Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 13, 1999, Fasc. 1-2

On Pairwise S–Closed Subspaces

Alev Kanibir

Presented by P. Kenderov

1. Introduction

The concept of bitopological spaces was first introduced by Kelly [3]. Levine [4] defined the notions of semiopen sets in topological spaces. Bose [1] extended the notions of semiopen sets to the bitopological settings. Thomson [7] initiated the notion of S–closed topological spaces. Mukherjee [5] generalized the notions of S–closed spaces to the bitopological settings.

In this paper we investigate the properties of pairwise S–closed subspaces.

Throughout this paper we shall denote by (X, τ_1, τ_2) a bitopological space. For any subset A, $\tau_i - \text{int}A$ and $\tau_i - \text{cl}A$ denotes the interior of A and the closure of A with respect to τ_i, where $i = 1, 2$. If $A \subset Y \subset X$, $\tau_{i_Y} - \text{cl}A$ will denote the τ_{i_Y}–closure of the subset A in the subspace $(Y, \tau_{1_Y}, \tau_{2_Y})$ of (X, τ_1, τ_2), where $i = 1, 2$. A subset A of (X, τ_1, τ_2) is called τ_i semiopen with respect to τ_j if there exists a τ_i–open set U of X such that $U \subset A \subset \tau_j - \text{cl}U$, where $i, j = 1, 2$, $i \neq j$.

Definition 1.1. A subset Y of a space (X, τ_1, τ_2) is said to be pairwise dense [6], if every nonempty subset of X which is the intersection of a τ_1–open and a τ_2–open subset of X has nonempty intersection with Y.

Definition 1.2. (X, τ_1, τ_2) is called τ_i S–closed with respect to τ_j [5], if for each cover $\{V_\alpha \mid \alpha \in \nabla\}$ of X by τ_i semiopen sets with respect to τ_j, there exists a finite subfamily ∇_0 of ∇ such that $X = \cup\{\tau_j - \text{cl}V_\alpha \mid \alpha \in \nabla_0\}$, $i, j, = 1, 2, i \neq j$. X is called pairwise S–closed if it is τ_i S–closedwith respect to τ_j for $i, j, = 1, 2, i \neq j$.

Definition 1.3. A subset Y of (X, τ_1, τ_2) will be called τ_i S–closed with respect to τ_j in X [5], if and only if for every cover $\{V_\alpha : \alpha \in \nabla\}$ of Y by

τ_i semiopen sets with respect to τ_j of X, there exists a finite subfamily ∇_0 of ∇ such that $Y \subset \cup\{\tau_j - \mathrm{cl}V_\alpha \mid \alpha \in \nabla_0\}$, where $i, j, = 1, 2, i \neq j$.

Theorem 1.4. [5] *A subset Y of (X, τ_1, τ_2) will be τ_{i_Y} S–closed with respect to τ_{j_Y}, if Y is τ_i S–closed with respect to τ_j in X and $Y \in \tau_i$, where $i, j = 1, 2, i \neq j$.*

Theorem 1.5. [5] *If a subset Y of (X, τ_1, τ_2) is τ_{i_Y} S–closed with respect to τ_{j_Y} and $Y \in \tau_1 \cap \tau_2$, then Y is τ_i S–closed with respect to τ_j in X, for $i, j = 1, 2, i \neq j$.*

2. Pairwise S–closed subspaces

Lemma 2.1. [2] *Let (X, τ_1, τ_2) be a bitopological space, $Y \subset X$ pairwise dense set. If U is nonempty τ_i–open set $(i = 1, 2)$, then $U \cap Y \neq \emptyset$.*

