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Finite Integral Transforms for Non-Local Boundary Value Problems in the Complex Case

Radka I. Petrova

Presented by V. Kiryakova

Finite integral transforms of first kind when the non-local functional Φ is of the form

$$1)\ \Phi(f) = \frac{1}{k+1}[f'(0) + kf'(1)], \quad k \neq -1$$
$$2)\ \Phi(f) = \frac{1}{k+1}\left[f'(1) + kf'\left(\tfrac{1}{2}\right)\right], \quad k \neq -1,$$

in the complex case are found. 111 The convolution property and the inversion formula of these transforms are found.

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The considered finite integral transforms are connected with the non-local boundary eigenvalue problem

$$y'' + \lambda^2 y = f(x)$$

$$\Phi(y) = 0 \quad \Psi(y) = 0 \tag{1}$$

where Φ and Ψ are linear functionals on $C^1[0,1]$ and Φ is a non-local functional. There is no loss of generality to assume that $\Phi\{x\} = 1$. Let $supp\,\Phi$ contain at least a point $x_0 \neq 0$. This condition $\{supp\,\Phi \neq 0\}$ ensures infinite number of distinct eigenvalues $\nu_n = -\lambda_n^2$, $n = 1, 2, \ldots$ of (1). All λ_n and $(-\lambda_n)$ are the zeros of the entire function of exponential type

$$E(\lambda) = \Phi_\xi\{X(\lambda,\xi)\},$$

where $X(\lambda_n, \xi)$ are the eigenvalue functions of the operator $D = \dfrac{d^2}{dx^2}$.

Definition 1. Let $f \in C[0,1]$ and $R_\lambda f = R(f;\lambda)$ be the resolvent operator of (1) defined as a solution of the boundary value problem (1). The correspondence

$$f \to T_n\{f\} = \frac{1}{2\pi i}\int_{C_n} R(f;\lambda)d\lambda,\ n = 1,2,\ldots, \tag{2}$$

where C_n is a simple contour containing λ_n only, is said to be a finite integral transform for the problem (1).

There are three types of finite integral transforms for problem (1). Now we will consider the first kind finite integral transforms, i.e.

$$\Psi(y) = y(0)\,.$$

In this case $X(\lambda,x) = \dfrac{\sin\lambda x}{\lambda}$ and the resolvent operator $y = R_\lambda f$ can be represented in the form

$$R_\lambda f(x) = \left\{\frac{\sin\lambda\xi}{\lambda E(\lambda)}\right\} * f, \tag{3}$$

where * denotes the convolution operation

$$(f * g)(x) = -\Phi l_\xi\left[\frac{1}{2}\int_x^\xi f(\xi + x - \eta)g(\eta)d\eta\right.$$

$$\left.-\frac{1}{2}\int_{-x}^\xi f(|\xi - x - \eta|)g(|\eta|)\,\mathrm{sgn}\eta(\xi - x - \eta)d\eta\right]. \tag{4}$$

Using representation (3) the n transform $T_n\{f\}$ can be written in the form

$$T_n f(x) = \vartheta_n(x) * f(x),\ n = 1,2,\ldots \tag{5}$$

with

$$\vartheta_n(x) = \frac{1}{2\pi i}\int_{C_n}\frac{X(\lambda;x)}{E(\lambda)}d\lambda. \tag{6}$$

The operator $R_0 f$ is a right inverse operator of D. It can be represented as a convolution operator in the form

$$R_0 f(x) = x * f. \tag{7}$$

The main properties of these transforms: convolution property, its differential law are considered in other papers. Now we are investigating the problem of the inversion formula.

Let us introduce for each $f(x) \in C[0,1]$ the formal expansion

$$f(x) \sim \sum_{1}^{+\infty} T_n\{f\}. \tag{8}$$

If the formal expansion (8) represents the function $f(x)$ in $C^2[0,1]$, i.e. if

$$f(x) = \sum_{1}^{\infty} T_n\{f\}, \tag{9}$$

then we can consider (9) as an inversion formula.

