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On the Automorphisms in the Commutant of the Square of the Integration in the Spaces C and L

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An explicit integral representation of the commutant of the square of the integration operator in the spaces of continuous and of integrable functions on a symmetric interval is found. Sufficient conditions for such operators to be continuous automorphisms in these spaces are given.

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1. Introduction

Let Δ be an arbitrary interval containing the zero point and let $C(\Delta)$ be the space of continuous on Δ functions and $L(\Delta)$ be the space of Lebesgue integrable or locally integrable functions on Δ. They are considered as a Banach spaces in the compact case of the interval Δ , or as a Fréchet spaces in the noncompact case.

The Volterra integration operator

$$lf(t) = \int_0^t f(\tau)\, d\tau$$

is a right inverse of the differentiation operator d/dt in the considered spaces. It is closely connected with the Duhamel convolution

$$(f * g)(t) = \int_0^t f(t-\tau) g(\tau)\, d\tau, \tag{1}$$

namely l is a convolutional operator $\{1\}*$, i.e. $lf = \{1\} * f$.

Definition 1.1. The *commutant of* l in $C(\Delta)$ (or $L(\Delta)$) is said to be the set of all linear continuous operators $L : C(\Delta) \to C(\Delta)$ (or, $L : L(\Delta) \to L(\Delta)$) such that $Ll = lL$.

By analogy, the commutant of l^2 in $C(\Delta)$ (or $L(\Delta)$) consists of all continuous operators $L : C(\Delta) \to C(\Delta)$ (or $L : L(\Delta) \to L(\Delta)$), such that $Ll^2 = l^2L$.

Definition 1.2. A linear operator $M : C(\Delta) \to C(\Delta)$ (or $M : L(\Delta) \to L(\Delta)$) is said to be a *multiplier of the Duhamel convolutional algebra* iff

$$M(f * g) = (Mf) * g = f * (Mg)$$

holds for all $f, g \in C(\Delta)$ (or $L(\Delta)$).

In [3] it is shown that the multipliers of the Duhamel convolutional algebra are linear continuous operators forming a commutative ring. This ring coincides with the commutant of the integration operator l.

In a more general sense, the following lemma is true.

Lemma 1.1. ([3]) *If $*$ is a separately continuous and annihilators-free convolution of an endomorphism $\mathcal{L} : X(\Delta) \to X(\Delta)$ of a Frèchet space $X(\Delta)$ with a cyclic element, then the multiplier ring of the convolutional algebra $\{X(\Delta), *\}$ coincides with the commutant of the operator $\mathcal{L}$ in $X(\Delta)$.*

The explicit integral representation of the commutant of l in $C(\Delta)$ is known.

Theorem 1.1. ([4]) *A linear continuous operator $M : C(\Delta) \to C(\Delta)$ commutes with the Volterra integration operator l in $C(\Delta)$ iff it admits a convolutional representation of the form*

$$Mf = \frac{d}{dt}(m * f), \quad or \tag{2}$$

$$Mf(t) = m(0)f(t) + \int_0^t f(t-\tau)\, dm(\tau), \tag{2'}$$

where $m(t) \stackrel{\text{def}}{=} M\{1\}$ is a continuous function of (locally) bounded variation in Δ.

Let now $C^0(\Delta)$ be the space of the continuous on Δ functions, with $f(0) = 0$. Then by formula (2′), $Mf(0) = m(0)f(0) = 0$ and every operator M commuting with l is an inner operator in the space $C^0(\Delta)$.

Here we quote the theorem of Weston [10] for reprezentation of the multipliers of the Duhamel convolutional algebra $\{C^0([0,b]), *\}$ with $b \in R^+$.

Theorem 1.2. ([10]) *Let M be a multiplier of the Duhamel convolutional algebra $\{C^0([0,b]), *\}$. Then there exists a function $\nu \in BV([0,b])$ with $\nu(0) = 0$ and $\nu(t-) = \nu(t)$ whenever $0 < t \le b$, such that M is given by the formula*

$$Mf(t) = \int_0^t f(t-\tau)\,d\nu(\tau),$$

for every $f \in C^0([0,b])$. *This function* ν *is uniquely determined by* M.

Taking into account Lemma 1.1 and the fact that the function $\{t\}$ is a cyclic element of the operator l in the space $C^0(\Delta)$, we may conclude the following.

Corollary 1.1. *A linear continuous operator* $M : C^0(\Delta) \to C^0(\Delta)$ *commutes with the integration operator* l *in* $C^0(\Delta)$ *iff it admits an integral representation of the form*

$$Mf(t) = \int_0^t f(t-\tau)\,dm(\tau), \tag{3}$$

with $m \stackrel{\text{def}}{=} (M\{t\})' \in BV_{norm}(\Delta)$.