Lemma 2.2. [2] *Let (X, τ_1, τ_2) be a bitopological space. Let U and Y be the subsets of X. If U is a τ_i–semiopen set with respect to τ_j and Y is pairwise dense set, then*

$$\tau_j - clU \subset \tau_j - cl(U \cap Y), \qquad i, j = 1, 2, \ i \neq j.$$

Lemma 2.3. *Let (X, τ_1, τ_2) be a bitopological space and V, Y subsets of X. If V is τ_i semiopen with respect to τ_j and Y is pairwise dense set in X, then $V \cap Y$ is τ_{i_Y} semiopen with respect to τ_{j_Y}, $i, j = 1, 2, i \neq j$.*

Proof. If V is an empty set, then it is clear. If V is a nonempty set then there exists a nonempty τ_i–open set U such that $U \subset V \subset \tau_j - \mathrm{cl}U$. Hence $U \cap Y \subset V \cap Y \subset \tau_j - \mathrm{cl}U \cap Y$. By Lemma 2.1 $U \cap Y \neq \emptyset$. Clearly, $U \cap Y$ is τ_{i_Y}–open set and if we use Lemma 2.2 we get

$$U \cap Y \subset V \cap Y \subset \tau_j - \mathrm{cl}U \cap Y \subset \tau_j - \mathrm{cl}(U \cap Y) \cap Y \subset \tau_{j_Y} - \mathrm{cl}(U \cap Y).$$

This completes the proof. ■

Lemma 2.4. *Let (X, τ_1, τ_2) be a bitopological space and $A \subset Y \subset X$. If A is τ_{i_Y} semiopen with respect to τ_{j_Y}, then there exists a τ_i semiopen set V with respect to τ_j in X such that $A = V \cap Y$, where $i, j = 1, 2, i \neq j$.*

Proof. Since A is τ_{i_Y} semiopen with respect to τ_{j_Y} we have $U \subset A \subset \tau_{j_Y} - \text{cl}U$ for a τ_{j_Y}–open set U. Let $U = O \cap Y$ for τ_i–open set O. Therefore, we obtain

$$O \subset A \cup O \subset (\tau_{j_Y} - \text{cl}U) \cup O \subset (\tau_j - \text{cl}U) \cup O \subset \tau_j - \text{cl}O.$$

Put $V = A \cup O$.

$$(A \cup O) \cap Y = (A \cap Y) \cup (O \cap Y) = A \cup U = A$$

This completes the proof. ■

Lemma 2.5. *Let (X, τ_1, τ_2) be a bitopological space and $A \subset Y \subset X$. Then, we have*

(a) If A is pairwise dense set in X then A is pairwise dense set in Y.

(b) If A is pairwise dense set in X then Y is pairwise dense set in X.

Proof. (a) Let $V' \in \tau_{1_Y}$, $W' \in \tau_{2_Y}$ and $V' \cap W' \neq \emptyset$. There exists τ_1–open V such that $V' = V \cap Y$ and there exists τ_2–open W such that $W' = W \cap Y$. Since $W' \cap V' \neq \emptyset$, then $W \cap V \neq \emptyset$. Since A is pairwise dense set in X, then $V \cap W \cap A \neq \emptyset$. Since $A \subset Y$, then $W' \cap V' \cap A \neq \emptyset$. This shows that A is pairwise dense in Y.

(b) Let $V \in \tau_1$, $W \in \tau_2$ and $V \cap W \neq \emptyset$. Since A is pairwise dense set in X, then $W \cap V \cap A \neq \emptyset$ and since $A \subset Y$, then $W \cap V \cap Y \neq \emptyset$. This shows that Y is pairwise dense set in X. ■

Remark 2.6. In a bitopological space (X, τ_1, τ_2), for any subset Y which is τ_i S–closed with respect to τ_j is not necessarily τ_{i_Y} S–closed with respect to τ_{j_Y} in Y, $i, j = 1, 2, i \neq j$.

Example 2.7. Let R be the set of real numbers, τ_1 the usual topology of R, τ_2 the topology of countable complements of R and N the set of positive integers. Then N is τ_1 S–closed with respect to τ_2. Infact N is τ_2 S–closed with respect to τ_1 in R but N is not τ_{1_N} S–closed with respect to τ_{2_N}.

