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A New Class of Analytic Functions with Negative Coefficients

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A subclass $P(n, \lambda, \alpha, r)$ of analytic functions in the unit disk is introduced. The object of the present paper is to show some properties of functions belonging to the class $P(n, \lambda, \alpha, r)$.

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1. Introduction

Let $A(n)$ denote the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0;\ n \in \mathcal{N} = \{1, 2, ...\})$$

that are analytic in the unit disk $U = \{z : |z| < 1\}$. A function $f(z) \in A(n)$ is said to be in the class $P(n, \lambda, \alpha, r)$ if it satisfies

$$Re\{z\frac{(\lambda r z^{r-1} + 1 - \lambda)f'(z) + \lambda z^r f''(z)}{\lambda z^r f'(z) + (1-\lambda)f(z)}\} > \alpha, \quad (r = 1, 2, \ldots)$$

for some $\alpha \quad (0 \leq \alpha < 1), \quad \lambda \quad (0 \leq \lambda \leq 1)$ and for all $z \in U$.

We note that $P(n, \lambda, \alpha, 1) \equiv P(n, \lambda, \alpha)$, $P(n, 0, \alpha, 1) \equiv T_\alpha(n)$, $P(n, 1, \alpha, 1) \equiv C_\alpha(n)$, $P(1, 0, \alpha, 1) \equiv T^*(\alpha)$ and $P(1, 1, \alpha, 1) \equiv C(\alpha)$. The class $P(n, \lambda, \alpha)$ was studied by Altıntaş [1]. The classes $T_\alpha(n)$ and $C_\alpha(n)$ were studied by Srivastava, Owa and Chatterjea [3], and the classes $T^*(\alpha)$ and $C(\alpha)$ were studied by Silverman [2]. Therefore, our class $P(n, \lambda, \alpha, r)$ is the generalization of $P(n, \lambda, \alpha)$ by Altıntaş [1], and of $T_\alpha(n)$ and $C_\alpha(n)$ by Srivastava, Owa and Chatterjea [3] and of $T^*(\alpha)$ and $C(\alpha)$ by Silverman [2].

2. Some results for the class $P(n, \lambda, \alpha, r)$

Theorem 1. *A function $f(z) \in A(n)$ is in the class $P(n, \lambda, \alpha, r)$ if and only if*

$$\sum_{k=n+1}^{\infty} [(k-\alpha)(\lambda k - \lambda + 1) + \lambda k(r-1)]a_k \leq (1-\alpha) + \lambda(r-1). \tag{1}$$

Proof. Let $f(z) \in P(n, \lambda, \alpha, r)$. We can write

$$Re\{z\frac{(\lambda r z^{r-1} + 1 - \lambda)f'(z) + \lambda z^r f''(z)}{\lambda z^r f'(z) + (1-\lambda)f(z)}\} > \alpha.$$

Then,

$$\frac{[\lambda r z^{r-1} + (1-\lambda)][z - \sum_{k=n+1}^{\infty} k a_k z^k] + \lambda z^{r+1}[-\sum_{k=n+1}^{\infty} k(k-1)a_k z^{k-2}]}{\lambda z^r[1 - \sum_{k=n+1}^{\infty} k a_k z^{k-1}] + (1-\lambda)[z - \sum_{k=n+1}^{\infty} a_k z^k]}$$

$$= \frac{\lambda r z^r + (1-\lambda)z - \sum_{k=n+1}^{\infty}[\lambda r k + \lambda k^2 - \lambda k]a_k z^{k+r-1} - \sum_{k=n+1}^{\infty}(1-\lambda)k a_k z^k}{\lambda z^r + (1-\lambda)z - \sum_{k=n+1}^{\infty}[\lambda k a_k z^{k+r-1} - \sum_{k=n+1}^{\infty}(1-\lambda)a_k z^k}.$$

If we choose z real and let $z \to 1^-$, we get

$$\sum_{k=n+1}^{\infty} [(k-\alpha)(\lambda k - \lambda + 1) + \lambda k(r-1)]a_k \leq (1-\alpha) + \lambda(r-1).$$

