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One Conjecture Concerning the Permutation Products on Manifolds

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1. Introduction

Let M be an arbitrary set and m be a positive integer. In the direct product M^m we define a relation $\approx$ as follows

$$(x_1, \cdots, x_m) \approx (y_1, \cdots, y_m) \Leftrightarrow$$

there exists a permutation $\vartheta : \{1, 2, \cdots, m\} \to \{1, 2, \cdots, m\}$ such that

$$y_i = x_{\vartheta(i)} \qquad (1 \leq i \leq m).$$

This is a relation of equivalence and the class represented by $(x_1, \cdots, x_m)$ will be denoted by $(x_1, \cdots, x_m)/ \approx$ and the set $M^m / \approx$ will be denoted by $M^{(m)}$. The set $M^{(m)}$ is called a *permutation product* of M.

If M is a topological space, then $M^{(m)}$ is also a topological space. The space $M^{(m)}$ was introduced rather early [1] but it was studied mainly in [5]. If M is an arbitrary manifold and $m > 1$, then in [1] it is proven that

$$\pi_1(M^{(m)}) \cong H_1(M, Z).$$

Another important result [5] is that $(R^n)^{(m)}$ is a manifold only for $n = 2$. Indeed, it is proven that if $n \neq 2$ and $m > 1$, then the tangent space is not homeomorphic to the Euclidean space R^{nm} and hence $(R^n)^{(m)}$ is not a manifold. If $n = 2$, then $(R^2)^{(m)} = C^{(m)}$ is homeomorphic to C^m. Indeed, using that C is an algebraically closed field, it is obvious that the mapping $\varphi : C^{(m)} \to C^m$ defined by

$$\varphi((z_1, \cdots, z_m)/ \approx) = (\sigma_1, \sigma_2, \cdots, \sigma_m)$$

is a bijection, where $\sigma_i (1 \leq i \leq m)$ is the i-th symmetric function of $z_1, \cdots, z_m$. The mapping φ is also a homeomorphism. Using this mapping, many examples of commutative vector-valued groups were constructed [3], and moreover this theory about permutation products has an important role in the theory of the topological commutative vector-valued groups [4].

2. One conjecture concerning the permutation product of complex manifolds

In this section we give a conjecture concerning the permutation products of complex 1-dimensional manifolds. It may find application in the research of compact complex manifolds.

Let M be a real manifold. It is proven in [5] that the permutation product $M^{(m)}$ is not a manifold if $m > 1$ and dim $M > 2$. It is a manifold with boundary for dim $M = 1$. The "best" case is dim $M = 2$ and it is very convenient to assume that M is a 1-dimensional complex manifold. In [2] it is proven that $M^{(m)}$ also admits a complex structure. In the case of permutation products each m-tuple $(x_1, \cdots, x_m)$ of M^m is identified by $(x_{\tau(1)}, x_{\tau(2)}, \cdots, x_{\tau(m)})$ for each permutation $\tau : \{1, \cdots, m\} \to \{1, \cdots, m\}$. Now let us consider a subgroup G of the permutation group S_m and define a relation $\approx$ in M^m by

$$(x_1, x_2, \cdots, x_m) \approx (x_{\tau(1)}, x_{\tau(2)}, \cdots, x_{\tau(m)})$$

if and only if $\tau \in G$. The factor-space $M^m / \approx$ will be denoted by M^m/G. The following problem arrises:

Problem. *Find all the subgroups $G \leq S_m$ such that M^m/G is a complex manifold.*

If $G = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_r}$ where $S_{m_1}, \cdots, S_{m_r}$ are permutation groups of partition of S into r subsets with $m_1, \cdots, m_r$ elements $(m_1 + \cdots + m_r = m)$, then G satisfies the required condition. But the subgroup G does not yield a new complex manifold, because then M^m/G is homeomorphic to $M^{(m_1)} \times M^{(m_2)} \times \cdots \times M^{(m_r)}$.

In [5] the case where G is the cyclic group of m elements is studied. It is proven there that M^m/G is not a manifold if $m > 2$. Some other subgroups of S_m can be also verified not to satisfy the required condition. Indeed, assuming that M^m/G is a manifold, it is verified that the tangent space at some points is not homeomorphic to a Euclidean space. Now we give the following conjecture.

Conjecture. *Let $G \leq S_m$ and M be a 2-dimensional real manifold. Then M^m/G is a manifold if and only if $G = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_r}$, where*

$S_{m_1}, \cdots, S_{m_r}$ are permutation groups of partition of S into r subsets with elements $m_1, \cdots, m_r$.

This conjecture is equivalent to the following consequence.

