Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 13, 1999, Fasc. 1-2

A Few Remarks On Convex Mappings [1]

Yusuf Avcı * *and Eligiusz Zlotkiewicz* **

Presented by V. Kiryakova

In this note we are concerned with various transformations of convex univalent mappings and with the crossratio problem.

AMS Subj. Classification: 30C45

Key Words: convex univalent mappings

0. Introduction

We denote by S_C the class of all functions $f(z) = z + a_2z^2 + \cdots$ that are analytic in the unit disc $D = \{z \in \mathbf{C} : |z| < 1\}$ and such that $f(D)$ is a convex domain. The class S_C consists of univalent mappings.

Some years ago T. Suffridge [9] established the following remarkable theorem.

Theorem. *If $f \in S_C$ and a, $0 \le |a| < 1$ is a fixed number, then the function*

$$g(z) = z\frac{f(z) - f(a)}{z - a}$$

is starlike of order 1/2.

We start our considerations with a simple proof of this result.

Let $w(z) = \dfrac{a + z}{1 + \bar{a}z}$, then if $f \in S_C$ so does the function

$$h(z) = \frac{f(\omega) - f(a)}{(1 - |a|^2)f'(a)} \ .$$

This yields the relation

$$g(z) = \frac{z}{z - a}h\left(\frac{z - a}{1 - \bar{a}z}\right)(1 - |a|^2)f'(a) \ . \tag{1}$$

[1]This research is supported by Tubitak-Doprog Research Grant Programme

We may assume that h is analytic in a disc $|z| < 1 + \varepsilon$, $\varepsilon > 0$. Then if $|z| = 1$, one obtains

$$Re\frac{zg'(z)}{g(z)} = Re\left\{\frac{\eta h'(\eta)}{h(\eta)}\right\}\frac{1-|a|^2}{|z-a|^2} - Re\frac{a}{z-a} \quad , \quad \eta = \frac{z-a}{1-\bar{a}z}$$

which, in view of the inequality $Re\left\{\frac{\eta h'(\eta)}{h(\eta)}\right\} > \frac{1}{2}$ gives

$$Re\frac{zg'(z)}{g(z)} > \frac{1}{2} \ .$$

This proves the statement. ∎

1. Transformation of convex mappings

It follows from (1) that if $f \in S_C$, then the function

$$f(z) - \frac{a}{z}f(z) = H(z) \tag{2}$$

is univalent and maps D onto a starlike domain. There is an interesting and non-elementary result of Ruscheweyh [6] which says that $H(z)$ is starlike for some non-convex functions f. We shall apply (2) to establish a Hummel type inequality for convex mappings.

Theorem 1. *If $f(z) = z + a_2z^2 + \cdots$ belongs to S_C, then*

$$|a_n - 1| \le (n-1)|a_2 - 1|$$

for all n.

Proof. The function $H(z)$ in (2) is starlike of order $1/2$, so if normalized, it has all coefficients bounded by 1. We find

$$|a_n - a_{n+1}| \le |1 - a_2| \quad , \quad n = 1, 2, \ldots \quad , \quad \text{and} \ (a \to 1) \ .$$

Now,

$$a_n - 1 = (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_2 - 1)$$

and the result follows. ∎

We note that, Hummel [3] established the inequality

$$|b_n - n| \le \frac{n(n^2-1)}{6}|b_2 - 2|,$$

which in view of the relation $b_n = na_n$ yields for convex maps, the result

$$|a_n - 1| \leq \frac{n+1}{3}(n-1)|a_2 - 1| \ .$$

Thus, our result is better.

It is known [5] that if $f \in S_C$, the function

$$g(z) = \frac{2}{z}\int_0^z f(u)du$$

belongs to S_C. The factor 2 is for normalization. Since $g(z) = 2\int_0^1 f(rz)dr$, we conclude that $g(D) \subset f(D)$.

Similarly , if $0 < \alpha < 2\pi$, one may consider the integral

$$h(z) = \int_0^\alpha f(ze^{i\theta})d\theta \ .$$

If we set $\ell(z) = \log\frac{1-z}{1-e^{i\theta}z}$, then $\ell(z)$ is convex univalent and $h(z)$ is the convolution

$$h(z) = f(z) * \ell(z) \ .$$

So by the famous result of Ruscheweyh and Sheil-Small [7], h is convex. Also $h(D) \subset f(D)$.

The above observations combined together give the following theorem.