Analogous representation of the commutant of l in $L(\Delta)$ holds.

Theorem 1.3. ([8]) *A linear continuous operator* $M : L(\Delta) \to L(\Delta)$ *commutes with the Volterra integration operator* l *in* $L(\Delta)$ *iff it admits a representation of the form:*

$$Mf = \frac{d}{dt}(m * f), \tag{4}$$

where $m(t) \stackrel{\text{def}}{=}_{a.e.} M\{1\}$ *is a function of (locally) bounded variation in* Δ, *or*

$$Mf(t) =_{a.e.} H_m(t)f(t) + \int_0^t f(t-\tau)\,d\mu_m(\tau), \tag{4'}$$

where the measure μ_m *is generated by* $m \in BV_{norm}(\Delta)$, *and* H_m *is the jump-function*

$$H_m(t) = \begin{cases} m(0+), & t \in \Delta \cap (-\infty; 0), \\ m(0-), & t \in \Delta \cap [0; +\infty). \end{cases}$$

R e m a r k 1.1. ([8]) In the case of one-sided interval Δ, the operator M commuting with l is represented in the form

$$Mf(t) = \int_0^t f(t-\tau)\,d\mu(\tau), \tag{5}$$

with a certain Radon measure μ in $(-\infty; +\infty)$, generated by $m =_{a.e.} M\{1\} \in BV(\Delta)$ whose support is contained in Δ.

2. An integral representation of the commutant of l^2 in $C(\Delta)$

The explicit representation of the commutant of l^2 depends on the type of the interval Δ. That is the reason to consider first the case of one-sided interval of the type $\Delta = [0, b]$, or $\Delta = [0, b)$ with $0 < b \leq +\infty$.

Theorem 2.1. *A linear continuous operator $M : C(\Delta) \to C(\Delta)$, for $\Delta = [0, b]$, or $\Delta = [0, b)$, commutes with the square of the integration operator l^2 in $C(\Delta)$ iff it commutes with the operator l. Then M is represented by formula (2).*

Proof. We will use the fact that the constant function $\{1\}$ is a cyclic element of l^2 in $C(\Delta)$, i.e. the span of the functions $l^{2n}\{1\} = \frac{t^{2n}}{(2n)!}$, $n = 0, 1, 2, \ldots$ is dense in the space $C([0, b])$, or $C([0, b))$.

Let M commute with the operator l^2 in $C(\Delta)$. Then according to Lemma 1.1, it is a multiplier of the Duhamel convolution in $C(\Delta)$. But it is known, that the commutative ring of the multipliers of (1) coincides with the commutant of l in $C(\Delta)$ in accordance with the same lemma. Thus, applying Theorem 1.1, representation (2) of the operator M is obtained. ■

The restriction to the interval $\Delta = [0, b]$, or $\Delta = [0, b)$ in Theorem 2.1 is essential, since the function $\{1\}$ is not a cyclic element of l^2 in the space $C(\Delta)$, when Δ is a two-sided interval.

In the next considerations let Δ be a symmetric interval with respect to the point zero, i.e. it has a form $[-b, b]$ or $(-b, b)$ with $0 < b \leq +\infty$. In this case, (1) also is a convolution of l^2 in $C(\Delta)$. Then formula (2) gives only a "part" of the operators $M : C(\Delta) \to C(\Delta)$ commuting with l^2. In order to make this clear, let us consider the splitting of $C(\Delta)$ into a direct sum of the spaces $C_e(\Delta)$ of the even functions in Δ and $C_o(\Delta)$ a subspace of the odd in Δ functions, i.e.

$$C(\Delta) = C_e(\Delta) \oplus C_o(\Delta).$$

This means that every continuous function f is uniquely representable as a sum of its even and odd components:

$$f(t) = f_e(t) + f_o(t),$$

with $f_e(t) = \frac{1}{2}[f(t) + f(-t)] \in C_e(\Delta)$ and $f_o(t) = \frac{1}{2}[f(t) - f(-t)] \in C_o(\Delta)$. Thus $C_e(\Delta)$ and $C_o(\Delta)$ are invariant subspaces of $C(\Delta)$ for the operator l^2.

Theorem 2.2. *Let Δ be a symmetric interval with respect to the zero. A linear continuous operator $M : C(\Delta) \to C(\Delta)$ commutes with l^2 iff it admits* *a representation of the form*

$$Mf = \frac{d}{dt}(m * f_e) + \frac{d^2}{dt^2}(n * f_o), \tag{6}$$

where $m \overset{\text{def}}{=} M\{1\}$ *is a continuous function of bounded variation on* Δ *and* $n \overset{\text{def}}{=} M\{t\}$ *is an absolutely continuous on* Δ *function with* $n(0) = 0$ *and* n' *is a function of bounded variation on* Δ.