Theorem 1. *Let $f(x) \in C^2[0,1]$, $f(0)=0$, $\Phi(f)=0$. If the expansion (9) for the function $f(x) \equiv x$ is uniformly convergent and represents it, then the expansion (9) is uniformly convergent in [0,1] and represents the function $f(x)$ too, i.e. (9) holds .*

Proof. Since $I = R_0 D - F$ and under these assumptions, the defining projector is $Ff = f(0)[1-\Phi(1)x] + x\Phi(f) = 0$.Then

$$f = R_0 Df = R_0 f'' = x * f'' = \sum_{1}^{\infty} R_0 \vartheta_n * f''$$

$$= \sum_{1}^{\infty} \vartheta_n * R_0 Df = \sum_{1}^{\infty} \vartheta_n * f = \sum_{1}^{\infty} T_n f.$$

<u>Case 1.</u> Let the functional $\Phi(f)$ have the following representation

$$\Phi(f) = \frac{1}{k+1}[f'(0) + kf'(1)], \text{ where } k \neq -1. \tag{10}$$

The operator R_0 takes the form

$$R_0 f(x) = \int_0^x d\tau \int_0^\tau f(\sigma)d\sigma - \frac{xk}{k+1}\int_0^1 f(\tau)d\tau \tag{11}$$

and

$$R_0\{x\} = \frac{x^3}{6} - \frac{k}{2(k+1)}x. \tag{12}$$

In this case the indicatrix $E(\lambda)$ has the representation

$$E(\lambda) = \Phi_\xi\left\{\frac{\sin\lambda\xi}{\lambda}\right\} = \frac{1}{k+1}(1 + k\cos\lambda). \tag{13}$$

It is an entire function of exponential type. It has simple zeros λ_n and $\mu_n = -\lambda_n$. When $k \in (-\infty,-1)\cup[1,+\infty)$, $\left(-\frac{1}{k}\right) \in [-1,1]$, then the function $E(\lambda)$ has real zeros. This case is considered in [2] and the case $k = -1$ in [4]. Now we will investigate the problem when $k \in (-1,0)$ and $k \in (0,1)$. In these cases $|-\frac{1}{k}| > 1$ and then the function $E(\lambda)$ has two series of complex zeros λ_n and μ_n.

Lemma 1. *The transform (5) has the form*

$$T_n\{f\} = a_n \sin \lambda_n x, \tag{14}$$

where $a_n = \dfrac{2}{\sin \lambda_n} \displaystyle\int_0^1 f(\tau) \cos \lambda_n(\tau - 1) d\tau.$

Proof. Since the function $E(\lambda)$ has simple zeros, then

$$\vartheta_n^{\mu}(x) = \frac{\sin \mu_n x}{E'(\mu_n)} = \frac{\sin -\lambda_n x}{-E'(\lambda_n)} = \frac{\sin \lambda_n x}{E'(\lambda_n)} = \vartheta_n^{\lambda}(x) = \vartheta_n(x)$$

and $\vartheta_n(x) * f$ can be easily transformed into (14).

Theorem 2. *Expansion (8) for the function $f(x) \equiv x$ gives an uniformly convergent series which represents it in [0,1], i.e.*

$$x = 2\frac{(k+1)}{k} \sum_{-\infty}^{+\infty} \frac{\sin \lambda_n x}{\lambda_n^2 \sin \lambda_n}. \tag{15}$$

Proof. Let $f(x) \equiv x$, then $x \sim 2\dfrac{(k+1)}{k} \displaystyle\sum_{-\infty}^{+\infty} \frac{\sin \lambda_n x}{\lambda_n^2 \sin \lambda_n}$.