Theorem 2. *If $f \in S_C$ and $0 < \alpha < 2\pi$, then*

$$H(z) = \int_0^\alpha \int_0^1 f(zre^{i\theta})d\theta dr$$

is convex and univalent and $H(D) \subset f(D)$.

Suppose now $G(z)$ is given by the formula

$$G(z) = \int_0^z [f'(w(x))]^\alpha dx \ , \ w(z) = b\frac{z+a}{1+\bar{a}z} , \tag{3}$$

where $0 < \alpha$, a and b ($|a| < 1$, $|b| < 1$) are given and $f \in S_C$. It is known that if $b = 1$ and $\alpha = 1$, then $G(z)$ may not be convex for any value of a and it is univalent if $|a| < 1/\sqrt{2}$.

We prove the following theorem which shows the influence of new parameters.

Theorem 3. *The function $G(z)$ in (3) is convex provided the condition*

$$|a|(1+|b|+2\alpha|b|) \leq 1+|b|-2\alpha|b|$$

is fullfilled. The result is sharp.

Proof. There is no loss of generality, if we set $|z|=1$. Thus we have the relation

$$Re\left\{1+z\frac{G''(z)}{G'(z)}\right\} = \alpha Re\left\{1+\omega\frac{f''(\omega)}{f'(\omega)}\right\}\frac{1-|a|^2}{|a+z|^2} - \alpha\frac{1-|a|^2}{|a+z|^2}+1 \ .$$

Now, if $|\omega| \leq |b| < 1$, then the right hand side is bounded below by

$$1-\alpha\frac{2|b|}{1+|b|}\cdot\frac{1+|a|}{1-|a|} \ .$$

If $|b|=1$, then the right hand side is bounded from below by

$$1-\alpha\frac{1+|a|}{1-|a|} \ .$$

Both estimates are sharp. This completes the proof. ■

Let us notice that if $1+|b|-2\alpha|b|<0$, then $G(z)$ may not be convex for any value of a. If $|b|=1$, then G is convex provided $|a|<\frac{1-\alpha}{1+\alpha}$. We now investigate the problem of univalence of G under the condition $b=1$. By changing variables we may write

$$G\left(\frac{z-a}{1-\bar{a}z}\right) = \int_a^z [f'(u)]^\alpha \frac{(1-|a|)^2 du}{(1-\bar{a}u)^2} \ .$$

So, the function G is univalent, provided

$$H(z) = \int_a^z (f'(u))^\alpha \frac{du}{(1-\bar{a}u)^2}$$

is univalent. Since

$$H'(z) = (f'(z))^\alpha \frac{1}{(1-\bar{a}z)^2} \ ,$$

and the function $h(z)=\int_0^z (f'(u))^\alpha du$ is convex univalent for $0 \leq \alpha \leq 1$ we have

$$\left|\arg\frac{H'(z)}{h'(z)}\right| = 2|\arg(1-\bar{a}z)| \leq 2\arcsin|a| \leq \frac{\pi}{2} \ ,$$

provided $|a| < 1/\sqrt{2}$. On the other hand, $|\arg f'(z)| \le 2\arcsin|z| < \pi$. So we also have

$$\left| \arg \frac{H'(z)}{\left(\dfrac{z}{1-\bar{a}z}\right)'} \right| = \alpha |\arg f'(z)| \le \pi\alpha < \frac{\pi}{2} \ ,$$

if $\alpha \in [0, 1/2]$. Both results are best possible. Thus, we conclude that $G(z)$ with $b = 1$ is univalent and close to convex for all values of a provided $0 \le \alpha \le 1/2$. $G(z)$ is univalent and close to convex if $\alpha \in (1/2, 1]$ and $|a| < 1/\sqrt{2}$. What happens if $\alpha > 1$ remains still an *open question*.

2. The crossratio problem

It is known that the crossratio is invariant under a homography. If $R = R(\omega_1, \omega_2, \omega_3, \omega_4)$, $r = r(z_1, z_2, z_3, z_4)$ and $\omega_k = f(z_k)$, $k = 1, 2, 3, 4$ then the quotient $\rho = R/r$ gives a "measure of deviation " of f from a homography. We are interested in finding the best bounds of $|\rho|$ over the whole class of convex mappings as in [1]. Once the best bounds of the crossratio are known, many results concerning convex functions will follow by passing to limits. This appears to be a hard question. We are able, so far, to get some explicit bounds but not the sharp ones. We obtain the following theorem.

