)\right\}^{\alpha-n}\cdot\left\{1+\frac{\lambda\omega}{t}\left(\frac{x-a}{b-a}+\frac{t}{\lambda}\right)\right\}^{-\alpha-\beta-1}$$

$$\times F_n\left(\alpha-n,\beta,\frac{\left(x+\frac{tb}{\lambda}\right)+\frac{b\omega\lambda}{t}\left(1+\frac{t}{\lambda}\right)\left(\frac{x-a}{b-a}+\frac{t}{\lambda}\right)}{\left(1+\frac{t}{\lambda}\right)\left\{1+\frac{\lambda\omega}{t}\left(\frac{x-a}{b-a}+\frac{t}{\lambda}\right)\right\}}\right)$$

$$= \sum_{k=0}^{\infty}\sum_{p=0}^{n+k}\frac{(-t)^p}{p!}\frac{(-\omega/t)^k}{k!}(n+1)_k(-\beta-n-k)_p\cdot F_{n+k-p}(\alpha-n-k+p,\beta,x) .$$

The above results are believed to be new.

Here we would like to mention that, by using the following symmetry relation:

(3.8) $$F_n(\alpha,\beta,a+b-x)=(-1)^n F_n(\beta,\alpha,x)\ ,$$

one can immediately get the following generating relations from the relations (3.4)-(3.7):

(3.9)
$$(1+\lambda t)^{\beta-n}\left\{1-\frac{\lambda}{b-a}(x-b)t\right\}^{-\alpha-\beta-1}\cdot F_n\left(\alpha,\beta-n,\frac{x-\dfrac{a\lambda}{b-a}(x-b)t}{1-\dfrac{\lambda}{b-a}(x-b)t}\right)$$
$$=\sum_{k=0}^{\infty}\frac{(n+1)_k}{k!}F_{n+k}(\alpha,\beta-n-k,x)t^k\ ,$$

(3.10) $$(1+\lambda t)^{\beta}\left\{1-\frac{\lambda}{b-a}(x-b)t\right\}^{-\alpha-\beta-1}=\sum_{k=0}^{\infty}F_k(\alpha,\beta-k,x)t^k\ ,$$

(3.11) $$\left(1+\frac{t}{\lambda}\right)^n F_n\left(\alpha,\beta-n,\frac{x+at/\lambda}{1+t/\lambda}\right)$$
$$=\sum_{p=0}^{n}\frac{(-\alpha-n)_p}{p!}F_{n-p}(\alpha,\beta-n+p,x)t^p\ ,$$

(3.12) $$\left(1+\frac{t}{\lambda}\right)^n\cdot\left\{1+\frac{\lambda\omega}{t}\left(1+\frac{t}{\lambda}\right)\right\}^{\beta-n}\cdot\left\{1+\frac{\lambda\omega}{t}\left(\frac{b-x}{b-a}+\frac{t}{\lambda}\right)\right\}^{-\alpha-\beta-1}$$
$$\times F_n\left(\alpha,\beta-n,\frac{x+\dfrac{at}{\lambda}+\dfrac{a\omega\lambda}{t}\left(1+\dfrac{t}{\lambda}\right)\left(\dfrac{b-x}{b-a}+\dfrac{t}{\beta}\right)}{\left(1+\dfrac{t}{\lambda}\right)\left\{1+\dfrac{\lambda\omega}{t}\left(\dfrac{b-x}{b-a}+\dfrac{t}{\lambda}\right)\right\}}\right)$$
$$=\sum_{k=0}^{\infty}\sum_{p=0}^{n+k}\frac{\omega^k(n+1)_k}{k!t^k}\,\frac{t^p(-\alpha-n-p)_p}{p!}\cdot F_{n+k-p}(\alpha,\beta-n-k+p,x)\ .$$

Now, if we replace $\alpha$ by $\alpha+n$ on both sides of (3.4)-(3.7) and $\beta$ by $\beta+n$ on both sides of (3.9)-(3.12), we get the results which could have been obtained by double interpretations to $(n, \alpha)$ and $(n, \beta)$ respectively while applying Weisner's group theoretic method in the study of $F_n(\alpha, \beta, x)$.

Some special cases are given below:

Special Case 1: Putting $a = 0$ , $\lambda = 1$ and $\beta = b$ in (3.4) and (3.6) and then simplifying and finally taking limit as $b \to \infty$, we get the following results on generating functions involving Laguerre polynomials:

$$(3.13) \quad (1+t)^{\alpha-n}\exp(-xt)L_n^{(\alpha-n)}(x(1+t)) = \sum_{k=0}^{\infty} \frac{(n+1)_k}{k!} L_{n+k}^{(\alpha-n-k)}(x)t^k ,$$

$$(3.14) \quad L_n^{(\alpha-n)}(x-t) = \sum_{p=0}^{n} \frac{1}{p!} L_{n-p}^{(\alpha-n+p)}(x)t^p .$$

Now putting $n = 0$ in (3.13), we get

$$(3.15) \quad (1+t)^{\alpha}\exp(-xt) = \sum_{k=0}^{\infty} L_k^{(\alpha-k)}(x)t^k .$$

The above results do not seem to have appeared in the earlier works.