If $k \in (-1,0)$, then

$$\frac{1+\sqrt{1-k^2}}{-k} > 1 \quad \text{and} \quad \lambda_n = 2n\pi + i \ln \frac{1+\sqrt{1-k^2}}{-k}, \; n = 0, \pm 1, \pm 2, \ldots$$

Under the substitution $\ln \dfrac{1+\sqrt{1-k^2}}{-k} = b$ we have $\lambda_n = 2n\pi + ib$.

Then, $\sin \lambda_n = i \sinh b = -i\dfrac{\sqrt{1-k^2}}{k}$. Using the known identity

$$\sum_{-\infty}^{+\infty} \frac{\cos(m+a)x}{m+a} = \frac{\pi}{\tan a\pi}, \tag{16}$$

valid for $x \in (0, 2\pi)$ (see [3]), the following identities can be obtained:

$$\sum_{-\infty}^{+\infty} \frac{\cos \lambda_n x}{\lambda_n} = \frac{-i}{2\tanh \frac{b}{2}} = \frac{-i}{2}\sqrt{\frac{1-k}{1+k}}. \tag{17}$$

Integrating this with respect to x we have

$$\sum_{-\infty}^{+\infty} \frac{\sin \lambda_n x}{\lambda_n^2} = \frac{-xi}{2\tanh\frac{b}{2}} = -\frac{x}{2}i\sqrt{\frac{1-k}{1+k}},$$

where $\tanh\frac{b}{2} = \sqrt{\frac{1+k}{1-k}}$ and $x \in (0,1)$. By means of these identities it can be easily be shown that (15) is valid in $0 < x < 1$. Evidently (15) is valid for $x = 0$. If $x = 1$ then the the identity

$$2\frac{(k+1)}{k}\sum_{-\infty}^{+\infty}\frac{1}{\lambda_n^2} = 1. \tag{18}$$

should be proved. Indeed,

$$\sum_{-\infty}^{+\infty}\frac{1}{\lambda_n^2} = \sum_{-\infty}^{+\infty}\frac{1}{(2n\pi+ib)^2} = \frac{1}{4\pi^2}\sum_{-\infty}^{+\infty}\frac{1}{(n+ia)^2}$$

$$= \frac{1}{4\pi^2}\left(-\frac{1}{a^2}+2\sigma_1-4a^2\sigma_2\right) = \frac{k}{2(k+1)},$$

where $\sigma_1 = \frac{1}{2a}\pi\cosh a\pi - \frac{1}{2a^2}$ and $\sigma_2 = \frac{1}{4\pi^2}\left(\frac{1}{a}\coth a\pi + \frac{\pi}{\sinh^2 a\pi}\right) - \frac{1}{2a^2}$, (see [3]). Then (18) is proved.

If $k \in (0,1)$, then $\frac{1+\sqrt{1-k^2}}{k} > 1$, and

$$\lambda_n = (2n+1)\pi + i\ln\frac{1+\sqrt{1-k^2}}{k}, \quad \mu_n = (2n+1)\pi - i\ln\frac{1+\sqrt{1-k^2}}{k},$$

$n = 0, \pm 1, \pm 2, \ldots$. Now, $\ln\frac{1+\sqrt{1-k^2}}{k} = b_1$ and $\mu_n = -\lambda_n$, $\sin\lambda_n = -\sin ib_1 = -i\sinh b_1 = -i\frac{\sqrt{1-k^2}}{k}$. Using (16), it is obtained (17). By integration with respect to x, finally we have (15) when $x \in (0,1)$. The proof of (15) when $x = 1$ and $k \in (0,1)$ is the same as the case $k \in (-1,0)$. Using σ_1 and σ_2, we obtain that

$$\sum_{-\infty}^{+\infty}\frac{1}{\lambda_n^2} = \sum_{-\infty}^{+\infty}\frac{1}{[(2n+1)\pi+ib_1]^2} = \frac{k}{2(k+1)}.$$