Theorem 4. *If $f \in S_C$ then the following inequalities hold for $|\rho|$:*

i) $$\frac{(1-|z_3|)^2(1-|z_4|)^2}{16} \le |\rho| \le \frac{16}{(1-|z_3|)^2(1-|z_4|)^2} \ ,$$

ii) $$\frac{(1-|z_1|^2)^2(1-|z_2|^2)^2}{(|1-z_1\bar{z}_2| + |z_1 - z_2|)^4} \le |\rho| \le \frac{(1-|z_1|^2)^2(1-|z_2|^2)^2}{(|1-z_1\bar{z}_2| - |z_1 - z_2|)^4} \ ,$$

iii) $$\frac{|1-\bar{z}_4 z_3| - |z_3 - z_4|}{|1-\bar{z}_4 z_3| + |z_3 - z_4|} \le |\rho| \le \frac{|1-\bar{z}_4 z_3| + |z_3 - z_4|}{|1-\bar{z}_4 z_3| - |z_3 - z_4|} \ .$$

Proof. **a)** We set $\omega_k = f(z_k)$, $k = 1, 2, 3, 4$ for $f \in S_C$ and define

$$R(\omega_1, \omega_2, \omega_3, \omega_4) = \frac{(\omega_1 - \omega_3)(\omega_2 - \omega_4)}{(\omega_2 - \omega_3)(\omega_1 - \omega_4)} \ .$$

It is known [2, p.129] that if $f \in S_C$, then

$$\left| \frac{1}{\sqrt{f'(z)}} - 1 \right| \le |z| \ \cdot$$

Denote by $\varphi(\omega)$ the inverse of $f(z)$. In view of convexity of $f(D)$ one has

$$|z_k - z_l| = \left|\int_{\omega_l}^{\omega_k} \varphi'(\omega)\, d\omega\right| \le 4|\omega_k - \omega_l| \,.$$

On the other hand,

$$|\omega_k - \omega_l| = \left|\int_{z_l}^{z_k} f'(z)\, dz\right| \le |z_k - z_l| \frac{1}{(1-|z_l|)^2}\,,$$

provided $|z_k| \le |z_l|$. To fix our attention, let us assume that $|z_1| \le |z_2| \le |z_3| \le |z_4|$. All this taken together gives the inequality i),

$$\frac{(1-|z_3|)^2(1-|z_4|)^2}{16} \le |\rho| \le \frac{16}{(1-|z_3|)^2(1-|z_4|)^2}\,.$$

b) We may obtain another inequality by using the Suffridge theorem. If we set

$$g(z) = z\frac{f(z)-f(z_3)}{z-z_3}\ ; \ h(z) = z\frac{f(z)-f(z_4)}{z-z_4}\,,$$

then

$$\rho = \frac{g(z_1)}{g(z_2)} \cdot \frac{h(z_2)}{h(z_1)}\,.$$

Should g and h vary independently in the class of functions starlike of order 1/2, we would get not only best bounds on ρ but also the variability region of this quantity. Since g and h related, we only get bounds that are not sharp. It is known [4] that for H being starlike of order 1/2,

$$\left(\frac{1-|z_2|^2}{|1-z_1\bar{z}_2| + |z_1-z_2|}\right)^2 \le \left|\frac{z_2 H(z_1)}{z_1 H(z_2}\right| \le \left(\frac{1-|z_2|^2}{|1-z_1\bar{z}_2| - |z_1-z_2|}\right)^2$$

and these bounds are sharp.

Thus we get the inequalities ii),

$$\frac{(1-|z_1|^2)^2(1-|z_2|^2)^2}{(|1-z_1\bar{z}_2| + |z_1-z_2|)^4} \le |\rho| \le \frac{(1-|z_1|^2)^2(1-|z_2|^2)^2}{(|1-z_1\bar{z}_2| - |z_1-z_2|)^4}\,.$$

c) Finally, we know that $f \in S_C$ if and only if the function $g(z)$ defined by

$$g(z) = \frac{zf'(z)}{f(z)-f(\zeta)} - \frac{\zeta}{z-\zeta}$$

has the real part grater than 1/2 for each $\zeta \in D$. Therefore, there exists a probability measure μ such that

$$\frac{zf'}{f(z)-f(\zeta)} - \frac{\zeta}{z-\zeta} = \int_0^{2\pi} \frac{d\mu(u,\zeta)}{1-uz} .$$