Proof. Let $M : C(\Delta) \to C(\Delta)$ be a linear continuous operator commuting with l^2 in $C(\Delta)$, i.e. $Ml^2 = l^2M$ holds. Let us find how the operator M acts in the spaces $C_e(\Delta)$ and $C_o(\Delta)$, having in mind that the operator l^2 has the function $\{1\}$, as a cyclic element in $C_e(\Delta)$ and the function $\{t\}$, as a cyclic element in $C_o(\Delta)$. The Duhamel convolution (1) is an inner operation only in the space $C_o(\Delta)$, but not in the space $C_e(\Delta)$. Therefore we define the operation

$$f \overset{e}{*} g = l(f * g),$$

which is a convolution of l^2 in $C_e(\Delta)$, moreover l^2 is a convolutional operator $\{1\} \overset{e}{*}$, i.e. $l^2 f = \{1\} \overset{e}{*} f$ in $C_e(\Delta)$. Since M and l^2 commute in $C(\Delta)$, then $Ml^2 f = l^2 Mf$ holds for all $f \in C_e(\Delta)$. It may easily be shown that equality

$$(Mf) \overset{e}{*} g = f \overset{e}{*} (Mg)$$

is an identity in this space and substituting $g = \{1\}$ we get

$$(M\{1\}) \overset{e}{*} f = \{1\} \overset{e}{*} (Mf) = l^2(Mf),$$

or $Mf = (d/dt)(m * f)$ with $m \overset{\text{def}}{=} M\{1\} \in C(\Delta)$ for all $f \in C_e(\Delta)$. Therefore for the even component f_e of each continuous on Δ function f we get a representation

$$Mf_e = \frac{d}{dt}(m * f_e), \quad \text{where } m \overset{\text{def}}{=} M\{1\} \in C(\Delta). \tag{7}$$

As we know, the function $\{t\}$ is a cyclic element of l^2 in $C_o(\Delta)$. Then according to Lemma 1.1 the operator M commuting with l^2 in $C_o(\Delta)$ is a multiplier of (1) in $C_o(\Delta)$, i.e. $(Mf) * g = f * (Mg)$ holds for all $f, g \in C_o(\Delta)$. Substituting $g = \{t\}$ in this equality we have

$$(M\{t\}) * f = t * (Mf) = l^2(Mf)$$

and therefore, $Mf = d^2/dt^2(n * f)$, with $n \overset{\text{def}}{=} M\{t\} \in C(\Delta)$ for all $f \in C_o(\Delta)$. Thus for the odd component f_o of each continuous on Δ function f we get a representation

$$(8) \qquad Mf_o = \frac{d^2}{dt^2}(n * f_o), \text{ where } n \overset{\text{def}}{=} M\{t\} \in C(\Delta).$$

The next denotations will be introduced, so that we make a more exact characterization of the continuous functions m and n. If Δ is the interval $[-b.b]$, then let m^+, m^- and n^+, n^- be the restrictions of m and n to the intervals $[0, b]$ and $[-b, 0]$ respectively. The following conclusions and denotations are valid in the compact as well as in the noncompact case of the interval Δ and without any loss of generality we may restrict our considerations to the interval of the type $[-b, b]$ with $b \in R^+$.

Let us introduce the auxiliary operators

$$M^+f = \frac{d}{dt}(m^+ * f), \quad \text{for all } f \in C([0, b]) \text{ and}$$

$$M^-f = \frac{d}{dt}(m^- * f), \quad \text{in the space } C([-b.0]).$$

It is clear that $M^+f = (M\tilde{f})|_{[0,b]}$, where $\tilde{f}$ is the even continuation of $f \in C([0, b])$ to the interval $[-b, b]$. Analogously $M^-f = (M\tilde{f})|_{[-b,0]}$, for $f \in C([-b, 0])$. Thus M^+ and M^- are continuous linear operators in the spaces $C([0, b])$ and $C([-b, 0])$ respectively, which commute with the operator l^2 in these spaces and according to Theorem 2.1 they commute with l. Hence, in conformity with Theorem 1.1, the representation of the form (2) follows, which coincides with equality (7), as functions m^+ and m^- are continuous functions of bounded variation on aech of the intervals $[0, b]$ and $[-b, 0]$ respectively. Therefore, the function $m = M\{1\}$ in (7) is a continuous function of bounded variation on Δ.