Special Case 2: Putting $\alpha = \beta$ , $-a = b = \sqrt{\alpha}$, recalling that $\lambda \to 2/\sqrt{\alpha}$ in (3.4) and (3.6) and then taking limit as $\alpha \to \infty$, we get the following results on Hermite polynomials:

$$(3.16) \quad \exp(2xt - t^2)H_n(x-t) = \sum_{k=0}^{\infty} \frac{1}{k!} H_{n+k}(x)t^k ,$$

$$(3.17) \quad H_n(x+t) = \sum_{p=0}^{n} \binom{n}{p} H_{n-p}(x)(2t)^p .$$

Now putting $n = 0$ in (3.16), we get

$$(3.18) \quad \exp(2xt - t^2) = \sum_{k=0}^{\infty} \frac{1}{k!} H_k(x)t^k .$$

The above results are due to L. Weisner [4].

Special Case 3: Writing $1+2x/s$ , $s\omega/\epsilon$ in place of $x$ and $t$ in (3.4) and (3.6) and then putting $-a=b=\lambda=1$ , $\alpha=\nu-\epsilon-1$ , $\beta=\epsilon-1$ and then taking limit as $\epsilon\to\infty$, we get the following results on Bessel polynomials:
(3.19)

$$(1-x\omega)^{1-\nu}\exp(s\omega)\cdot y\left(\frac{x}{1-x\omega},\nu-n,s\right)=\sum_{k=0}^{\infty}y_{n+k}(x,\nu-n-k,s)\frac{(s\omega)^k}{k!},$$

(3.20)
$$(1+t)^n y\left(\frac{x}{1-t},\nu-n,s\right)=\sum_{p=0}^{\infty}\binom{n}{p}y_{n-p}(x,\nu-n+p,s)t^p.$$

Putting $n=0$ in (3.19), we get

(3.21)
$$(1-x\omega)^{1-\nu}\exp(s\omega)=\sum_{k=0}^{\infty}y_k(x,\nu-k,s)\frac{(s\omega)^k}{k!}.$$

The above results involving Bessel polynomials introduced by H.L. Krall and O. Frink [11] are found derived in [12-13].

Special Case 4: Putting $-a=b=1$ and $\lambda=1$ in (3.4) and (3.6) we get the following results of Jacobi polynomials:

(3.22)
$$(1-t)^{\alpha-n}\left\{1-\frac{t}{2}(1+x)\right\}^{-\alpha-\beta-1}\cdot P_n^{(\beta,\alpha-n)}\left(\frac{x-\frac{t}{2}(1+x)}{1-\frac{t}{2}(1+x)}\right)$$
$$=\sum_{k=0}^{\infty}\frac{(n+1)_k}{k!}P_{n+k}^{(\beta,\alpha-n-k)}(x)t^k,$$

(3.23)
$$(1-t)^n P_n^{(\beta,\alpha-n)}\left(\frac{x-t}{1-t}\right)=\sum_{p=0}^{n}\frac{(-\beta-n)_p}{p!}P_{n-p}^{(\beta,\alpha-n+p)}(x)t^p.$$

Now putting $n=0$ in (3.22) and then interchanging $\alpha,\beta$, we get the following well known result of Feldheim [14]:

(3.24)
$$(1-t)^{\beta}\left\{1-\frac{t}{2}(1+x)\right\}^{-\alpha-\beta-1}=\sum_{k=0}^{\infty}P_k^{(\alpha,\beta-k)}(x)t^k,$$

and the same is also derived by W.A. Al-Salam [15] and V.K. Verma [16] in different way.

Here we would like to mention that, by using the symmetry relation [17]:

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x), \tag{3.25}$$

one can get the following generating relations from (3.22) and (3.23):

$$(1+t)^{\alpha-n}\left\{1+\frac{t}{2}(1-x)\right\}^{-\alpha-\beta-1} \cdot P_n^{(\alpha-n,\beta)}\left(\frac{x-\frac{t}{2}(1-x)}{1+\frac{t}{2}(1-x)}\right) = \sum_{k=0}^{\infty}\frac{(n+1)_k}{k!}P_{n+k}^{(\alpha-n-k,\beta)}(x)t^k, \tag{3.26}$$

and

$$(1+t)^n P_n^{(\alpha-n,\beta)}\left(\frac{x-t}{1+t}\right) = \sum_{p=0}^{n}\frac{(-\beta-n)_p}{p!}P_{n-p}^{(\alpha-n+p,\beta)}(x)t^p. \tag{3.27}$$

If we replace $\alpha$ by $\alpha+n$ in (3.13)-(3.14), (3.16)-(3.17), (3.26)-(3.27) we get the results found derived in [18,19,20,21] and if we interchange $\alpha,\beta$ and then replace $\beta$ by $\beta+n$ in (3.22)-(3.23), we get the results of A.B. Chakraborti [22].