As a sum of real series, the function x has the form

$$x = 2\sqrt{\frac{1+k}{1-k}}\left\{\sinh bx\left[\frac{1}{b^2} - 2\sum_{1}^{\infty}\frac{\cos 2nx\pi}{(2n\pi)^2+b^2}\right.\right.$$

$$+4b^2\sum_1^\infty \frac{\cos 2nx\pi}{[(2n\pi)^2+b^2]^2}\Bigg] + 4b\cosh bx \sum_1^\infty \frac{2n\pi \sin 2nx\pi}{[(2n\pi)^2+b^2]^2}\Bigg\}, \tag{19}$$

if $k \in (-1,0)$, $x \in [0,1]$ and

$$x = 4\sqrt{\frac{1+k}{1-k}}\Bigg\{\sinh b_1x\Bigg[2b_1^2\sum_0^\infty \frac{\cos(2n+1)x\pi}{[(2n+1)^2\pi^2+b_1^2]^2} - \sum_0^\infty \frac{\cos(2n+1)x\pi}{(2n+1)^2\pi^2+b_1^2}\Bigg]$$

$$+2b_1\cosh b_1x\sum_0^\infty \frac{(2n+1)\pi\sin(2n+1)x\pi}{[(2n+1)^2\pi^2+b_1^2]^2}\Bigg\}, \tag{20}$$

if $k \in (0,1)$ and $x \in [0,1]$.

Finally, applying Theorem 1 for the function $f(x)$, we obtain the inversion formula

$$f(x) = \sum_{-\infty}^{+\infty} \frac{2}{\sin\lambda_n}\int_0^1 f(\tau)\cos\lambda_n(\tau-1)d\tau.\sin\lambda_n x,$$

where $\lambda_n = 2n\pi + i\ln\dfrac{1+\sqrt{1-k^2}}{-k}$ as $k \in (-1,0)$ and

$\lambda_n = (2n+1)\pi + i\ln\dfrac{1+\sqrt{1-k^2}}{k}$ as $k \in (0,1)$ and $x \in [0,1]$.

Case 2. Let $\Phi(f)$ have the form

$$\Phi(f) = \frac{1}{k+1}\left[f'(1) + kf'\left(\frac{1}{2}\right)\right],\ k \neq -1. \tag{21}$$

Then

$$R_0 f(x) = \int_0^x d\tau\int_0^\tau f(\sigma)d\sigma - \frac{x}{k+1}\left[\int_0^1 f(\tau)d\tau + k\int_0^{\frac{1}{2}} f(\tau)d\tau\right] \tag{22}$$

and

$$R_0\{x\} = \frac{x^3}{6} - \frac{k+4}{8(k+1)}x.$$

The entire function of exponential type $E(\lambda)$ has the following representation

$$E(\lambda) = \Phi_\xi\left\{\frac{\sin\lambda\xi}{\lambda}\right\} = \frac{1}{k+1}\left(\cos\lambda + k\cos\frac{\lambda}{2}\right)$$

$$= \frac{1}{k+1}\left(2\cos^2\frac{\lambda}{2} + k\cos\frac{\lambda}{2} - 1\right). \tag{23}$$

It has simple zeros λ_n and $\mu_n = -\lambda_n$, since k is a real number. The substitution $\cos\frac{\lambda}{2} = z$ into (23) gives the quadratic equation

$$2z^2 + kz - 1 = 0\,,.$$

having two solutions:

$$z_1 = -\frac{k+\sqrt{k^2+8}}{4} \quad , \quad z_2 = \frac{\sqrt{k^2+8}-k}{4}.$$

Let $\cos\frac{\lambda}{2} = z_1$ and $\cos\frac{\mu}{2} = z_2$. We obtain two series of solutions λ_n and μ_n.