Using functions in $C^2(\Delta)$ and applying the continuity of the operator M and convolution (1) in $C(\Delta)$, $n(0) = 0$ may be proved and the relation $Mf(0) = 0$ for all odd functions $f \in C(\Delta)$ is resulted.

Let us now introduce the next operators:

$$N^+f = \frac{d^2}{dt^2}(n^+ * f) \text{ and } N^-f = \frac{d^2}{dt^2}(n^- * f),$$

defined correspondingly in the spaces $C^0([0, b])$ and $C^0([-b, 0])$. Here it is clear that $N^+f = (M\bar{f})|_{[0,b]}$ and $N^-f = (M\bar{f})|_{[-b,0]}$, with $\bar{f}$, an odd continuation of the function $f \in C^0([0, b])$ or $f \in C^0([-b, 0])$ to the interval $[-b, b]$. Then N^+ and N^- are continuous operators, commuting with l^2 in corresponding spaces and in conformity with Theorem 2.1 they commute with l. Therefore, according to Corollary 1.1 each of them has a representation of the form (3). For example,

$$N^+f(t) = \int_0^t f(t-\tau)\,d\nu^+(\tau),$$

with $\nu^+ \in BV_{norm}([0,b])$. Thus, the next conversions take place:

$$l^2N^+f(t) = N^+l^2f(t) = -\int_0^t \left[\int_0^\tau \left(\int_t^{t-\sigma} f(x)\,dx\right) d\sigma\right] d\nu^+(\tau)$$

$$= \int_0^t f(x)\,d_x \int_{t-x}^t \left(\int_\sigma^t \nu^+(\tau)\,d\tau\right) d\sigma \quad \text{for all } f \in C^\circ([0,b]), \text{ or}$$

$$l^2N^+f(t) = \int_0^t f(t-\tau)\,d_\tau \int_\tau^t \left(\int_\sigma^t \nu^+(u)\,du\right) d\sigma. \tag{9}$$

On the other hand, according to (8) we have:

$$l^2N^+f(t) = n^+ * f = \int_0^t f(t-\tau)\,d_\tau \int_0^\tau n^+(\sigma)\,d\sigma. \tag{10}$$

Comparing equalities (9) and (10),

$$\int_0^\tau n^+(\sigma)\,d\sigma = \int_\tau^t \left(\int_\sigma^t \nu^+(u)\,du\right) d\sigma$$

is obtained. After differentiation on τ, the formula

$$n^+(\tau) = \int_t^\tau \nu^+(u)\,du \tag{11}$$

is found. This means that n^+ is an absolutely continuous function on the finite subinterval $[0,t] \subset [0,b]$. Analogously, the fact that n^- is an absolutely continuous function on each finite subsegment of $[-b,0]$, may be proved.

After differentiation of the formula (11) the equality

$$\frac{d}{d\tau} n^+(\tau) =_{a.e.} \nu^+(\tau) \quad \text{for} \quad \tau \in [0,t] \subset \Delta$$

is found, but $\nu^+ \in BV_{norm}(\Delta)$ according to Corollary 1.1. Therefore, the function n' is a function of bounded variation on Δ.

Finally, after summation of equalities (7) and (8) we get for the operator M a representation of the form (6) with the functions $m \in BV \cap C(\Delta)$ and $n \in AC(\Delta)$, $n(0) = 0$.

Conversely, if m and n are functions in the desired spaces, then expression (6) defines an operator $M : C(\Delta) \to C(\Delta)$, according to Mikusinski [7] and Bozhinov ([1], p.137). It may be verified directly that M is in fact a multiplier of the Duhamel convolutional algebra and so it is a continuous operator, commuting with l^2 in $C(\Delta)$. ∎

Corollary 2.1. *A continuous linear operator $M : C(\Delta) \to C(\Delta)$ commutes with the square of the Volterra integration operator l^2 in $C(\Delta)$ iff it admits an integral representation of the form*

$$(12)\quad Mf(t) = m(0)f_e(t) + \nu(0)f_o(t) + \int_0^t f_e(t-\tau)\,dm(\tau) + \int_0^t f_o(t-\tau)\,d\nu(\tau),$$

where $m \stackrel{\text{def}}{=} M\{1\} \in BV \cap C(\Delta)$ and $\nu \stackrel{\text{def}}{=} (M\{t\})' \in BV_{norm}(\Delta)$.

Proof. The representation (12) is an immediate consequence of (6), obtained after accomplishing of the denoted differentiations, and Corollary 2.1 is another formulation of Theorem 2.2. ∎

The idea to look for two functions m and n, needed for a representation of the commutant of l^2 in $C(\Delta)$, arrises out of the paper of Raichinov [9].

