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Matrix Theorems for the de Branges Weight Functions

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On the basis of Theorems 1, 2, 3 and Corollaries 1, 2 proved in this paper, the de Branges weight functions (4) or system (6)-(7) can be examined with the help of the matrix theory.

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Let S denote the class of all functions

(1) $$f(z) = \sum_{n=1}^{\infty} a_n z^n, \; a_1 = 1,$$

analytic and univalent in the unit disc $|z| < 1$. Bieberbach [1] conjectured that the inequalities

(2) $$|a_n| \leq n, \quad n = 2, 3, \ldots,$$

hold, where the equalities hold only for the Koebe function

(3) $$f_0(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n \in S$$

and its rotations $e^{-i\alpha} f_0\left(ze^{i\alpha}\right) \in S$ where α is real.

De Branges [2,3] proved the Bieberbach conjecture (2) and (3) for the functions (1). Fitzgerald and Pommerenke [4,5], Weinstein [6], and the author

[7,8,9,10] simpler proved the same conjecture, respectively (see the Grinshpan reviews in [11]).

For any positive integer $n \geq 1$, de Branges [2,3] introduced and used the system of weight functions

$$\sigma_k(t) = k\sum_{\nu=0}^{n-k}(-1)^{\nu}\frac{(2k+\nu+1)_{\nu}(2k+2\nu+2)_{n-k-\nu}}{(k+\nu)\nu!(n-k-\nu)!}e^{-\nu t-kt} \tag{4}$$

for $0 \leq t < +\infty$ and $k = 1,2,\ldots,n$ where $(a)_{\nu}$ for an arbitrary number a denotes

$$(a)_{\nu} = a(a+1)\ldots(a+\nu-1), \quad \nu = 1,2,\ldots; \quad (a)_0 = 1. \tag{5}$$

The functions (4), having in mind (5), are the unique solution of the de Branges system of differential equations

$$\sigma_k(t) - \sigma_{k+1}(t) = -\frac{\sigma_k'(t)}{k} - \frac{\sigma_{k+1}'(t)}{k+1}, \tag{6}$$

$$0 \leq t < +\infty, \quad k = 1,2,\ldots,n, \quad \sigma_{n+1}(t) = 0,$$

with initial conditions

$$\sigma_k(0) = n-k+1, \quad k = 1,2,\ldots,n. \tag{7}$$

The author [7,9,10] introduced and used the full notations $\sigma_{nk}(t) \equiv \sigma_k(t), n \geq k, k = 1,2,\ldots,n, 0 \leq t < +\infty$, of the de Branges function (4). Further, we shall use the notation $\sigma_{nk}(t)$ $(1 \leq k \leq n+1)$ for the kth de Branges weight function $\sigma_k(t)$ which satisfies (6)-(7) for any fixed value of $n (n = 1,2,\ldots)$.

Now we shall prove the following

Theorem 1. *For all fixed integer $n \geq 2$, let*

$$T_n = \begin{bmatrix} -1 & 2 & -2 & 2 & \ldots & \ldots & \ldots & \ldots & (-1)^n 2 \\ 0 & -2 & 4 & -4 & \ldots & \ldots & \ldots & \ldots & (-1)^{n-1}4 \\ 0 & 0 & -3 & 6 & \ldots & \ldots & \ldots & \ldots & (-1)^{n-2}6 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & -k & 2k & -2k & \ldots & (-1)^{n-k+1}2k \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & -(n-1) & 2(n-1) \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & 0 & -n \end{bmatrix} \tag{8}$$

and

$$P_n(t) = \begin{bmatrix} \sigma_{n1}(t) \\ \sigma_{n2}(t) \\ \vdots \\ \sigma_{nk}(t) \\ \vdots \\ \sigma_{nn}(t) \end{bmatrix}, \qquad P_n'(t) = \begin{bmatrix} \sigma_{n1}'(t) \\ \sigma_{n2}'(t) \\ \vdots \\ \sigma_{nk}'(t) \\ \vdots \\ \sigma_{nn}'(t) \end{bmatrix} \tag{9}$$

be $n \times n$ matrix and column matrices, where $\sigma_{nk}(t) \equiv \sigma_k(t)$, $k = 1, 2, \ldots, n$, $0 \leq t < +\infty$, and $\sigma_{nk}'(t) \equiv \sigma_k'(t)$, $k = 1, 2, \ldots, n$, $0 \leq t < +\infty$, are the de Branges weight functions (4) and their derivatives with the full notations, respectively.