Lemma 2. *The transform (5) can be represented in the form*

$$T_n\{f\} = a_n \sin\lambda_n x - b_n \sin\mu_n x\,, \tag{24}$$

where $a_n = c_n^{\lambda}$, $b_n = c_n^{\mu}$ *and*

$$c_n^{\theta} = -\frac{4}{\sqrt{k^2+8}}\frac{1}{\sin\frac{\theta_n}{2}}\left[\int_0^1 f(\tau)\cos\theta_n(\tau-1)d\tau + k\int_0^{\frac{1}{2}} f(\tau)\cos\theta_n(\tau-\frac{1}{2})d\tau\right].$$

Proof. Analogous to that of Lemma 1. Can be done by direct check.

Now we can introduce a formal eigenexpansion for the problem (1),

$$f \sim \sum_1^{\infty} T_n\{f\} = \sum_1^{\infty}(a_n \sin\lambda_n x - b_n \sin\mu_n x).$$

Then for the function $f(x) \equiv x$ we have

$$x \sim \frac{4(k+1)}{\sqrt{k^2+8}}(B-A), \tag{25}$$

$$\text{where} \quad A = \sum_1^{\infty}\frac{\sin\lambda_n x}{\lambda_n^2 \sin\frac{\lambda_n}{2}} \quad , \quad B = \sum_1^{\infty}\frac{\sin\mu_n x}{\mu_n^2\sin\frac{\mu_n}{2}}.$$

Let us prove Theorem 2 in this case.

Case 2^a. Let $k \in (-\infty,-1)$. The solutions of the quadratic equation are $z_1 \in \left[-\frac{1}{6},1\right]$ and $z_2 > 1$. The equation $\cos\frac{\lambda}{2} = z_1$ has real solutions. If $\beta = \arccos z_1$, $\beta \in \left[0,\frac{7\pi}{12}\right)$ then $\lambda_n = \pm 2\beta + 4n\pi$,. We will take only the positive zeros $\lambda_n = |2\beta + 4n\pi|$, $n = 0, \pm 1, \pm 2, \ldots$. The equation $\cos\frac{\mu}{2} = z_2 > 1$ has not real solutions. For μ_n we obtain two series of complex numbers $\mu_n^{(1)} =$

$4n\pi - 2i\ln y_1$ and $\mu_n^{(2)} = 4n\pi + 2i\ln y_1$, where $y_1 = z_2 + \sqrt{z_2^2 - 1} > 1$, $n = 0, \pm 1, \pm 2, \ldots$, $\mu_n^{(1)} = -\mu_n^{(2)}$ and further, we consider for

$$\mu_n = \mu_n^{(2)} = 4n\pi + 2i\ln y_1,\ n \in (-\infty, +\infty) \text{ and } n \text{ is an integer} .$$

Now let us consider the series A and B. Since

$$\sin\frac{\lambda_n}{2} = \sin|\beta + 2n\pi| = \sin[(\beta + 2n\pi)\mathrm{sgn}n] = \mathrm{sgn}n.\sin\beta;$$

$$\sum_{-\infty}^{+\infty}\frac{\cos\lambda_n x}{\lambda_n \mathrm{sgn}n} = \sum_{-\infty}^{+\infty}\frac{\cos(2\beta + 4n\pi)x}{2\beta + 4n\pi} = \frac{1}{4\pi}\sum_{-\infty}^{+\infty}\frac{\cos(n+a)4x\pi}{n+a} = \frac{1}{4}\frac{1}{\tan\frac{\beta}{2}}.$$

Here $a = \frac{\beta}{2\pi}$ is not an integer and $x \in (0, 1/2)$ (see [3]). By integration with respect to x, we obtain

$$\sum_{-\infty}^{+\infty}\frac{\sin\lambda_n x}{\lambda_n^2 \mathrm{sgn}n} = \frac{1}{4\tan\frac{\beta}{2}}x \quad \text{and} \quad A = \frac{1}{\sin\beta}\frac{x}{4\tan\frac{\beta}{2}} = \frac{x}{4(1 - z_1)}.$$