Corollary 2.2. *A linear continuous operator $M : C(\Delta) \to C(\Delta)$, commuting with the operator l^2, commutes with the Volterra integration operator l iff $M\{1\} = (M\{t\})'$.*

Proof. Let $M : C(\Delta) \to C(\Delta)$ commute with l^2 in $C(\Delta)$ and $M\{1\} = (M\{t\})'$, i.e. $m(t) = n'(t) = \nu(t)$ for all $t \in \Delta$, in particular $m(0) = \nu(0)$ holds. Then from (12) we get a representation

$$Mf(t) = m(0)f(t) + \int_0^t f(t-\tau)\,dm(\tau),$$

which is just representation (2′) and according to Theorem 1.1 the operator M commutes with l in $C(\Delta)$.

Let now M commute with l^2 in $C(\Delta)$, but $M\{1\} \not\equiv (M\{t\})'$ in Δ, as $m(0) = \nu(0)$. Then the equality (12) may be written in the form

$$Mf(t) = m(0)f(t) + \int_0^t f(t-\tau)\,dm(\tau) + \int_0^t f_o(t-\tau)\,d\nu_0(\tau),$$

where $\nu_0 = \nu - m = (M\{t\})' - M\{1\} \not\equiv 0$. If the operator

$$M_1 f(t) = m(0)f(t) + \int_0^t f(t-\tau)\,dm(\tau)$$

is denoted, then according to Theorem 1.1, it commutes with l in $C(\Delta)$. Thus, M commutes with l^2 and M_1, in conformity with its representation, also is of the commutant of l^2 in $C(\Delta)$. Therefore, the operator

$$M_0 f(t) = \int_0^t f_o(t-\tau)\, d\nu_0(\tau),$$

with a function of bounded variation ν_0, which does not vanish almost everywhere in Δ, is an example of an operator commuting with l^2, which does not commute with l in $C(\Delta)$. ■

3. Sufficient conditions for automorphisms in the commutant of the operator l^2 in $C(\Delta)$

In this section we consider the question: Which conditions must satisfy the functions m and n, respectively ν, in order the map $M : C(\Delta) \to C(\Delta)$ of the form (6) or (12) to be an one-to-one correspondence of the space $C(\Delta)$ onto itself, i.e. to exist a continuous inverse M^{-1} of the operator M?

Lemma 2, [4] shows that when M^{-1} exists, it also is a multiplier of the convolutional algebra and in conformity with Larsen [6] it is a continuous operator.

Let us consider carefully an operator of the form (12), denoting:

$$M_e f_e(t) = m(0) f_e(t) + \int_0^t f_e(t-\tau)\, dm(\tau), \quad \text{and} \tag{13}$$

$$M_o f_o(t) = \nu(0) f_o(t) + \int_0^t f_o(t-\tau)\, d\nu(\tau), \tag{14}$$

where $m = M\{1\} \in BV \cap C(\Delta)$ and $\nu = (M\{t\})' \in BV(\Delta)$.

Lemma 3.1. *If m and ν are even functions respectively in $C(\Delta)$ and $BV(\Delta)$, then the operators $M_e : C_e(\Delta) \to C_e(\Delta)$ and $M_o : C_o(\Delta) \to C_o(\Delta)$, i.e. they are inner operators in the corresponding spaces.*

P r o o f. By direct calculations we find

$$\int_0^{-t} f_e(-t-\tau)\, dm(\tau) = \int_0^t f_e(t-\tau)\, dm(\tau), \quad \text{when } m \text{ is an even function and}$$

$$\int_0^{-t} f_o(-t-\tau)\, d\nu(\tau) = -\int_0^t f_o(t-\tau)\, d\nu(\tau), \quad \text{with even function } \nu \text{ in } BV(\Delta).$$

Thus, the operator of the form (13) is an inner operator in the space $C_e(\Delta)$ and this one, of the form (14) is an inner operator in the subspace $C_o(\Delta)$. ■

Now the question for invertability of the operator of the form (12) is easily solved. The problem is closely connected with the existence of a solution of the integral equation

$$(15)\quad m(0)f_e(t)+\nu(0)f_o(t)+\int_0^t f_e(t-\tau)\,dm(\tau)+\int_0^t f_o(t-\tau)\,d\nu(\tau)=g(t)$$

for each function $g \in C(\Delta)$ and given functions $m \in BV \cap C(\Delta)$ and $\nu \in BV_{norm}(\Delta)$.