Then the column matrix $P_n(t)$ satisfies the matrix linear homogeneous differential equation of the first order:

$$P_n'(t) = T_n P_n(t) \tag{10}$$

with the initial condition

$$P_n(0) = \begin{bmatrix} n \\ n-1 \\ \vdots \\ n-k+1 \\ \vdots \\ 1 \end{bmatrix}. \tag{11}$$

P r o o f. For a fixed integer $n \geq 2$ with the help of the full notations $\sigma_{nk}(t) \equiv \sigma_k(t)$, $k = n+1, n, n-1, \ldots, 2, 1$, $0 \leq t < +\infty$, from (6) we successively obtain

$$-\frac{\sigma_{nn}'(t)}{n} = \sigma_{nn}(t), \tag{12}$$

$$-\frac{\sigma_{n,n-1}'(t)}{n-1} = \sigma_{n,n-1}(t) - 2\sigma_{nn}(t), \tag{13}$$

$$-\frac{\sigma_{n,n-2}'(t)}{n-2} = \sigma_{n,n-2}(t) - 2\sigma_{n,n-1}(t) + 2\sigma_{nn}(t), \tag{14}$$

etc. If we assume that the formula

$$-\frac{\sigma_{n,k}'(t)}{k} = \sigma_{n,k}(t) + 2\Sigma_{\nu=k+1}^{n}(-1)^{\nu-k}\sigma_{n\nu}(t), \tag{15}$$

is true for any positive integer k with $2 \leq k \leq n-1$ $(n \geq 3)$, then from (6) and (15) we obtain that

$$
\begin{aligned}
-\frac{\sigma'_{n,k-1}(t)}{k-1} &= \sigma_{n,k-1}(t) - \sigma_{nk}(t) + \frac{\sigma'_{nk}(t)}{k} \\
&= \sigma_{n,k-1}(t) - 2\sigma_{n,k}(t) + 2\Sigma_{\nu=k+1}^{n}(-1)^{\nu-k+1}\sigma_{n,\nu}(t) \qquad (16) \\
&= \sigma_{n,k-1}(t) + 2\Sigma_{\nu=k}^{n}(-1)^{\nu-k+1}\sigma_{n,\nu}(t).
\end{aligned}
$$

From the comparison of (15) and (16) it follows that the formula (15) is true for all $k = n-1, n-2, \ldots, 2, 1$ $(n \geq 2)$. Therefore, the relations (8)-(10) follow from (12)-(15). The relation (11) follows from (7).

This completes the proof of Theorem 1. ■

Further, we shall prove the following

Theorem 2. *For a fixed integer $n \geq 2$, the inverse $n \times n$ matrix T_n^{-1} of the matrix (8) is*

$$
(17)\, T_n = \begin{bmatrix}
-\frac{1}{1} & -\frac{2}{2} & -\frac{2}{3} & -\frac{2}{4} & \cdots & \cdots & \cdots & \cdots & -\frac{2}{n} \\
0 & -\frac{1}{2} & -\frac{2}{3} & -\frac{2}{4} & \cdots & \cdots & \cdots & \cdots & -\frac{2}{n} \\
0 & 0 & -\frac{1}{3} & -\frac{2}{4} & \cdots & \cdots & \cdots & \cdots & -\frac{2}{n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & -\frac{1}{k} & -\frac{2}{k+1} & -\frac{2}{k+2} & \cdots & -\frac{2}{n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & -\frac{1}{n-1} & -\frac{2}{n} \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & -\frac{1}{n}
\end{bmatrix}
$$

