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Invariants for Commutative Group Algebras of Mixed and Torsion Groups

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In the present research it is found a full set of invariants (defining the latter up to an isomorphism) of the algebra FG in the terms of a group G and of a field F, when G belongs to the some general classes of abelian groups. In particular, the results of Karpilovsky and May are confirmed.

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0. Introduction

Let F be a field of characteristic $p \neq 0\,(\mathrm{char}F = p)$ with an algebraic cover $\overline{F}$ and let G be an abelian group with torsion part G_0 and p-component G_p.

In the paper [3], a simplified necessary and sufficient condition for the group algebra FH to be F-isomorphic to the algebra FG over an algebraically closed field $F = \overline{F}$ is given, in the cases when G belongs to the classes

(*) G is a direct sum of cyclic groups.

(**) G have an algebraic compact p-primary component.

(***) G is a torsion simply presented group.

In this work we obtain a complete system of invariants for the F-algebra FG (as a condition $F = \overline{F}$ may be not valid) in the following three cases

(o) G is a direct sum of cyclic groups such that G_0/G_p is finite.

(oo) G_p is algebraic compact and G_0/G_p is finite.

(ooo) G is simply presented such that G/G_p is finite.

In the proofs of the central results we shall use the next criterion of Berman–Mollov, in a more convenient Mollov's form (see [9]) for an isomorphism of semisimple group algebras of finite abelian groups over an arbitrary field.

Theorem (Berman–Mollov). *Let A be a finite abelian group and F is a field whose characteristic not divide the cardinality $|A|$ of A. Then $FA \cong FC$ as F-algebras for any group C if and only if $|A| = |C|$ and $|A^{q^i}| = |C^{q^i}|$, for all primes $q \neq \mathrm{char}F$ and $i \in s_q(F)$.*

In the above statement under $\mathrm{spec}_q(F) = s_q(F)$ we denote the spectrum of a field F with respect to $q \neq p$ (see also [9]). If $F = \overline{F}$, then $s_q(F)$ is an empty number set, i.e. $s_q(F) = \varnothing$.

Besides we will use the following (see [7], p.487 or [1], p.12) proposition.

Proposition (Berman–Mollov, May). *Let F be a field with a characteristic which does not divide the orders of the torsion elements in an abelian group A. Then the F-isomorphism $FA \cong FC$ for any group C yields $FA_0 \cong FC_0$.*

Now we can formulate the main results in the next two sections.

1. Mixed groups

Denote by $G^m = \{g^m \mid g \in G\}$, where $m \in \mathbb{N}$. We begin the results with the following simple lemma.

Lemma. *If C is a pure subgroup in the abelian group A, then for each natural m*

$$(A/C)^m \cong A^m/C^m\,.$$

Proof. It is evident that $(A/C)^m = A^mC/C \cong A^m/(C \cap A^m) = A^m/C^m$, as required. ■

We can now state the first theorem.

Theorem (Invariants). *Let G be a direct sum of cyclic groups so that G_0/G_p is finite. Then $FH \cong FG$ as F-algebras for any group H if and only if the following holds:*

(1) H is abelian,

(2) $H_p \cong G_p$,

(3) $H/H_0 \cong G/G_0$,

(4) $|H_0/H_p| = |G_0/G_p|$,

(5) $|(H_0)^{q^i}/(H_p)^{q^i}| = |(G_0)^{q^i}/(G_p)^{q^i}|$, for all primes $q \neq p$ and $i \in s_q(F)$.

Proof. **Necessity.** We shall consider three subcases.

First. Obviously $FG \cong FH$ is abelian, hence H is the same. That is why by [6], $G/G_0 \cong H/H_0$.

Second. From [6] it follows also too that, $FG \cong FH$ implies $F(G/G_p) \cong F(H/H_p)$ and by virtue of above Proposition, $F[(G/G_p)_0] \cong F[(H/H_p)_0]$. But $(G/G_p)_0 = G_0/G_p$ and by symmetry $(H/H_p)_0 = H_0/H_p$. Thus $F(G_0/G_p) \cong F(H_0/H_p)$. According to the above theorem for an isomorphism of finite groups, we conclude $|G_0/G_p| = |H_0/H_p|$ and $|(G_0/G_p)^{q^i}| = |(H_0/H_p)^{q^i}|$ for all primes $q \neq p$, $i \in s_q(F)$, since p is not divided $|G_0/G_p|$ using the important Caushy theorem (cf. [5],p.37, Theorem 1). Moreover G_p is pure in G_0, consequently in view of the Lemma, $(G_0/G_p)^{q^i} \cong (G_0)^{q^i}/(G_p)^{q^i}$. Analogously $(H_0/H_p)^{q^i} \cong (H_0)^{q^i}/(H_p)^{q^i}$, so the condition (5) is true.