Let us remind, that f_e and f_o are the uniquely determined even and odd components of an arbitrary function $f \in C(\Delta)$. Thus for the function g there exists analogous unique expansion $g = g_e + g_o$. Then, according to Lemma 3.1, the equation (15) may be uniquely decomposed into two integral equations:

$$\mu f_e(t)+\int_0^t f_e(t-\tau)\,dm(\tau)=g_e(t) \quad \text{and}$$

$$\lambda f_o(t)+\int_0^t f_o(t-\tau)\,d\nu(\tau)=g_o(t),$$

where $g_e \in C_e(\Delta)$, $g_o \in C_o(\Delta)$ and $m \in BV \cap C(\Delta)$, $\nu \in BV(\Delta)$ are given functions and $\mu = m(0)$ and $\lambda = \nu(0)$ are given numbers. In conformity with Lemma 3.[4] and Lemma 3.1, each of them has a unique continuous solution in the corresponding subspace, by the conditions $\nu \in BV \cap C(\Delta)$ and $\mu = m(0) \neq 0$ and $\lambda = \nu(0) \neq 0$.

In this way we proved the following lemma.

Lemma 3.2. *Let g be a continuous function on Δ and m and ν are even continuous functions of bounded variation in the same interval. Then the equation (15) has a unique continuous solution in Δ, provided $m(0) \neq 0$ and $\nu(0) \neq 0$.*

Here we can state the main result of this paper.

Theorem 3.1. *A linear operator $M : C(\Delta) \to C(\Delta)$, which commutes with the integration operator l^2 is a linear continuous automorphism in the space $C(\Delta)$ if the functions $m(t) \stackrel{\text{def}}{=} M\{1\}$ and $\nu(t) \stackrel{\text{def}}{=} (M\{t\})'$ are even continuous functions of (locally) bounded variation in Δ, such that $m(0) \neq 0$ and $\nu(0) \neq 0$.*

P r o o f. According to Lemma 2, [4] it must be shown that the conditions $m(0) \neq 0$ and $\nu(0) \neq 0$ are sufficient for invertability of the operator M in the space $C(\Delta)$.

Since m and ν are even continuous functions of (locally) bounded variation in Δ, Lemma 3.1 and Lemma 3.2 prove that this conditions are sufficient for the operator M of the form (12) to be an one-to-one mapping of $C(\Delta)$ onto itself.

This proves the theorem. ∎

4. Representation of the commutant of l^2 in $L(\Delta)$

In the case of one-sided interval of the type $\Delta = [0,b]$ or $\Delta = [0,b)$ with $0 < b \leq +\infty$, since the constant function $\{1\}$ is a cyclic element of the operator l^2 in $L(\Delta)$, it can be proved that the commutant of l coincides with the commutant of l^2. Then, the following theorem is true.

Theorem 4.1. *A linear continuous operator $M : L(\Delta) \to L(\Delta)$, for $\Delta = [0,b]$ or $\Delta = [0,b)$, commutes with the square of the integration operator l^2 in $L(\Delta)$ iff it commutes with the operator l. Then M is represented in the form (4).*

Proof. It is analogous of the proof of Theorem 2.1 and we omit it.

More attention will be paid to the case of a symmetric interval Δ with respect to the point 0, i.e. $\Delta = [-b,b]$ or $\Delta = (-b,b)$ with $0 < b \leq +\infty$. Here, an expansion of the integrable functions $f \in L(\Delta)$ as a sum of their even and odd components $f(t) = f_e(t) + f_o(t)$ again will be used , as equality holds almost everywhere, i.e. the decomposition $L(\Delta) = L_e(\Delta) \oplus L_o(\Delta)$ is true.

Theorem 4.2. *Let Δ be a symmetric interval with respect to the zero. A linear continuous operator $M : L(\Delta) \to L(\Delta)$ commutes with l^2 iff it admits a representation of the form*

$$(16) \qquad Mf(t) = \frac{d}{dt}(m * f_e) + \frac{d^2}{dt^2}(n * f_o),$$

where $m =_{a.e.} M(\{1\})$ is a function of (locally) bounded variation on Δ and $n =_{a.e.} M(\{t\})$ is a (locally) absolutely continuous function on Δ with $n' \in BV_{(loc)}(\Delta)$.

