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Necessary Optimality Conditions for Systems of Neutral Type Obtained by a Special Variation

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Using a special variation of control, introduced by Zabello, we obtain necessary conditions for optimality of first order in a problem of minimization of a functional for a system of neutral type.

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1. Statement of the problem

Let us consider the system of neutral type

(1) $$\dot{x}(t) = f(x(t), \dot{x}(t-h), u(t), t), \quad t \in [t_0, t_1] = T,$$

(2) $$x_\circ(.) = \{\Phi(\tau), \tau \in [t_\circ - h, t_\circ[, \quad x(t_\circ) = x_\circ\},$$

where $x \in R^n$; h, $h > 0$, is a constant delay; $u \in R^r$ is a piece-wise continuous, piece-wise smooth r-vector function of control; $\Phi(\tau)$ is a piece-wise continuous n-vector function, $\tau \in [t_\circ - h, t_\circ[$, ; $x_\circ$ - a constant n-vector; the n-vector-function $f(x, y, u, t)$ is continuous with its derivatives $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial u$ ($y(t) = \dot{x}(t-h)$, $t \in T$).

The following constraints on the control are given:

(3) $$g_i(\dot{u}(t), u(t)) = 0, \ i = \overline{1,k}, \ g_i(\dot{u}(t), u(t)) \leq 0, \ i = \overline{k+1, m},$$

(4) $$u(t) \in U \subset R^r, \quad \dot{u}(t) \in \overset{\circ}{U} \subset R^r, t \in T,$$

where scalar functions $g_i(v,u)$, $i = \overline{1,m}$, are piece-wise continuous ($v(t) = \dot{u}(t)$, $t \in T$), U and $\mathring{U}$ are compact sets.

On the trajectories of the system (1) - (4) the following cost functional

$$J(u) = \varphi(x(t_1)) \to \min \tag{5}$$

is to be minimized, where φ is a continuous with $\partial\varphi/\partial x$ scalar function.

2. Necessary optimality condition

Let $u^\circ(t)$, $t \in T$, be the optimal control in the problem (1) - (5) and $x^\circ(t)$, $t \in T$, - the corresponding trajectory of (1) - (2).

We consider the following variation $\Delta u^\circ(t)$ of the control $u^\circ(t)$ introduced in [1], and called there an inner variation:

$$\Delta u^\circ(t) = \begin{cases} 0, t\overline{\in}[\underline{\Theta},\overline{\Theta}), & [\underline{\Theta},\overline{\Theta}] \subset (t_\circ, t_1), \\ u^\circ(t+\varepsilon\sigma) - u^\circ(t), & t \in [\underline{\Theta},\overline{\Theta}), \end{cases} \tag{6}$$

where $\underline{\Theta}$, $\overline{\Theta}$ are not points of discontinuance of $u^\circ(t)$ or $\dot{u}^\circ(t)$, $t \in T$, $\overline{\Theta} - \underline{\Theta} < h$, $|\sigma| \le 1$, $\varepsilon > 0$ and sufficiently small.

Every piece-wise continuous, piece-wise smooth r-vector-function $u(t)$, $t \in T$, satisfying the constraints (3),(4) will be called an admissible control. If $\varepsilon > 0$ is sufficiently small and σ is arbitrary, $|\sigma| \le 1$, then the control $\overline{u}(t) = u^\circ(t) + \Delta u^\circ(t)$, $t \in T$, will obviously be admissible. Let $\overline{x}(t)$, $t \in T$, be the corresponding trajectory of (1) - (2) and $\Delta_{\overline{u}} x^\circ(t) = \overline{x}(t) - x^\circ(t)$.

Lemma. *For the deviation $\Delta_{\overline{u}} x^\circ(t)$, corresponding to the variation (6), the next evaluation holds*

$$||\Delta_{\overline{u}} x^\circ(t)|| \le \mathcal{K}(\overline{\Theta} - \underline{\Theta})\varepsilon, \quad \mathcal{K} = \text{const} \tag{7}$$

(/ $||\cdot||$ *is in the space* $C(R^n, T)$).