Third. Actually, $G_p \subseteq G$ is a direct sum of cyclics, therefore $G_p = \bigcup_{n=1}^{\infty} M_n$, $M_n \subseteq M_{n+1}$ and $M_n \cap G_p^{p^n} = 1$.

Suppose that $N(FG)$ is a nilradical of FG and $N(I(FG;G))$ is a nilradical of the fundamental ideal $I(FG;G)$ of a ring FG. Further, $FG \cong FH$ does imply $1 + N(FG) = 1 + N(FH)$. But $N(FG) = N(I(FG;G)) = I(FG;G_p)$, where $I(FG;G_p)$ is a relative augmentation ideal in FG with respect to the subgroup G_p. Similarly for $N(FH)$. Thus finally $1 + I(FH;H_p) = 1 + I(FG;G_p)$, where the last group is a direct sum of cyclics (cf. [3]). Really $1 + I(FG;G_p) = \bigcup_{n=1}^{\infty} [1 + I(FG;M_n)]$, $1 + I(FG;M_n) \subseteq 1 + I(FG;M_{n+1})$ and $[1+I(FG;M_n)] \cap [1+I(FG;G_p)]^{p^n} = [1+I(FG;M_n)] \cap [1+I(F^{p^n}G^{p^n};G_p^{p^n})] = 1+I(F^{p^n}G^{p^n};M_n \cap G_p^{p^n}) = 1$, and we need only apply the Kulikov criterion (see [10], p.106, Theorem 17.1). As a final, $H_p \subseteq 1 + I(FH;H_p)$ is one also. Besides, $FG \cong FH$ implies that G_p and H_p have equal Ulm–Kaplansky–Mackey invariants (see [6]). That is why $G_p \cong H_p$ ([11]).

Sufficiency. The quotient G_0/G_p is finite, hence bounded. Moreover G_0 is pure in G, consequently G_0/G_p is pure in G/G_p by [10]. From one assertion of Kulikov ([10], p.140, Theorem 27.5) G_0/G_p is a direct factor of G/G_p, say $G/G_p \cong G_0/G_p \times G/G_0$. Anallogicly $H/H_p \cong H_0/H_p \times H/H_0$. So we conclude $F(G/G_p) \cong F(G_0/G_p) \otimes_F F(G/G_0)$ and $F(H/H_p) \cong F(H_0/H_p) \otimes_F F(H/H_0)$. From the conditions (4), (5) follows that $F(G_0/G_p) \cong F(H_0/H_p)$. Then together with the isomorphism (3), we have $F(G/G_0) \cong F(H/H_0)$, so $F(G/G_p) \cong F(H/H_p)$.

But G is a direct sum of cyclics, thus $G/G_0 \cong H/H_0$ are both free and we observe that, $G \cong G_0 \times G/G_0 \cong G_p \times G/G_p$ and $H \cong H_0 \times H/H_0 \cong H_p \times H/H_p$. Further $FG \cong [F(G/G_p)]G_p \cong [F(H/H_p)]H_p \cong FH$, as desired. The proof is fulfilled. ■

Corollary (ISOMORPHISM [2]). *Let G be a direct sum of cyclic groups for which G_0 is p-torsion. Then $FH \cong FG$ as F-algebras if and only if $H \cong G$.*

Now we can state the second important theorem.

Theorem INVARIANTS). *Let G be a group such that G_p is algebraically compact and G_0/G_p is finite. Then $FH \cong FG$ as F-algebras for any group H if and only if (1)–(5) hold.*

Proof. **Necessity.** Using the technics from the previous theorem, we will prove only that $G_p \cong H_p$.

Indeed, G_p is algebraic compact if and only if $G_p^{p^k}$ is divisible for some natural k (see [10]). As we see that $I(FG; G_p) = I(FH; H_p)$, hence $I(FH; H_p^{p^k}) = I(FG; G_p^{p^k})$ and $I(FH; H_p^{p^{k+1}}) = I(FG; G_p^{p^{k+1}}) = I(FG; G_p^{p^k})$, because $FG = FH$. Thus $I(FH; H_p^{p^k}) = I(FH; H_p^{p^{k+1}})$, i.e. $H_p^{p^k} = H_p^{p^{k+1}}$ and furthermore $H_p^{p^k}$ is divisible. This is equivalent to the fact H_p is algebraic compact. But if FG and FH are F-isomorphic, then G_p and H_p have isomorphic divisible parts and equal functions of Ulm–Kaplansky–Mackey (see [6]). So we establish, $G_p \cong H_p$ (cf. [10,11]).
Sufficiency. Following step by step the idea in the previous theorem, we will prove only that G_p is a direct factor of G. Similarly for H_p. These conjectures are true, since G_p and H_p are both algebraically compact and pure subgroups of G and H, respectively (see [10]). The statement is completely proved.