Proof. Let $M : L(\Delta) \to L(\Delta)$ be a linear continuous operator commuting with l^2 in $L(\Delta)$, i.e. $Ml^2 = l^2M$. By analogy with the proof of Theorem 2.2,

$$(17) \qquad Mf_e(t) = \frac{d}{dt}(m * f_e), \text{ with } m =_{a.e.} M\{1\} \in L(\Delta),$$

may be shown, because $f \overset{e}{*} g = l(f * g)$ is a convolution of l^2 in $L_e(\Delta)$ with a cyclic element the constant function $\{1\}$. As we know, the Duhamel convolution is a convolution of l^2 in $L_o(\Delta)$ with the function $\{t\}$ as a cyclic element, therefore

$$(18) \qquad Mf_o(t) = \frac{d^2}{dt^2}(n * f_o), \text{ holds with } n =_{a.e.} M\{t\} \in L(\Delta).$$

Let now $f \in L([0,b])$ and let be denoted $\tilde{f}$ as an even and $\bar{f}$ as an odd continuation of the function f in the interval $[-b,b]$. The evident equalities $\widetilde{l^2 f} = l^2 \tilde{f}$ and $\overline{l^2 f} = l^2 \bar{f}$ hold for all $f \in L([0,b])$. Then, the known denotations will be used: m^+ is a restriction of the function m to the interval $[0,b]$ and $M^+ f = (d/dt)(m^+ * f)$ for all $f \in L([0,b])$ is a restriction of the operator M to $[0,b]$. Then we have

$$M^+ l^2 f = M(\widetilde{l^2 f})|_{[0,b]} = M(l^2 \tilde{f})|_{[0,b]} = l^2 M \tilde{f}|_{[0,b]} = l^2 (M \tilde{f}|_{[0,b]}) = l^2 M^+ f,$$

i.e. M^+ commutes with l^2 in the space $L([0,b])$. Thus according to Theorem 4.1, M^+ commutes with l in $L([0,b])$ and in conformity with Theorem 1.3, it is represented in the form (4) with a function $m^+ =_{a.e.} M^+\{1\} \in BV([0,b])$. In the same manner, the representation of the operator $M^- f = (d/dt)(m^- * f)$, with the function
$m^- =_{a.e.} M^-\{1\} \in BV([-b,0])$ may be obtained. But (4) coincides with equality (17), therefore $M(\{1\}) =_{a.e.} m \in BV_{(loc)}(\Delta)$.

Analogously, denoting by n^+ a restriction of n to the interval $[0,b]$ and by $N^+ f = (d^2/dt^2)(n^+ * f)$ a restriction of the operator M to the space $L([0,b])$, the fact that N^+ commutes with l^2 in $L([0,b])$ may be shown. Thus, according to Theorem 4.2 and Remark 1.1, the operator N^+ admits a representation of the form (5):

$$N^+ f(t) =_{a.e.} \int_0^t f(t-\tau)\, d\nu^+(\tau),$$

with Radon measure ν^+, generated by the function of bounded variation on $[0,b]$. For the sake of uniqueness of the representation, let us suppose that ν^+ is normalized, as $\nu(t+0) = \nu(t)$ for all $t \in [0,b)$ and $\nu(b-0) = \nu(b)$. Then

$$l^2 N^+ f(t) = \int_0^t l^2 f(t-\tau)\, d\nu^+(\tau) = n^+ * f = \int_0^t f(t-\tau) n^+(\tau)\, d\tau \text{ for all } f \in L([0,b]).$$

When $f \in C([0,b])$, from the upper equality

$$(19) \qquad \int_0^t f(t-\tau)\, d_\tau \int_\tau^t \left(\int_\sigma^t \nu^+(u)\, du \right) d\sigma = \int_0^t f(t-\tau)\, d_\tau \int_0^\tau n^+(\sigma)\, d\sigma$$

may be obtained. But the space $C([0,b])$ is dense in $L([0,b])$, then equality (19) holds almost everywhere for all $f \in L([0,b])$. Therefore

$$\int_0^\tau n^+(\sigma)\, d\sigma = \int_\tau^t \left(\int_\sigma^t \nu^+(u)\, du \right) d\sigma,$$

because corresponding measures are absolutely continuous. By differentiation on τ the equality

$$n^+(\tau) =_{a.e.} \int_t^\tau \nu^+(u)\,du, \quad \text{with } \nu^+ \in BV_{norm}([0,t])$$

is obtained, i.e. n^+ is an absolutely continuous function on each finite interval $[0,t] \subset [0,b]$. Thus the function n in the formula (18) is a (locally) absolutely continuous in Δ and its derivative is a function of (locally) bounded variation on Δ.

At last, additing expressions (17) and (18), the representation (16) with the functions $m \in BV(\Delta)$ and $n \in AC(\Delta)$ is obtained.