## 4. Variant of the result (3.3)

By interchanging the order of operators $A_2$ and $A_3$ in $e^{a_3A_3}e^{a_2A_2}e^{a_1A_1}$, we get

$$e^{a_2A_2}e^{a_3A_3}e^{a_1A_1}f(x,y) = (1+a_2\lambda y)^{\alpha}\left\{1+a_2\lambda\frac{x-a}{b-a}y\right\}^{-\alpha-\beta-1}\cdot f(\xi,\eta), \tag{4.1}$$

where

$$\xi = \frac{x-\frac{a_3}{\lambda y}b(1+a_2\lambda y)+a_2\lambda b\frac{x-a}{b-a}y\left\{1-\frac{a_3}{\lambda y}(1+a_2\lambda y)\right\}}{\left(1+a_2\lambda\frac{x-a}{b-a}y\right)\left\{1-\frac{a_3}{\lambda y}(1+a_2\lambda y)\right\}},$$

$$\eta = e^{a_1}\frac{y\left\{1-\frac{a_3}{\lambda y}(1+a_2\lambda y)\right\}}{1+a_2\lambda y}.$$

Putting $a_1=0$, we get

$$e^{a_2A_2}e^{a_3A_3}(F_n(\alpha-n,\beta,x)y^n) \tag{4.2}$$

$$= (1+a_2\lambda y)^{\alpha-n}\left\{1+a_2\lambda\frac{x-a}{b-a}y\right\}^{-\alpha-\beta-1}\cdot\left\{1-\frac{a_3}{\lambda y}(1+a_2\lambda y)\right\}^n y^n$$

$$\times F_n\left(\alpha-n,\beta,\frac{x-\frac{a_3}{\lambda y}b(1+a_2\lambda y)+a_2\lambda b\frac{x-a}{b-a}y\left\{1-\frac{a_3}{\lambda y}(1+a_2\lambda y)\right\}}{\left(1+a_2\lambda\frac{x-a}{b-a}y\right)\left\{1-\frac{a_3}{\lambda y}(1+a_2\lambda y)\right\}}\right).$$

But

$$\text{(4.3)}\qquad e^{a_2A_2}e^{a_3A_3}(F_n(\alpha-n,\beta,x)y^n)$$

$$=\sum_{k=0}^{\infty}\sum_{p=0}^{n+k}\frac{(-a_2y)^k}{k!}\frac{(a_3/y)^p}{p!}(-\beta-n)_p(n-p-1)_k\cdot F_{n-p+k}(\alpha-n+p-k,\beta,x)y^n.$$

Equating (4.2) and (4.3), we get

$$\text{(4.4)}\qquad (1+a_2\lambda y)^{\alpha-n}\left\{1+a_2\lambda\frac{x-a}{b-a}y\right\}^{-\alpha-\beta-1}\cdot\left\{1-\frac{a_3}{\lambda y}(1+a_2\lambda y)\right\}^n$$

$$\times F_n\left(\alpha-n,\beta,\frac{x-\frac{a_3}{\lambda y}b(1+a_2\lambda y)+a_2\lambda b\frac{x-a}{b-a}y\left\{1-\frac{a_3}{\lambda y}(1+a_2\lambda y)\right\}}{\left(1+a_2\lambda\frac{x-a}{b-a}y\right)\left\{1-\frac{a_3}{\lambda y}(1+a_2\lambda y)\right\}}\right)$$

$$=\sum_{k=0}^{\infty}\sum_{p=0}^{n+k}\frac{(-a_2y)^k}{k!}\frac{(a_3/y)^p}{p!}(-\beta-n)_p(n-p+1)_k\cdot F_{n-p+k}(\alpha-n+p-k,\beta,x).$$

**Observation.** From the discussion of the paper it may be concluded that the double interpretations in the application of Weisner's method while studying a special function $p_n^{(\alpha)}(x)$ for deriving generating functions, may be replaced and the desired results obtained by double interpretations of $\alpha$ and $n$ may be easily derived from the results obtained by single interpretation to the index ($n$) in the Weisner's method while studying $p_n^{(\alpha-n)}(x)$, a modified form of $p_n^{(\alpha)}(x)$ only by a mere replacement of $\alpha$ by $\alpha+n$.

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