Corollary. *Let $G \leq S_m$ and M be a 1-dimensional complex manifold. Then M^m/G is a complex manifold if and only if $G = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_r}$, where $S_{m_1}, \cdots, S_{m_r}$ are permutation groups of partition of S into r subsets with elements $m_1, \cdots, m_r$.*

3. Verification of the conjecture for $m = 4$

First, note that the conjecture is trivially satisfie if $m = 1, 2$. If $m = 3$, then the subgroups of S_3 are the cyclic group Z_3 of three elements and the groups of type $S_2 \times S_1$. Then M^3/Z_3 is not a manifold according to [5], and $M^3/S_2 \times S_1 \cong M^{(2)} \times M$ is a manifold. Hence the conjecture also holds for $m = 3$.

The non-trivial case of the conjecture is $m = 4$. It is sufficient to consider all non-trivial subgroups of S_4 up to isomorphism induced by a permutation in S_4. Further on an "isomorphism" will mean such a special kind of isomorphism. For the sake of convenience we denote the cycle $a_{i_1} \to a_{i_2} \to \cdots \to a_{i_p} \to a_{i_1}$ by $(a_{i_1} a_{i_2} \cdots a_{i_p})$.

1. The subgroups of S_4 of order 2 up to isomorphism are:

$$G_1 = \{\epsilon, (12)\} \qquad \text{and} \qquad G_2 = \{\epsilon, (12)(34)\}.$$

Since $G_1 \cong S_2 \times S_1 \times S_1$, then

$$M^4/G_1 \cong M^4/S_2 \times S_1 \times S_1 \cong M^2/S_2 \times M/S_1 \times M/S_1 \cong M^{(2)} \times M^2$$

is a manifold.

The factor-space M^4/G_2 is homeomorphic to $(C^2)^{(2)}$ and hence it is not a manifold.

2. The subgroup of S_4 of order 3 up to isomorphism is unique:

$$G_3 = \{\epsilon, (123), (132)\}.$$

Hence $M^4/G_3 \cong M^4/(Z_3 \times S_1) \cong M^3/Z_3 \times M/S_1$ and it is not a manifold because M^3/Z_3 is not a manifold according to [5].

3. The subgroups of S_4 of order 4 up to isomorphism are:

$$G_4 = \{\epsilon, (12), (34), (12)(34)\}, G_5 = \{\epsilon, (12)(34), (13)(24), (14)(23)\}$$

$$\text{and} \quad G_6 = \{\epsilon, (1234), (13)(24), (1432)\}.$$

Note that $G_6 \cong Z_4$ and hence M^4/G_6 is not a manifold according to [5]. Further $G_4 \cong S_2 \times S_2$ and hence $M^4/G_4 \cong M^2/S_2 \times M^2/S_2 \cong M^{(2)} \times M^{(2)}$ is a manifold. Next we will prove that M^4/G_5 is not a manifold. Indeed, the tangent space at the point $(a,a,c,c) \in M^4/G_5 (a \neq c)$ is given by

$$C^4/\rho = \{(x,y,z,t)/\rho : x,y,z,t \in C \text{ and}$$

$$(x,y,z,t)\rho(p,q,r,s) \Leftrightarrow (\exists \tau \in G_5)\tau((a,a,c,c)) = (a,a,c,c) \text{ and}$$

$$\tau((x,y,z,t)) = (p,q,r,s)\}$$

$$= \{(x,y,z,t)/\rho : x,y,z,t \in C \text{ and}$$

$$(x,y,z,t)\rho(p,q,r,s) \Leftrightarrow (p,q,r,s) = (x,y,z,t) \text{ or } (p,q,r,s) = (y,x,t,z)\} \cong (C^2)^{(2)}$$

which is not homeomorphic to R^8 and thus M^4/G_5 is not a manifold.

4. The subgroup of S_4 of order 6 up to isomorphism is unique:

$$G_7 = \{\epsilon, (12), (13), (23), (123), (132)\}.$$

Since $G_7 \cong S_3 \times S_1$, $M^4/G_7 \cong M^3/S_3 \times M/S_1 \cong M^{(3)} \times M$ is a manifold.

5. The subgroup of S_4 of order 8 up to isomorphism is unique:

$$G_8 = \{\epsilon, (1234), (1432), (13)(24), (13), (24), (12)(34), (14)(23)\}.$$

Then $M^4/G_8 \cong (C^{(2)})^{(2)} \cong (C^2)^{(2)}$ and hence it is not a manifold.

6. The subgroup of S_4 of order 12 up to isomorphism is unique:

$$G_9 = \{\epsilon, (123), (132), (124), (142), (134), (143), (234), (243),$$

$$(12)(34), (13)(24), (14)(23)\},$$

i.e. the subgroup of even permutations. In this case M^4/G_9 is not a manifold just as M^4/G_5 was not a manifold. Indeed, the tangent space at the point $(a,a,c,c) \in M^4/G_9 (a \neq c)$ is homeomorphic to $(C^2)^{(2)} \neq R^8$ and hence M^4/G_9 is not a manifold.

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