Conversely, if m and n are respectively a function of bounded variation and an absolutely continuous function on Δ with $n' \in BV_{(loc)}(\Delta)$, then equality (16) defines an operator $M : L(\Delta) \to L(\Delta)$ according to Bozhinov ([1], p.137). It may be verified directly that it is a continuous operator, commuting with l^2 in $L(\Delta)$. ∎

Corollary 4.1. *A continuous linear operator $M : L(\Delta) \to L(\Delta)$ commutes with l^2 iff it admits an integral representation of the form*

(20)

$$Mf(t) =_{a.e.} H_m(t)f_e(t)+H_\nu(t)f_o(t)+\int_o^t f_e(t-\tau)\,d\mu_m(\tau)+\int_0^t f_o(t-\tau)\,d\mu_\nu(\tau),$$

where μ_m and μ_ν are measures, generated by the functions $m \overset{\text{def}}{=}_{a.e.} M\{1\}$, and $\nu \overset{\text{def}}{=}_{a.e.} M(\{t\})'$, as $m,\nu \in BV_{norm}(\Delta)$ and the functions H_m and H_ν are respectively

$$H_m(t) = \begin{cases} m(0+), & t \in \Delta \cap (-\infty;0) \\ m(0-), & t \in \Delta \cap [0;+\infty), \end{cases}$$

$$H_\nu(t) = \begin{cases} \nu(0+), & t \in \Delta \cap (-\infty;0) \\ \nu(0-), & t \in \Delta \cap [0;\infty) \end{cases}$$

P r o o f. Representation (20) is obtained from (16) after accomplishing of the denoted differentiations, taking into account that $(M\{t\})' =_{a.e.} n'(t) = \nu(t) \in BV_{norm}(\Delta)$ and conforming with formula (4'). ∎

5. Automorphisms in the commutant of l^2 in the space $L(\Delta)$

Our aim here is to characterize the functions m and ν in representation (20) in order the operator M to be one-to-one continuous mapping of the space $L(\Delta)$ onto itself, i.e. to exist a continuous inverse M^{-1}. The continuity of the operator M^{-1}, when it exists, is ensured by Lemma 3.1, [8]. Thus the problem of invertability of the operator M is reduced to the question of solvability of the integral equation

$$(21)\quad H_m(t)f_e(t)+H_\nu(t)f_o(t)+\int_o^t f_e(t-\tau)\,dm(\tau)+\int_0^t f_o(t-\tau)\,d\nu(\tau)=g(t),$$

for each function $g\in L(\Delta)$ and given functions $m,\nu\in BV_{norm}(\Delta)$.

Lemma 5.1. *Let Δ be a symmetric interval with respect to the zero, g be a Lebesgue integrable on Δ function and m and ν be normalized even functions of (locally) bounded variation in Δ, as $m(0+)\neq 0$, $m(0-)\neq 0$ and $\nu(0+)\neq 0$, $\nu(0-)\neq 0$. Then the integral equation (21) has a unique solution, which is an integrable on Δ function .*

Proof. As in the continuous case, let us consider two operators

$$M_e f_e(t)=H_m(t)f_e(t)+\int_0^t f_e(t-\tau)\,dm(\tau)\quad\text{and}$$

$$M_o f_o(t)=H_\nu(t)f_o(t)+\int_0^t f_o(t-\tau)\,d\nu(\tau),$$

with even functions $m=_{\text{a.e.}}M\{1\}$ and $\nu=_{\text{a.e.}}(M\{t\})'$, $m,\nu\in BV_{norm}(\Delta)$. They are inner operators in the spaces $L_e(\Delta)$ and $L_o(\Delta)$ respectively. Using the expansion of the function $g(t)=g_e(t)+g_o(t)$, the integral equation (21) may be uniquely decomposed into two integral equations

$$H_m(t)f_e(t)+\int_0^t f_e(t-\tau)\,dm(\tau)=g_e(t)\quad\text{and}$$

$$H_\nu(t)f_o(t)+\int_0^t f_o(t-\tau)\,d\nu(\tau)=g_o(t).$$

Each of them has a unique integrable solution when $m(0+)\neq 0$ and $m(0-)\neq 0$, respectively $\nu(0+)\neq 0$, $\nu(0-)\neq 0$, according to Corollary 3.1, [8]. ■

Theorem 5.1. *A linear operator $M: L(\Delta)\to L(\Delta)$, which commutes with the integration operator l^2 is a continuous automorphism in the space $L(\Delta)$ if the functions $m\overset{\text{def}}{=}_{\text{a.e.}}M\{1\}$ and $\nu\overset{\text{def}}{=}_{\text{a.e.}}(M\{t\})'$ are normalized even functions of (locally) bounded variation on Δ and satisfy the conditions $m(0+)\neq 0$, $m(0-)\neq 0$ and $\nu(0+)\neq 0$, $\nu(0-)\neq 0$.*

Proof. Analogous to the proof of Theorem 3.1 and based on Lemma 3.1 [8] and Lemma 4.1. That is why we omit it.

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