P r o o f. For $t \in [\overline{\Theta}, \underline{\Theta}]$ we get

$$\Delta_{\overline{u}} \dot{x}^\circ(t) = \dot{\overline{x}}(t) - \dot{x}^\circ(t) = f(\overline{x}(t), \overline{y}(t), \overline{u}(t), t) \tag{8}$$

$$-f(x^\circ(t), y^\circ(t), u^\circ(t), t) = \frac{\partial f(x^\circ(t), y^\circ(t), u^\circ(t), t)}{\partial u}(\overline{u}(t) - u^\circ(t))$$

$$+\frac{\partial f(x^\circ(t), y^\circ(t), u^\circ(t), t)}{\partial x}(\overline{x}(t) - x^\circ(t)) + \frac{\partial f(x^\circ(t), y^\circ(t), u^\circ(t), t)}{\partial y}(\overline{y}(t) - y^\circ(t))$$

$$+o_1(||\Delta u^\circ(t)||, ||\Delta_{\overline{u}} x^\circ(t)||, ||\Delta_{\overline{u}} y^\circ(t)||), (\Delta_{\overline{u}} y^\circ(t) = \overline{y}(t) - y^\circ(t))$$

$$= \dot{\overline{x}}(t-h) - \dot{x}^\circ(t-h) = \Delta_{\overline{u}}\dot{x}^\circ(t-h)).$$

As

(9) $$\Delta u^\circ(t) = \overline{u}(t) - u^\circ(t) = \varepsilon \dot{u}^\circ(t)\sigma + o_2(\varepsilon)$$

and using the integral continuance of the solution $\Delta_{\overline{u}}x^\circ(t)$, we obtain

(10) $$\Delta_{\overline{u}}x^\circ(t) = \epsilon_\sigma \frac{\partial f(x^\circ(t), y^\circ(t), u^\circ(t), t)}{\partial u}\dot{u}^\circ(t)$$
$$+\frac{\partial f(x^\circ(t), y^\circ(t), u^\circ(t), t)}{\partial x}\Delta_{\overline{u}}x^\circ(t)$$
$$+\frac{\partial f(x^\circ(t), y^\circ(t), u^\circ(t), t)}{\partial y}\Delta_{\overline{u}}y^\circ(t) + o_3(\varepsilon).$$

Integrate (10) on $[\underline{\Theta}, t]$:

$$\Delta_{\overline{u}}x^\circ(t) = \varepsilon \int_{\underline{\Theta}}^t \sigma \frac{\partial f(x^\circ(\tau), y^\circ(\tau), u^\circ(\tau), \tau)}{\partial u}\dot{u}^\circ(\tau)d\tau$$
$$+ \int_{\underline{\Theta}}^t \frac{\partial f(x^\circ(\tau), y^\circ(\tau), u^\circ(\tau), \tau)}{\partial x}\Delta_{\overline{u}}x^\circ(\tau)d\tau$$
$$+ \int_{\underline{\Theta}}^t \frac{\partial f(x^\circ(\tau), y^\circ(\tau), u^\circ(\tau), \tau)}{\partial y}\Delta_{\overline{u}}\dot{x}^\circ(\tau-h)d\tau + o_4(\varepsilon).$$

From the continuance of functions $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial u$ and properies of $u(\tau)$ it follows that there exist such constants $\mathcal{K}_i$, $i = 1,2,3$, that :

$$||\Delta_{\overline{u}}x^\circ(t)|| \leq \varepsilon \int_{\underline{\Theta}}^t |\sigma|\mathcal{K}_1 d\tau + \mathcal{K}_2 \int_{\underline{\Theta}}^t ||\Delta_{\overline{u}}x^\circ(\tau)||d\tau$$
$$+\mathcal{K}_3 \int_{\underline{\Theta}}^t ||\Delta_{\overline{u}}\dot{x}^\circ(\tau-h)||d\tau$$

and as from (6) $\Delta_{\overline{u}}x^\circ(\tau) \equiv 0$ when $\tau < \underline{\Theta}$, then