Substructing 1, dividing by z and integrating with respect to z both sides it follows that

$$\log \frac{f(z)-f(\zeta)}{z-\zeta} \cdot \frac{\zeta}{f(\zeta)} = - \int_{|u|=1} \log(1-uz)\, d\mu(u,\zeta) .$$

Note that if $\zeta = z_1$, $\eta = z_2$ and $z = z_3$, then

$$\begin{aligned} \log \frac{R(\omega_1,\omega_2,\omega_3,0)}{r(z_1,z_2,z_3,0)} &= \log \frac{\zeta}{f(\zeta)} \frac{f(z)-f(\zeta)}{z-\zeta} - \log \frac{\eta}{f(\eta)} \frac{f(z)-f(\eta)}{z-\eta} \\ &= \int_{|u|=1} \log(1-uz)[d\mu(u,\eta) - d\mu(u,\zeta)] . \end{aligned}$$

By a result of Shober [7, p.23] we know that

$$\exp \int_{|u|=1} \log(1-uz) d(\mu_1 - \mu_2) < \frac{1+cz}{1-z}$$

with $|c| = 1$. It follows that

$$\frac{R(\omega_1,\omega_2,\omega_3,0)}{r(z_1,z_2,z_3,0)} = \frac{1+c\,\omega(z)}{1-\omega(z)}$$

with $|\omega(z)| \le |z|$. Ultimately

$$\frac{1-|z_3|}{1+|z_3|} \le \frac{R(\omega_1,\omega_2,\omega_3,0)}{r(z_1,z_2,z_3,0)} \le \frac{1+|z_3|}{1-|z_3|} .$$

Also note that given z_i $(i = 1,2,3,4)$, we define $z_k^* = (z_k - z_4)/(1-\bar{z}_4 z_k)$ and

$$\omega_k^* = \frac{f(z_k)-f(z_4)}{f'(z_4)(1-|z_4|^2)} ,$$

then it follows that

$$\frac{R(\omega_1^*,\omega_2^*,\omega_3^*,0)}{r(z_1^*,z_2^*,z_3^*,0)} = \frac{R(\omega_1,\omega_2,\omega_3,\omega_4)}{r(z_1,z_2,z_3,z_4)} = \rho .$$

Combining this with the preceeding inequalities, we arrive at the inequalities iii),

$$\frac{1-\left|\dfrac{z_3-z_4}{1-\bar{z}_4 z_3}\right|}{1+\left|\dfrac{z_3-z_4}{1-\bar{z}_4 z_3}\right|} \le |\rho| \le \frac{1+\left|\dfrac{z_3-z_4}{1-\bar{z}_4 z_3}\right|}{1-\left|\dfrac{z_3-z_4}{1-\bar{z}_4 z_3}\right|}.$$

This completes the proof of the theorem. ■

References

[1] Y. A v c ı, E. Z l o t k i e w i c z. An extremal problem for univalent functions. *Ist. Univ. Fen Fak. Mat. Derg.* **50**, 1991, 159-164.

[2] A. W. G o o d m a n. *Univalent Functions*, Vol. **I** and Vol. **II**, Mariner Publ. Co. Inc., Tampa-Florida, 1983.

[3] J. A. H u m m e l. The coefficients of starlike functions. *Proc. Amer. Math. Soc.*, **22**, 1969, 311-315.

[4] Z. L e w a n d o w s k i. Sur certaines classes de fonctions univalentes dans le cercle unité. *Ann. Univ. M. Curie-Sklodowska*, **13**, 1959, 115-126.

[5] R. J. L i b e r a. Some classes of regular univalent functions. *Proc. Amer. Math. Soc.*, **16**, 1965, 755-758.

[6] S. R u s c h e w e y h. Linear operators between classes of prestarlike functions. *Comment. Math. Helvetici*, **52**, 1977, 497-509.

[7] S. R u s c h e w e y h, T. S h e i l - S m a l l. Hadamard product of schlicht functions and the Polya-Schoenberg conjecture. *Comment. Math. Helv.*, **48**, 1973, 119-135.

[8] G. S c h o b e r. *Univalent Functions - Selected Topics. = Lecture Notes in Maths.*, No. **478**, Springer Verlag, New York, 1975.

[9] T. J. S u f f r i d g e. Convolutions of convex functions. *J. Math. Mech.*, **15**, 1966, 795-804.

* *Dept. of Mathematics*
Istanbul University
34459 Vezneciler, Istanbul, TURKEY

** *Institute of Mathematics*
Marie Curie - Sklodowska University
20-031 Lublin, POLAND

Received: 02.12.1996