Since $\sin\frac{\mu_n 2}{=}\sin(i\ln y_1) = i\sinh\ln y_1 = i\sinh = b$, where $b = \ln y_1$,we consider

$$\sum_{-\infty}^{+\infty}\frac{\cos\mu_n x}{\mu_n} = \frac{1}{4\pi}\sum_{-\infty}^{+\infty}\frac{\cos(n + i\frac{b}{2\pi})4x\pi}{n + i\frac{b}{2\pi}}, \text{ where } a = i\frac{b}{2\pi},\ x \in (0, 1/2).$$

Using (16) we find $\sum_{-\infty}^{+\infty}\frac{\cos\mu_n x}{\mu_n} = \frac{1}{4i\tanh\frac{b}{2}}$ and by integration, $\sum_{-\infty}^{+\infty}\frac{\sin\mu_n x}{\mu_n^2} = \frac{x}{4i\tanh\frac{b}{2}}$. Finally, for B we obtain

$$B = \frac{x}{4(1 - \cosh b)} = \frac{x}{4(1 - z_2)}$$

and for $B - A$,

$$B - A = \frac{x}{4(1 - z_2)} - \frac{x}{4(1 - z_1)} = \frac{x}{4}\frac{z_2 - z_1}{(1 - z_1)(1 - z_2)} = \frac{x}{4}\frac{\sqrt{k^2 + 8}}{k + 1}.$$

Then

$$x = \frac{4(k+1)}{\sqrt{k^2 + 8}}\left[\sum_{-\infty}^{+\infty}\frac{\sin\mu_n x}{\mu_n^2\sin\frac{\mu_n}{2}} - \sum_{-\infty}^{+\infty}\frac{\sin\lambda_n x}{\lambda_n^2\sin\frac{\lambda_n}{2}}\right] \tag{26}$$

as asserted. Evidently, identity (26) is valid for $x = 0$. Now we prove that (26) is valid for $x = 1/2$ too. Let $x = 1/2$. The right side of (26) has the form

$$S = \frac{4(k+1)}{\sqrt{k^2+8}}\left[\sum_{-\infty}^{+\infty}\frac{1}{\mu_n^2} - \sum_{\infty}^{+\infty}\frac{1}{\lambda_n^2}\right]$$

$$= \frac{1}{4\pi^2}\frac{k+1}{\sqrt{k^2+8}}\left[-\frac{1}{a^2} + 2\sigma_1 - 4a^2\sigma_2 - \frac{1}{b^2} - 2\overline{\sigma_1} - 4b^2\overline{\sigma_2}\right],$$

where $b = \dfrac{\beta}{2\pi}$, $a = \dfrac{\ln y_1}{2\pi}$. As we use σ_1, σ_2 and the sums of the well known numerical series of [3],

$$\overline{\sigma_1} = \frac{1}{2x}\left(\frac{1}{x} - \pi\cot\pi x\right) \quad \text{and}$$

$$\overline{\sigma_2} = -\frac{1}{2a^4} + \frac{\pi}{4a^2}\left(\frac{1}{a}\cot a\pi + \frac{\pi}{\sin^2 a\pi}\right),$$

where $\overline{\sigma_k} = \sum_{1}^{+\infty}\dfrac{1}{(n^2-a^2)^k}$, it is obtained that $S = 1/2$. Therefore, (26) is valid for $x \in [0, 1/2]$.

<u>Case 2^b.</u> $k \in (-1, 1]$. Now $z_1 \in [-1, -1/2]$. The equation $\cos\frac{\lambda}{2} = z_1$ has real solutions. If $\alpha = \arccos z_1$, $\alpha \in \left[\frac{2\pi}{3}, \pi\right]$, $\lambda_n = \pm 2\alpha + 4n\pi$. We will take only the positive solutions $\lambda_n = |2\alpha + 4n\pi|$, $n = 0, \pm1, \pm2, \ldots$, $z_2 \in [\frac{1}{2}, 1]$, $\beta = \arccos z_2$, $\beta \in [0, \frac{\pi}{3}]$, $\mu_n = \pm 2\beta + 4n\pi$. As $\mu_n > 0$, then $\mu_n = |2\beta + 4n\pi|$. In this case λ_n and μ_n are real numbers.