Theorem 2.8. *Let Y be a pairwise dense set in the bitopological space (X, τ_1, τ_2), Then Y is τ_{i_Y} S–closed with respect to τ_{j_Y} iff Y is τ_i S–closed with respect to τ_j in X, where $i, j = 1, 2, i \neq j$.*

Proof. Let Y be τ_{i_Y} S–closed with respect to τ_{j_Y} and $\{V_\alpha \mid \alpha \in \nabla\}$ a cover of Y, where each V_α is τ_i semiopen with respect to τ_j. Then by Lemma 2.3, $V_\alpha \cap Y$ is τ_{i_Y} semiopen with respect to τ_{j_Y} for each α. By hypotesis we have $Y \subset \cup\{\tau_{j_Y} - \mathrm{cl}(V_\alpha \cap Y) \mid \alpha \in \nabla_0\}$ where ∇_0 is a finite subfamily of ∇. Therefore we have $Y \subset \cup\{\tau_j - \mathrm{cl}V_\alpha \mid \alpha \in \nabla_0\}$. This shows that Y is τ_i S–closed with respect to τ_j in X.

Let Y be τ_i S–closed with respect to τ_j in X and $\{V_\alpha \mid \alpha \in \nabla\}$ a cover of Y, where each V_α is τ_{i_Y} semiopen with respect to τ_{j_Y}. By Lemma 2.4, for each V_α, there exists a τ_i semiopen set U_α with respect to τ_j such that $V_\alpha = U_\alpha \cap Y$. Then $Y \subset \cup\{U_\alpha \mid \alpha \in \nabla\}$. Since Y is τ_i S–closed with respect to τ_j in X then $Y \subset \cup\{\tau_j - \mathrm{cl}U_\alpha \mid \alpha \in \nabla_0\}$, where ∇_0 is a finite subfamily of ∇. By Lemma 2.2 $\tau_j - \mathrm{cl}U_\alpha \subset \tau_j - \mathrm{cl}(U_\alpha \cap Y)$ and hence $\tau_j - \mathrm{cl}U_\alpha \cap Y \subset \tau_{j_Y} - \mathrm{cl}(U_\alpha \cap Y)$. Therefore, we have $Y = \cup\{\tau_{j_Y} - \mathrm{cl}V_\alpha \mid \alpha \in \nabla_0\}$. Then Y is τ_{i_Y} S–closed with respect to τ_{j_Y}. ∎

Remark 2.9. Let Y be a subset of the bitopological space (X, τ_1, τ_2). Any subset of Y which is τ_i S–closed with respect to τ_j in X is not necessarily a τ_{i_Y} S–closed with respect to τ_{j_Y}, where $i, j = 1, 2, i \neq j$.

Example 2.10. Let R be the set of real numbers, τ_1 the usual topology of R, τ_2 the topology of countable complements of R, N the set of positive integers and Z the set of integers. Then N is τ_1 S–closed with respect to τ_2 and τ_2 S–closed with respect to τ_1 in X but N is not τ_{1_Z} S–closed with respect to τ_{2_Z}.

Theorem 2.11. *Let Y be a pairwise dense subset of the bitopological space (X, τ_1, τ_2) and $A \subset Y$. Then A is τ_{i_Y} S–closed with respect to τ_{j_Y} iff A is τ_i S–closed with respect to τ_j in X, where $i, j = 1, 2, i \neq j$.*

Proof. Let A be τ_{i_Y} S–closed with respect to τ_{j_Y} and $\{V_\alpha \mid \alpha \in \nabla\}$ a cover of A, where each V_α is τ_i semiopen with respect to τ_j. By Lemma 2.3 $V_\alpha \cap Y$ is τ_{i_Y} semiopen with respect to τ_{j_Y} for each α. Since A is τ_{i_Y} S–closed with respect to τ_{j_Y} then $A \subset \cup\{\tau_{j_Y} - \mathrm{cl}(V_\alpha \cap Y) \mid \alpha \in \nabla_0\}$. Therefore, $A \subset \cup\{\tau_j - \mathrm{cl}V_\alpha \mid \alpha \in \nabla_0\}$. Hence A is τ_i S–closed with respect to τ_j.