The indicated calculation of the invariants for a F-algebra FG in the terms of G and of F extends the following result of Karpilovsky (1982).

Corollary (Karpilovsky [4]). *Let G be a group such that G_0 is algebraic compact p-torsion. Then $FH \cong FG$ as F-algebras for any group H if and only if $H \cong G$.*

2. Torsion groups

Theorem (INVARIANTS). *Let G be a simply presented group for which G/G_p is finite. Then a full system of invariants for the F-algebra FG consists the isomorphic class of G_p, the cardinality of G/G_p and the cardinality of $G^{q^i}/(G_p)^{q^i}$ for all primes $q \neq p$, $i \in s_q(F)$.*

Proof. Trivially, we observe that G is torsion.
Necessity. By the used scheme, we will prove only that $G_p \cong H_p$.

Indeed, apparently $S(FG) \cong S([F(G/G_p)]G_p)$ (see [2]), where G/G_p is p-divisible. We can may assume that F is perfect. Therefore $F(G/G_p)$

is perfect without nilpotent elements, that is why by [8], $S(FG)$ is simply presented if and only if G_p is, too. But $FG \cong FH$ does imply $S(FG) \cong S(FH)$, hence H_p is simply presented. Moreover, $FG \cong FH$ ensures that G_p and H_p have isomorphic maximal divisible subgroups and equal functions of Ulm–Kaplansky–Mackey (cf. [6]). Thus, $G_p \cong H_p$ (see [11]).
Sufficiency. The groups G and H are both torsion, so $G \cong G_p \times G/G_p$, $H \cong H_p \times H/H_p$ and the proof is easy, following the preceding method. The assertion is verified. ■

Corollary. *Let G be a finite group. Then a complete set of invariants for the F-algebra FG consists the isomorphic class of G_p, the cardinality of G/G_p and the cardinality of $G^{q^i}/(G_p)^{q^i}$ for all primes $q \neq p$, $i \in s_q(F)$.*

Corollary (May [8]). *Let G be a simply presented p-primary group. Then $FH \cong FG$ as F-algebras if and only if $H \cong G$.*

3. Concluding discussion

In the present research the fully invariants for a group algebra FG were computed in the terms of a group G and of a field F with $\mathrm{char}F = p > 0$, when G belongs to the some major group classes. As we see, the complete set of invariants in these cases is $\{G_p, G/G_0, |G_0/G_p|, |(G_0)^{q^i}/(G_p)^{q^i}|, q \neq p, i \in s_q(F)\}$ by the restriction $|G_0/G_p| < \aleph_0$. If $|G_0/G_p| \geq \aleph_0$, then probably the elements of this set are not enough.

References

[1] С. Д. Б е р м а н, Т. Ж. М о л л о в. Расщепляемость базисов и изоморфизм групповых колец абелевых групп. *Плиска*, **2**, 1981, 6-22.
[2] P. V. D a n c h e v. Isomorphism of commutative modular group algebras. *Serdica*, **23**, No 2-3, 1997, 211-224.
[3] P. V. D a n c h e v. Invariants of commutative group algebras. *Math. Balcanica*, **12**, No 1-2, 1998, 13-22.
[4] G. K a r p i l o v s k y. *Unit Groups of Group Rings.* Marcel Dekker Inc., New York, 1989.
[5] С. Л е н г. *Алгебра.* Мир, Москва, 1968.
[6] W. M a y. Commutative group algebras. *Trans. Amer. Math. Soc.*, **136**, 1969, 139-149.

[7] W. M a y. Group algebras over finitely generated rings. *J. of Algebra*, **39**, 1976, 483-511.

[8] W. M a y. Modular group algebras of simply presented abelian groups. *Proc. Amer. Math. Soc.*, **104**, 1988, 403-409.

[9] Т. Ж. М о л л о в. Мультипликативные группы полупростых групповых алгебр. *Плиска*, **8**, 1986, 54-64.

[10] Л. Ф у к с. *Бесконечные абелевы группы*, т.**1**, Мир, Москва, 1974.

[11] Л. Ф у к с. *Бесконечные абелевы группы*, т. **2**, Мир, Москва, 1977.

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