Now, let the function $f(z) \in A(n)$ satisfy the condition (1). Then, we show that $f(z) \in P(n, \lambda, \alpha, r)$. First, we can write

$$|z\frac{(\lambda r z^{r-1} + 1 - \lambda)f'(z) + \lambda z^r f''(z)}{\lambda z^r f'(z) + (1-\lambda)f(z)} - 1|$$

$$\leq \Big\{|\sum_{k=n+1}^{\infty} [(r-1)\lambda k + \lambda k(k-1)]a_k z^{k+r-1}$$

$$+ \sum_{k=n+1}^{\infty} [(1-\lambda)k - (1-\lambda)]a_k z^k - (r-1)\lambda z^r|\Big\}/$$

$$\Big\{|\lambda z^r + (1-\lambda)z| - |\sum_{k=n+1}^{\infty} \lambda k a_k z^{k+r-1} + \sum_{k=n+1}^{\infty} (1-\lambda)a_k z^k|\Big\}.$$

If we take limit for $z \to 1^-$ in the inequality, we obtain

$$|z\frac{(\lambda r z^{r-1} + 1 - \lambda)f'(z) + \lambda z^r f''(z)}{\lambda z^r f'(z) + (1-\lambda)f(z)} - 1| \leq 1 - \alpha,$$

or

$$Re\{z\frac{(\lambda r z^{r-1}+1-\lambda)f'(z)+\lambda z^r f''(z)}{\lambda z^r f'(z)+(1-\lambda)f(z)}\}\geq\alpha.$$

Then $f(z)\in P(n,\lambda,\alpha,r)$.

Theorem 2. *Let the functions*

$$f(z)=z-\sum_{k=n+1}^{\infty}a_k z^k$$

and

$$g(z)=z-\sum_{k=n+1}^{\infty}b_k z^k$$

be in the class $P(n,\lambda,\alpha,r)$. Then for $0\leq\xi\leq 1$

$$h(z)=(1-\xi)f(z)+\xi g(z)=z-\sum_{k=n+1}^{\infty}c_k z^k \quad (c_k\geq 0)$$

is in the class $P(n,\lambda,\alpha,r)$.

Proof. Suppose that $f(z),g(z)\in P(n,\lambda,\alpha,r)$. Then we have from Theorem 1,

$$\sum_{k=n+1}^{\infty}[(k-\alpha)(\lambda k-\lambda+1)+\lambda k(r-1)]a_k\leq(1-\alpha)+\lambda(r-1)$$

and

$$\sum_{k=n+1}^{\infty}[(k-\alpha)(\lambda k-\lambda+1)+\lambda k(r-1)]b_k\leq(1-\alpha)+\lambda(r-1).$$

We can see that

$$\begin{aligned}&\sum_{k=n+1}^{\infty}[(k-\alpha)(\lambda k-\lambda+1)+\lambda k(r-1)]c_k\\=&\sum_{k=n+1}^{\infty}[(k-\alpha)(\lambda k-\lambda+1)+\lambda k(r-1)][(1-\xi)a_k+\xi b_k]\\=&(1-\xi)\sum_{k=n+1}^{\infty}[(k-\alpha)(\lambda k-\lambda+1)+\lambda k(r-1)]a_k\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{k=n+1}^{\infty} [(k-\alpha)(\lambda k - \lambda + 1) + \lambda k(r-1)]b_k \\
&\leq (1-\xi)[(1-\alpha) + \lambda(r-1)] + \xi[(1-\alpha) + \lambda(r-1)] \\
&= (1-\alpha) + \lambda(r-1)
\end{aligned}
$$

which completes the proof of Theorem 2.

Theorem 3. *If $f(z) \in P(n, \lambda, \alpha, r)$, then $f(z) \in P(n, \lambda, \beta)$, where*

$$\beta = 1 - \frac{(\lambda n + 1)[(1-\alpha) + \lambda(r-1)]}{(1-r)\lambda^2 + (n+r-1)\lambda + 1}.$$

P r o o f. If $f(z) \in P(n, \lambda, \beta)$, then

$$\sum_{k=n+1}^{\infty} [(k-\beta)(\lambda k - \lambda + 1)a_k \leq 1 - \beta. \tag{2}$$

If $f(z) \in P(n, \lambda, \alpha, r)$, then

$$\sum_{k=n+1}^{\infty} [(k-\alpha)(\lambda k - \lambda + 1) + \lambda k(r-1)]a_k \leq (1-\alpha) + \lambda(r-1). \tag{3}$$

It is sufficient to show that (3) implies (2), that is, that

$$\frac{(k-\beta)(\lambda k - \lambda + 1)}{1-\beta} \leq \frac{(k-\alpha)(\lambda k - \lambda + 1) + \lambda k(r-1)}{(1-\alpha) + \lambda(r-1)} \quad (k \geq n+1) \tag{4}$$