Proof. For a fixed integer $n \geq 2$, let $T_n^{k\nu}$ denote the cofactor of the element on the kth row $(1 \leq k < n)$ and the νth column $(1 \leq \nu \leq n)$ of the determinant of the matrix T_n in (8). The values of this determinant and these cofactors are

$$\det T_n = (-1)^n n! \tag{18}$$

and

$$\begin{aligned} T_n^{k\nu} &= (-1)^{n-1} 2\frac{n!}{k}, \quad 1 \le \nu \le k-1 \ (2 \le k \le n), \\ T_n^{k\nu} &= (-1)^{n-1}\frac{n!}{k}, \quad \nu = k \ (1 \le k \le n), \\ T_n^{k\nu} &= 0, \quad k \le \nu \le n \ (1 \le k \le n-1). \end{aligned} \tag{19}$$

Then the inverse $n \times n$ matrix T_n^{-1} of the matrix (8) is

$$T_n^{-1} = \frac{1}{\det T_n}\begin{bmatrix} T_n^{11} & T_n^{21} & \dots & T_n^{n1} \\ T_n^{12} & T_n^{22} & \dots & T_n^{n2} \\ \dots & \dots & \dots & \dots \\ T_n^{1n} & T_n^{2n} & \dots & T_n^{nn} \end{bmatrix}. \tag{20}$$

Thus from (18)-(20) we obtain (17) which competes the proof of Theorem 2. ∎

Finally, we prove the following

Theorem 3. *For a fixed integer $n \ge 2$, the inverse matrix equation of (10) is*

$$P_n(t) = T_n^{-1} P_n'(t), \tag{21}$$

where $P_n(t), P_n'(t)$ and T_n^{-1} are determined by (9) and (17).

P r o o f. For a fixed integer $n \ge 2$, if we multiply (10) from the left by T_n^{-1}, then we shall obtain (21), having in mind (9) and (17).

This completes the proof of Theorem 3. ∎

Corollary 1. *For a fixed integer $n \ge 2$, the matrix equations (10)-(11) and (21) are corresponding equivalent to the system of equations*

$$\begin{aligned} \sigma_{nk}'(t) &= -k\sigma_{nk}(t) + 2k\Sigma_{\nu=k+1}^{n}(-1)^{\nu-k+1}\sigma_{n\nu}(t), \quad 1 \le k \le n-1, \\ \sigma_{nn}'(t) &= -n\sigma_{nn}(t) \end{aligned} \tag{22}$$

and

$$\begin{aligned} \sigma_{nk}(t) &= -\frac{1}{k}\sigma_{nk}'(t) - 2\Sigma_{\nu=k+1}^{n}\frac{1}{\nu}\sigma_{n\nu}'(t), \quad 1 \le k \le n-1, \\ \sigma_{nn}(t) &= -\frac{1}{n}\sigma_{nn}'(t) \end{aligned} \tag{23}$$

which are mutually inverse.

Corollary 2. *For a fixed integer $n \geq 1$, every de Branges weight function (4) satisfies the linear homogeneous differential equation of order $n-k+1$:*

$$\left(\frac{d}{dt}+k\right)\left(\frac{d}{dt}+k+1\right)\cdots\left(\frac{d}{dt}+n\right)\sigma_{nk}(t)$$

$$(24) \quad \equiv \sigma_{nk}^{n-k+1}(t)+\Sigma_{\nu=1}^{n-k+1}C_{n-k+1}^{n}[k,k+1,\ldots,n]\,\sigma_{nk}^{(n-k+1-\nu)}(t) \quad =0,$$

$$0 \leq t < +\infty, \quad k=1,2,\ldots,n, \quad \sigma_{nk}^{(0)}(t)\equiv\sigma_{nk}(t),$$

where $C_{n-k+1}^{\nu}[k,k=1,\ldots,n]$ denotes the sum of all products of the numbers $k,k+1,\ldots,n$ taken as the combinations without permutation of order ν.

Thus, on the basis of the equations (10)-(11) and (21) or their equivalent (22) and (23) as well as the equation (24), the de Branges weight functions (4) or system (6)-(7) can be examined with the help of the matrix theory.

The differential equation (24) was given, some years earlier, by Schmersau [12], directly deduced from the recursion (6) and discussed for arbitrary initial values.

See other methods in Henrici [13, pp. 592-611], Koepf and Schmersau [14,15] and Xie Ming-Qin [16].

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