We are to prove the identity (26). As in the case 2^a, $\sin\frac{\lambda_n}{2} = \operatorname{sgn} n \sin\alpha$ and $\sin\frac{\mu_n}{2} = \operatorname{sgn} n \sin\beta$, then

$$A = \sum_{-\infty}^{+\infty}\frac{\sin\lambda_n x}{\lambda_n^2\sin\frac{\lambda_n}{2}} = \frac{1}{\sin\alpha}\frac{x}{4\tan\frac{\alpha}{2}} = \frac{x}{4(1-z_1)} \quad \text{and}$$

$$B = \sum_{-\infty}^{+\infty}\frac{\sin\mu_n x}{\mu_n^2\sin\frac{\mu_n}{2}} = \frac{1}{\sin\beta}\frac{x}{4\tan\frac{\beta}{2}} = \frac{x}{4(1-z_2)}.$$

Therefore, if $x \in [0, 1/2)$, (26) is valid in this case too. Let $x = \dfrac{1}{2}$,

$$S = \frac{4(k+1)}{\sqrt{k^2+8}}\left[-\sum_{-\infty}^{+\infty}\frac{1}{(2\alpha+4n\pi)^2} + \sum_{-\infty}^{+\infty}\frac{1}{(2\beta+4n\pi)^2}\right]$$

$$= \frac{1}{4\pi^2}\frac{4(k+1)}{\sqrt{k^2+8}}\left[-\frac{1}{a^2} - 2\sigma_1 - 4a^2\sigma_2 + \frac{1}{b^2} + 2\overline{\sigma_1} + 4b^2\overline{\sigma_2}\right]$$

$$= \frac{1}{2}\frac{(k+1)}{\sqrt{k^2+8}}\left(\frac{1}{1-z_2} - \frac{1}{1-z_1}\right) = \frac{1}{2}.$$

Case2^c. Let $k \in (1, +\infty)$. The solution z_1 is negative and z_2 is positive and $z_1 < -1$, $z_2 \in (0, 1/2)$. The equation $\cos\lambda = z_1$ has no real solutions. Its solutions are the complex numbers

$$\lambda_n^{(1)} = 2(2n+1)\pi + 2i\ln y_2 \quad \text{and} \quad \lambda_n^{(2)} = 2(2n+1)\pi - 2i\ln y_2,$$

where $y_2 = \sqrt{z_1^2 - 1} - z_1 > 0$, $\lambda_n^{(1)} = -\lambda_n^{(2)}$. We take only

$$\lambda_n = \lambda_n^{(1)} = 2(2n+1)\pi + 2i\ln y_2,\ n = 0, \pm1, \pm2, \ldots .$$

The equation $\cos\frac{\mu}{2} = z_2$ has two series real solutions. If $\beta = \arccos z_2$, $\beta \in (\frac{\pi}{3}, \frac{\pi}{2})$, then $\mu_n = \pm 2\beta + 4n\pi$. We consider only the positive solutions

$$\mu_n = |2\beta + 4n\pi|, \quad n = 0, \pm1, \pm2, \ldots$$

As in the previous case, using (16) we receive

$$B - A = \sum_{-\infty}^{+\infty} \frac{\sin\mu_n x}{\mu_n^2 \sin\frac{\mu_n}{2}} - \sum_{-\infty}^{+\infty} \frac{\sin\lambda_n x}{\lambda_n^2 \sin\frac{\lambda_n}{2}}$$