(11) $$||\Delta_{\overline{u}}x^\circ(t)|| \leq \mathcal{K}_1\varepsilon(\overline{\Theta} - \underline{\Theta}) + \mathcal{K}_2 \int_{\underline{\Theta}}^t ||\Delta_{\overline{u}}x^\circ(\tau)||d\tau,$$

Hence from (11) and Gronwall-Bellman's inequality it follows:

(12) $$||\Delta_{\overline{u}}x^\circ(t)|| \leq \mathcal{K}_1\varepsilon(\overline{\Theta} - \underline{\Theta})e^{\mathcal{K}_2(t-\underline{\Theta})} \leq \mathcal{K}_1\varepsilon(\overline{\Theta} - \underline{\Theta})e^{\mathcal{K}_2(\overline{\Theta}-\underline{\Theta})}$$

$$= \mathcal{K}\varepsilon(\overline{\Theta} - \underline{\Theta}), \quad t \in [\underline{\Theta}, \overline{\Theta},],$$

if $\mathcal{K} = \mathcal{K}_1 e^{\mathcal{K}_2(\overline{\Theta} - \underline{\Theta})}$.

With (12) the lemma is proved for $t \in [\overline{\Theta}, \underline{\Theta}]$. For $t_\circ < t < \Theta$ obviously $\Delta_{\overline{u}} x^\circ(t) \equiv 0$, and for $\overline{\Theta} < t \leq t_1$ we will obtain (7) using (12) when $t = \overline{\Theta}$. The lemma is proved. ■

Consider the conjugated system

$$\frac{d\psi(t)}{dt} = -\frac{\partial f'(x(t), \dot{x}(t-h), u(t), t)}{\partial x}\psi(t)$$

$$+\frac{d}{dt}\Big[\frac{\partial f'(x(t+h), \dot{x}(t), u(t+h), t+h)}{\partial y}\psi(t+h)\Big],$$

(13) $$t_\circ \leq t \leq t_1 - h;$$

$$\frac{d\psi(t)}{dt} = -\frac{\partial f'(x(t), \dot{x}(t-h), u(t), t)}{\partial x}\psi(t),$$

$$t_1 - h \leq t \leq t_1;$$

(14) $$\psi(t_1) = -\frac{\partial \varphi(x(t_1))}{\partial x}, \quad \psi(t) \equiv 0,\ t \in (t_1, t_1 + h],$$

$$\psi^-(\tau_i) = \psi^+(\tau_i) - \Big[\frac{\partial f'(x(\tau_i + h), \dot{x}(\tau_i), u(\tau_i + h), \tau_i + h)}{\partial y}\psi(\tau_i + h)\Big]^+$$

(15)

$$+\Big[\frac{\partial f'(x(\tau_i + h), \dot{x}(\tau_i), u(\tau_i + h), \tau_i + h)}{\partial y}\psi(\tau_i + h)\Big]^-,$$

where points of discontinuance τ_i of $\psi(t)$ are those of the control $u(t)$, and also points of the kind $t_1 - jh$, $\tau_i - jh$ $(\psi^\pm(\tau_i) = \psi(\tau_i \pm 0))$.

Let $\psi^\circ(t)$ be the trajectory of (13) - (15) corresponding to $u^\circ(t)$ and $x^\circ(t)$, $t \in T$. We will obtain a formula for the deviation of the criterion (5):

$$\Delta J(u^\circ) = J(\overline{u}) - J(u^\circ) = \varphi(\overline{x}(t_1)) - \varphi(x^\circ(t_1))$$

(16)

$$= \frac{\partial \varphi'(x^\circ(t_1))}{\partial x}\Delta_{\overline{u}} x^\circ(t_1) + 0^{(1)}(||\Delta_{\overline{u}} x^\circ(t_1)||).$$