Let A be τ_i S–closed with respect to τ_j in X and $\{V_\alpha \mid \alpha \in \nabla\}$ a cover of A, where each V_α is τ_{i_Y} semiopen with respect to τ_{j_Y}. By Lemma 2.4, for each V_α, there exists a τ_i semiopen set U_α with respect to τ_j such that $V_\alpha = U_\alpha \cap Y$. Since $A \subset \cup\{U_\alpha \mid \alpha \in \nabla\}$ and A is τ_i S–closed with respect to τ_j, then $A \subset \cup\{\tau_j - \mathrm{cl}U_\alpha \mid \alpha \in \nabla_0\}$, where ∇_0 is a finite subfamily of ∇. By Lemma 2.2. $\tau_j - \mathrm{cl}U_\alpha \subset \tau_j - \mathrm{cl}(U_\alpha \cap Y)$ and hence $\tau_j - \mathrm{cl}U_\alpha \cap Y \subset \tau_{j_Y} - \mathrm{cl}(U_\alpha \cap Y)$. Therefore, $A \subset \cup\{\tau_{j_Y} - \mathrm{cl}V_\alpha \mid \alpha \in \nabla_0\}$. Hence A is τ_{i_Y} S–closed with repect to τ_{j_Y}.

Corollary 2.12. *Let A be pairwise dense subset of a bitopological space (X, τ_1, τ_2) such that $A \subset Y \subset X$. Then A is a pairwise S–closed subspace of Y iff A is a pairwise S–closed subspace of X.*

P r o o f. This follows from Lemma 2.5, Theorems 2.8 and 2.11. ■

Theorem 2.13. *Let Y be a pairwise open set in the bitopological space (X, τ_1, τ_2) and $A \subset Y$. Then A is τ_{i_Y} S–closed with respect to τ_{j_Y} iff A is τ_i S-closed with respect to τ_j in X, where $i, j = 1, 2, i \neq j$.*

P r o o f. Similar to the proof of Theorem 2.11. ■

Corollary 2.14. *Let A and Y be pairwise open subsets of a bitopological space (X, τ_1, τ_2) such that $A \subset Y \subset X$. Then A is a pairwise S–closed subspace of Y iff A is a pairwise S–closed subspace of X.*

P r o o f. This follows from Theorems 1.4, 1.5 and 2.13. ■

References

[1] S. B o s e. Semi open sets, semi continuity and semi open mappings in bitopological spaces. *Bull. Cal. Math. Soc.*, **73**, 1981, 237-246.

[2] A. K a n i b i r. A note on a pairwise α-continuous functions. *Bull. Cal. Math. Soc.*, **88**, 1996, 49-52.

[3] J.C. K e l l y. Bitopological spaces. *Proc. London Math. Soc.*, **13**, 1963, 71-89.

[4] N. L e v i n e. Semi open sets and semi continuity in topological spaces. *Amer. Math. Monthly*, **70**, 1963, 36-41.

[5] M. N. M u k h e r j e e. On pairwise S-closed bitopological spaces. *Int. J. Math. & Math. Sci.*, **8(4)**, 1985, 729-745.

[6] A.R. S i n g a l, S.P. A r y a. On pairwise almost regular spaces. *Glasnik Matematicki*, **6(26)**, 1971, 335-343.

[7] T. T h o m p s o n. S-closed spaces. *Proc. Amer. Math. Soc.*, **60**, 1976, 335-338.

Zonguldak Karaelmas University
Faculty of Art and Science
Department of Mathematics
67100 Zonguldak, TURKEY

Received: 22.12.1995