Since (4) is equivalent to

$$
\begin{aligned}
\beta &\leq 1 - \frac{(\lambda k - \lambda + 1)[(1-\alpha) + \lambda(r-1)]}{(\lambda k - \lambda + 1) + \lambda(1-\lambda)(r-1)} \quad (k \geq n+1) \\
&\equiv \phi(k)
\end{aligned}
\tag{5}
$$

and $\phi(k) \geq \phi(n+1)$ $(k \geq n+1)$, (4) holds true for any $k \geq n+1$, $0 \leq \alpha < 1$ and $0 \leq \lambda \leq 1$. Then we obtain

$$\beta = 1 - \frac{(\lambda n + 1)[(1-\alpha) + \lambda(r-1)]}{(1-r)\lambda^2 + (n+r-1)\lambda + 1}$$

for $k = n + 1$.

Theorem 4. *If* $f(z) \in P(n, \lambda, \alpha, r)$, *then*

$$
\begin{aligned}
&|z| - \frac{(1-\alpha)+\lambda(r-1)}{(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)}|z|^{n+1} \\
(6) \qquad &\leq |f(z)| \leq |z| + \frac{(1-\alpha)+\lambda(r-1)}{(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)}|z|^{n+1}
\end{aligned}
$$

and

$$
\begin{aligned}
&1 - \frac{(n+1)(1-\alpha)+\lambda(r-1)}{(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)}|z|^{n} \\
(7) \qquad &\leq |f'(z)| \leq 1 + \frac{(n+1)(1-\alpha)+\lambda(r-1)}{(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)}|z|^{n}
\end{aligned}
$$

for $z \in U$.

Proof. We note that

$$
\begin{aligned}
&[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)] \sum_{k=n+1}^{\infty} a_k \\
&\leq \sum_{k=n+1}^{\infty} [(k-\alpha)(\lambda k-\lambda+1)+\lambda k(r-1)]a_k \\
(8) \qquad &\leq (1-\alpha)+\lambda(r-1)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)]}{n+1} \sum_{k=n+1}^{\infty} k a_k \\
&\leq \sum_{k=n+1}^{\infty} [(k-\alpha)(\lambda k-\lambda+1)+\lambda k(r-1)]a_k \\
(9) \qquad &\leq (1-\alpha)+\lambda(r-1).
\end{aligned}
$$

Therefore, (6) follows (8) and (7) follows (9). Equalities in (6) and (7) are attained for the functions $f(z)$ given by

$$
f(z) = z - \frac{(1-\alpha)+\lambda(r-1)}{(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)} z^{n+1}.
$$

Definition: Modified Hadamard product. For functions

$$
f_j(z) = z - \sum_{k=n+1}^{\infty} a_{k,j} z^k \quad (j = 1, 2)
$$

in the class $A(n)$, we define the modified product $f_1 * f_2(z)$ of $f_1(z)$ and $f_2(z)$ by

$$f_1 * f_2(z) = z - \sum_{k=n+1}^{\infty} a_{k,1}a_{k,2}z^k.$$

Then, we prove

Theorem 5. *If $f_j(z) \in P(n, \lambda, \alpha, r), j = 1, 2$, then $f_1 * f_2(z) \in P(n, \lambda, \alpha, r)$, where*

$$\beta = \frac{(\lambda r - \lambda + 1)[(n+1-\alpha)(\lambda n + 1) + \lambda(n+1)(r-1)]^2}{[(n+1-\alpha)(\lambda n+1) + \lambda(n+1)(r-1)]^2 - (\lambda n + 1)[(1-\alpha) + \lambda(r-1)]^2}$$

$$- \frac{[(n+1)(\lambda n + 1) + \lambda(n+1)(r-1)][(1-\alpha) + \lambda(r-1)]^2}{[(n+1-\alpha)(\lambda n+1) + \lambda(n+1)(r-1)]^2 - (\lambda n + 1)[(1-\alpha) + \lambda(r-1)]^2}. \tag{10}$$

P r o o f. Since, for $f_j(z) \in P(n, \lambda, \alpha, r)$

$$\sum_{k=n+1}^{\infty} \frac{(k-\alpha)(\lambda k - \lambda + 1) + \lambda k(r-1)}{(1-\alpha) + \lambda(r-1)} a_{k,j} \leq 1 \quad (j = 1, 2) \tag{11}$$

the Cauchy-Schwarz inequality leads to

$$\sum_{k=n+1}^{\infty} \frac{(k-\alpha)(\lambda k - \lambda + 1) + \lambda k(r-1)}{(1-\alpha) + \lambda(r-1)} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{12}$$