$$= \frac{1}{16\pi^2}\left[\frac{1}{\sin\beta}\sum_{-\infty}^{+\infty}\frac{\sin y(n+\frac{\beta}{2\pi})}{(n+\frac{\beta}{2\pi})^2} - \frac{i}{\sinh b}\sum_{-\infty}^{+\infty}\frac{\sin y(n+\frac{1}{2}+\frac{ib}{2\pi})}{(n+\frac{1}{2}+\frac{ib}{2\pi})^2}\right]$$

$$= \frac{x}{4(1-z_2)} - \frac{x}{4(1-z_1)} = \frac{x}{4}\frac{\sqrt{k^2+8}}{k+1},$$

where $b = \ln y_2$, $y = 4\pi x$. Then (26) is asserted, if $x \in (0, \frac{1}{2})$. By analogy with the previous cases, it is proved that the identity is valid for $x = 0$ and $x = \frac{1}{2}$ too.

Finally, let us prove the identity (26), if $x \in [\frac{1}{2}, 1]$. Setting $\overline{x} = x + \frac{1}{2}$ it is evident that if $x \in [0, \frac{1}{2}]$, then $\overline{x} \in [\frac{1}{2}, 1]$. The series

$$\sum_{1}^{+\infty} \frac{\sin\mu_n\overline{x}}{\mu_n^2\sin\frac{\mu_n}{2}} \quad \text{and} \quad \sum_{1}^{+\infty} \frac{\sin\lambda_n\overline{x}}{\lambda_n^2\sin\frac{\lambda_n}{2}}$$

as $n \in (-\infty, +\infty)$ and n is an integer have the following representations

$$\sum_{1}^{+\infty} \frac{\sin\mu_n\overline{x}}{\mu_n^2\sin\frac{\mu_n}{2}} = z_2\sum_{1}^{+\infty}\frac{\sin\mu_n x}{\mu_n^2\sin\frac{\mu_n}{2}} + \sum_{1}^{+\infty}\frac{\cos\mu_n x}{\mu_n^2} = z_2 B + S_\mu \text{ and}$$

$$\sum_1^{+\infty} \frac{\sin \lambda_n \overline{x}}{\lambda_n^2 \sin \frac{\lambda_n}{2}} = z_1 \sum_1^{+\infty} \frac{\sin \lambda_n x}{\lambda_n^2 \sin \frac{\lambda_n}{2}} + \sum_1^{+\infty} \frac{\cos \lambda_n x}{\lambda_n^2} = z_1 A + S_\lambda ,$$

where

$$S_\theta = \sum_1^{+\infty} \frac{\cos \theta_n x}{\theta_n^2}.$$

As we use the well known series of [3] for S_λ and S_μ, we obtain

$$S_\mu = \sum_1^{+\infty} \frac{\cos \mu_n x}{\mu_n^2} = -\frac{x}{4} + \frac{1}{8(1 - z_2)} \quad \text{and}$$

$$S_\lambda = \sum_1^{+\infty} \frac{\cos \lambda_n x}{\lambda_n^2} = -\frac{x}{4} + \frac{1}{8(1 - z_1)}$$

when $x \in [0, \frac{1}{2}]$.

Using the sums of the same series if $x \in [0, \frac{1}{2}]$ and the sum found above, finally we receive that

$$\frac{4(k+1)}{\sqrt{k^2+8}} \left(\sum_1^{+\infty} \frac{\sin \mu_n \overline{x}}{\mu_n^2 \sin \frac{\mu_n}{2}} - \sum_1^{+\infty} \frac{\sin \lambda_n \overline{x}}{\lambda_n^2 \sin \frac{\lambda_n}{2}} \right)$$

$$= \frac{4(k+1)}{\sqrt{k^2+8}} \left[\frac{x}{4} \left(\frac{z_2}{1 - z_2} - \frac{z_1}{1 - z_1} \right) + \frac{1}{8} \left(\frac{1}{1 - z_2} - \frac{1}{1 - z_1} \right) \right] = \overline{x}.$$

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Institute of Applied Mathematics and Informatics
Technical University of Sofia
P.O.Box 384, Sofia 1000, BULGARIA