Using (8), the system (1),(13) - (15) and the obvious identity

$$\int_{t_0}^{t_1} \dot{\psi}^{\circ\prime}(t)\Delta_{\overline{u}}\dot{x}^\circ(t)dt = \psi^{\circ\prime}(t_1)\Delta_{\overline{u}}x^\circ(t_1) - \psi^{\circ\prime}(t_\circ)\Delta_{\overline{u}}x^\circ(t_\circ)$$

$$-\int_{t_\circ}^{t_1} \psi^{\circ\prime}(t)\Delta_{\overline{u}}x^\circ(t)dt + \sum_{\tau_i}[\psi^{\acute{o}-}(\tau_i) - \psi^{\acute{o}+}(\tau_i)]\Delta_{\overline{u}}x^\circ(\tau_i)$$

$$= \psi^{\circ\prime}(t_1)\Delta_{\overline{u}}x^\circ(t_1) - \int_{t_\circ}^{t_1} \dot{\psi}^{\circ\prime}(t)\Delta_{\overline{u}}x^\circ(t)dt$$

we express

$$-\psi^{\acute{o}}(t_1)\Delta_{\overline{u}}x^\circ(t_1) - \int_{t_\circ}^{t_1} \psi^{\acute{o}}(t)\Big[\frac{\partial f(x^\circ(t),\dot{x}^\circ(t-h),u^\circ(t),t)}{\partial x}\Delta_{\overline{u}}x^\circ(t)$$

$$+\frac{\partial f(x^\circ(t),\dot{x}^\circ(t-h),u^\circ(t),t}{\partial y}\Delta_{\overline{u}}\dot{x}^\circ(t-h) + \frac{\partial f(x^\circ(t),\dot{x}^\circ(t-h),u^\circ(t),t}{\partial u}\Delta u^\circ(t)$$

$$+0_1(||\Delta_{\overline{u}}x^\circ(t)||,\ ||\Delta_{\overline{u}}\dot{x}^\circ(t-h)||,\ ||\Delta u^\circ(t)||)\Big]dt$$

$$+\int_{t_\circ}^{t_1} \psi^{\circ\prime}(t)\frac{\partial f(x^\circ(t),\dot{x}^\circ(t-h),u^\circ(t),t)}{\partial x}\Delta_{\overline{u}}x^\circ(t)dt$$

$$-\psi^{\circ\prime}(t+h)\frac{\partial f(x^\circ(t+h),\dot{x}^\circ(t),u^\circ(t+h),t+h)}{\partial y}\Delta_{\overline{u}}x^\circ(t)\Big|_{t_\circ}^{t_1}$$

$$+\int_{t_\circ}^{t_1} \psi^{\circ\prime}(t+h)\frac{\partial f(x^\circ(t+h),\dot{x}^\circ(t),u^\circ(t+h),t+h)}{\partial y}\Delta_{\overline{u}}\dot{x}^\circ(t)dt$$

$$= -\int_{t_\circ}^{t_1} \psi^{\circ\prime}(t)\frac{\partial f(x^\circ(t),\dot{x}^\circ(t-h),u^\circ(t),t)}{\partial u}\Delta u^\circ(t)dt$$

$$-\int_{t_\circ}^{t_1} \psi^{\circ\prime}(t)\frac{\partial f(x^\circ(t),\dot{x}^\circ(t-h),u^\circ(t),t)}{\partial y}\Delta_{\overline{u}}\dot{x}^\circ(t-h)dt$$

$$-\int_{t_\circ}^{t_1} \psi^{\circ\prime}(t)0_1(||\Delta_{\overline{u}}x^\circ(t)||,\ ||\Delta_{\overline{u}}\dot{x}^\circ(t-h)||,\ ||\Delta u^\circ(t)||dt$$

$$+\int_{t_\circ+h}^{t_1} \psi^{\acute{o}}(\tau)\frac{\partial f(x^\circ(\tau),\dot{x}^\circ(\tau-h),u^\circ(\tau),\tau)}{\partial y}\Delta_{\overline{u}}\dot{x}^\circ(\tau-h)d\tau.$$