Note that we have to find the largest β such that

$$\sum_{k=n+1}^{\infty} \frac{(k-\beta)(\lambda k - \lambda + 1) + \lambda k(r-1)}{(1-\beta) + \lambda(r-1)} a_{k,1}a_{k,2} \leq 1 \tag{13}$$

Therefore, in view of (12) and (13), if

$$\begin{aligned} &\frac{(k-\beta)(\lambda k - \lambda + 1) + \lambda k(r-1)}{(1-\beta) + \lambda(r-1)} \sqrt{a_{k,1}a_{k,2}} \\ &\leq \frac{(k-\alpha)(\lambda k - \lambda + 1) + \lambda k(r-1)}{(1-\alpha) + \lambda(r-1)}, \end{aligned} \tag{14}$$

then (13) is satisfied. We have, from (12)

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(1-\alpha) + \lambda(r-1)}{(k-\alpha)(\lambda k - \lambda + 1) + \lambda k(r-1)} \quad (k \geq n+1). \tag{15}$$

Thus, if

$$\frac{(k-\beta)(\lambda k-\lambda+1)+\lambda k(r-1)}{(1-\beta)+\lambda(r-1)}\left[\frac{(1-\alpha)+\lambda(r-1)}{(k-\alpha)(\lambda k-\lambda+1)+\lambda k(r-1)}\right]$$
$$\leq \frac{(k-\alpha)(\lambda k-\lambda+1)+\lambda k(r-1)}{(1-\alpha)+\lambda(r-1)}, \quad (k\geq n+1) \tag{16}$$

or, if

$$\beta \leq \frac{(\lambda r-\lambda+1)\,[(k-\alpha)(\lambda k-\lambda+1)+\lambda k(r-1)]^2}{[(k-\alpha)(\lambda k-\lambda+1)+\lambda k(r-1)]^2-(\lambda k-\lambda+1)[(1-\alpha)+\lambda(r-1)]^2}$$
$$-\frac{[k(\lambda k-\lambda+1)+\lambda k(r-1)][(1-\alpha)+\lambda(r-1)]^2}{[(k-\alpha)(\lambda k-\lambda+1)+\lambda k(r-1)]^2-(\lambda k-\lambda+1)[(1-\alpha)+\lambda(r-1)]^2}$$

then (13) is satisfied. Letting

$$\phi(k)=\frac{(\lambda r-\lambda+1)[(k-\alpha)(\lambda k-\lambda+1)+\lambda k(r-1)]^2}{[(k-\alpha)(\lambda k-\lambda+1)+\lambda k(r-1)]^2-(\lambda k-\lambda+1)[(1-\alpha)+\lambda(r-1)]^2}$$
$$-\frac{[k(\lambda k-\lambda+1)+\lambda k(r-1)][(1-\alpha)+\lambda(r-1)]^2}{[(k-\alpha)(\lambda k-\lambda+1)+\lambda k(r-1)]^2-(\lambda k-\lambda+1)[(1-\alpha)+\lambda(r-1)]^2},$$

we see that $\phi(k)$ is increasing on k. This implies that

$$\beta\leq\phi(n+1)$$
$$=\frac{(\lambda r-\lambda+1)[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)]^2}{[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)]^2-(\lambda n+1)[(1-\alpha)+\lambda(r-1)]^2}$$
$$-\frac{[(n+1)(\lambda n+1)+\lambda(n+1)(r-1)][(1-\alpha)+\lambda(r-1)]^2}{[(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)]^2-(\lambda n+1)[(1-\alpha)+\lambda(r-1)]^2},$$

which completes the proof of our theorem.

Finally, by taking the functions

$$f_j(z)=z-\frac{(1-\alpha)+\lambda(r-1)}{(n+1-\alpha)(\lambda n+1)+\lambda(n+1)(r-1)}z^{n+1} \quad (j=1,2),$$

we can see that the result is sharp.

Remark: If we take $r=1$ in Theorem 5, then we have the result by Altıntaş [1, Theorem 4]. If we take $r=1$ and $\lambda=0$ in Theorem 5, then we have the result by Srivastava, Owa and Chatterjea [3, Theorem 4].

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