As $\Delta x^\circ(t) = 0,\ t \le t_\circ$ and $\psi^\circ(t) \equiv 0,\ t > t_1$, then

$$-\psi^{\circ\prime}(t_1)\Delta_{\overline{u}}x^\circ(t_1) = -\int_{t_\circ}^{t_1} \psi^{\circ\prime}(t)\frac{\partial f(x^\circ(t),\dot{x}^\circ(t-h),u^\circ(t)t)}{\partial u}\Delta u^\circ(t)dt$$

$$-\int_{t_0}^{t_1}\psi^{\circ\prime}(t)0_1(||\Delta_{\overline{u}}x^\circ(t)||,\ ||\Delta_{\overline{u}}\dot{x}^\circ(t-h)||,\ ||\Delta u^\circ(t)||)dt.$$

Then from (16) and (6) we get

$$\Delta J(u^\circ) = -\int_{\underline{\Theta}}^{\overline{\Theta}} \psi^{\circ\prime}(t)\frac{\partial f(x^\circ(t),\dot{x}^\circ(t-h),u^\circ(t),t)}{\partial u}\Delta u^\circ(t)dt$$

$$-\int_{t_0}^{t_1}\psi^{\circ\prime}(t)0_1(||\Delta_{\overline{u}}x^\circ(t)||,\ ||\Delta_{\overline{u}}\dot{x}^\circ(t-h)||,\ ||\Delta u^\circ(t)||)dt$$

$$+0^{(1)}(||\Delta_{\overline{u}}x^\circ(t_1)||).$$

If we use formula (9) and the integral continuity of solutions $x(t)$, the deviation is:

$$\Delta J(u^\circ) = -\varepsilon\int_{\underline{\Theta}}^{\overline{\Theta}} \psi^{\circ\prime}(t)\frac{\partial f(x^\circ(t),\dot{x}^\circ(t-h),u^\circ(t),t)}{\partial u}\dot{u}^\circ(t)\sigma dt + o(\varepsilon),$$

which with Hamilton's function

$$H(x(t),\dot{x}(t-h),\psi(t),u(t),t) = \psi'(t)f(x(t),\dot{x}(t-h),u(t),t)$$

can be written as in [1]:

$$\Delta J(u^\circ) = -\varepsilon\int_{\underline{\Theta}}^{\overline{\Theta}} \frac{\partial H'(x^\circ(t),\dot{x}^\circ(t-h),\psi^\circ(t)u^\circ(t)t)}{\partial u}\dot{u}^\circ(t)\sigma dt + o(\varepsilon). \quad (17)$$

Theorem. *If $u^\circ(t)$ and $x^\circ(t)$, $t \in T$, are the optimal control and trajectory in the problem (1) - (5) and $\psi^\circ(t)$, $t \in T$, is the corresponding solution of the conjugated system (13) - (15), then next relations hold:*

$$\alpha^\circ(t) = \frac{\partial H'(x^\circ(t),\dot{x}^\circ(t-h),\psi^\circ(t),u^\circ(t),t)}{\partial u}\dot{u}^\circ(t) \equiv 0, \quad (18)$$

$$t \in T\backslash\Omega;$$

$$\alpha^\circ(\omega+0) = \alpha^\circ(\omega-0) = 0,\ \omega \in \Omega, \quad (19)$$

where Ω is the set of points of discontinuity of functions $u^\circ(t), \dot{u}^\circ(t)$, $t \in T$.

Proof. If we recall the assumptions about $\varepsilon, \underline{\Theta}, \overline{\Theta}, \sigma$, then (18) follows from (17), as $\Delta J(u^\circ) \geq 0$. The equalities (19) can be proved, assuming the opposite is true and using the continuity of $\alpha 6 \circ (t)$ and (18), as in [1]. ∎

The identity (18) can be considered as a generalized Euler's condition, and equalities (19) - as an analogue of Weierstrass-Erdman's condition.

The results of this paper were partly reported in [2].

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