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Subgroups of the Group of the General Similitudes in the Galilean Plane [1]

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In this paper the subgroups of the group of the general similitudes in the Galilean plane are determined.

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1. Introduction

In the affine version the Galilean plane Γ_2 is an affine plane with a special direction which may be taken coincident with the y-axis of the basic affine coordinate system Oxy [5,6,8,9]. The affine transformations leaving invariant the special direction Oy can be written in the form

$$\begin{aligned} \bar{x} &= a_1 + a_2 x, \\ \bar{y} &= a_3 + a_4 x + a_5 y, \end{aligned} \tag{1.1}$$

where $a_2 a_5 \neq 0$ and $a_1, \ldots, a_5$ are real parameters. The transformations (1.1) map a line segment and an angle of Γ_2 into a proportional ones with the coefficients of proportionality $|a_2|$ and $|a_2^{-1} a_5|$, respectively. Thus they form the group H_5 of the general similitudes of Γ_2. The infinitesimal operators of H_5 are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = x\frac{\partial}{\partial y}, \quad X_5 = y\frac{\partial}{\partial y}$$

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and satisfy the system

$$
(1.2)\qquad
\begin{aligned}
&[X_1,X_2]=X_1,\quad [X_1,X_3]=0,\quad [X_1,X_4]=X_3,\quad [X_1,X_5]=0,\\
&[X_2,X_3]=0,\quad [X_2,X_4]=X_4,\quad [X_2,X_5]=0,\\
&[X_3,X_4]=0,\quad [X_3,X_5]=X_3,\\
&[X_4,X_5]=X_4,
\end{aligned}
$$

where $[\cdot,\cdot]$ is the bracket of Poisson.

For the necessities of some applications the natural problem which arises is to classify the subgroups of H_5. That is the aim of this paper and we give the subgroups of H_5 which are different up to a Galilean general similitude. The results have been announced without ptoofs in [1], which are given here.

2. Four-parametric subgroups of H_5

A four-parametric subgroup of H_5 can be defined by four linearly independent infinitesimal operators [3,4]

$$
(2.1)\qquad Y_h=\sum_{k=1}^{5} a_{hk}X_k,\qquad h=1,\dots,4,
$$

satisfying the conditions

$$
(2.2)\qquad [Y_i,Y_j]=\sum_{k=1}^{4} c_{ij}^{k}Y_k,\qquad i\neq j;\, i,j=1,\dots,4,
$$

where a_{hk} and c_{ij}^k are real numbers. We shall now consider the possible cases.

1. $a_{12}=a_{13}=a_{14}=a_{15}=0$. Since $Y_1\neq 0$ it follows that $a_{11}\neq 0$ and we can assume without loss of generality that $a_{11}=1$, $a_{21}=a_{31}=a_{41}=0$.

1.1. $a_{23}=a_{24}=a_{25}=0$. Now $a_{22}\neq 0$ and using similar arguments as above, we can choose $a_{22}=1$, $a_{32}=a_{42}=0$.

1.1.1. $a_{34}=a_{35}=0$. Then $a_{33}\neq 0$ and we can suppose that $a_{33}=1$, $a_{43}=0$. Thus we obtain the operators

$$
(2.3)\qquad Y_1=X_1,\ Y_2=X_2,\ Y_3=X_3,\ Y_4=a_{44}X_4+a_{45}X_5
$$

and applying (1.2) we find

$$(2.4)\qquad \begin{aligned} &[Y_1, Y_2] = X_1, \quad [Y_1, Y_3] = 0, \quad [Y_1, Y_4] = a_{44}X_3, \\ &\qquad\qquad\qquad\quad [Y_2, Y_3] = 0, \quad [Y_2, Y_4] = a_{44}X_4, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad [Y_3, Y_4] = a_{45}X_3. \end{aligned}$$

The operators (2.3) define a group iff $|a_{44}| + |a_{45}| \neq 0$ and the systems (2.2) and (2.4) are equivalent. It is possible if and only if $a_{44} \neq 0, a_{45} = 0$ or $a_{44} = 0, a_{45} \neq 0$.

1.1.1.1. $a_{44} \neq 0$, $a_{45} = 0$. Putting $a_{44} = 1$ we get the subgroup

$$H_4^1 = \{X_1, X_2, X_3, X_4\}.$$

1.1.1.2. $a_{44} = 0, a_{45} \neq 0$. Now we choose $a_{45} = 1$ and we find the subgroup

$$H_4^2 = \{X_1, X_2, X_3, X_5\}.$$

1.1.2. $a_{34} \neq 0$, $a_{35} = 0$. We can take $a_{34} = 1$, $a_{44} = 0$. The operators (2.1) have the form

$$(2.5)\qquad Y_1 = X_1,\ Y_2 = X_2,\ Y_3 = a_{33}X_3 + X_4,\ Y_4 = a_{43}X_3 + a_{45}X_5$$

and according to (2.2) they define a group iff $a_{45} = 0$. Then $a_{43} \neq 0$ and we can take $a_{43} = 1, a_{33} = 0$. Replacing in (2.5) we obtain $Y_1 = X_1, Y_2 = X_2, Y_3 = X_4, Y_4 = X_3$ and therefore we get again the subgroup H_4^1.

1.1.3. $a_{35} \neq 0$. We put $a_{35} = 1, a_{45} = 0$ and consequently

$$(2.6)\quad Y_1 = X_1,\ Y_2 = X_2,\ Y_3 = a_{33}X_3 + a_{34}X_4 + X_5,\ Y_4 = a_{43}X_3 + a_{44}X_4.$$

Since $|a_{43}| + |a_{44}| \neq 0$, we distinguish the following cases:

1.1.3.1. $a_{43} \neq 0, a_{44} = 0$. Choosing $a_{43} = 1, a_{33} = 0$ we obtain that the operators define a group iff $a_{34} = 0$ and it is H_4^2.

1.1.3.2. $a_{43} = 0, a_{44} \neq 0$. It is easy to verify that in this case the corresponding operators do not define a group.

1.1.3.3. $a_{43} \neq 0, a_{44} \neq 0$. Now we have

$$\begin{aligned} &[Y_1, Y_2] = X_1, \quad [Y_1, Y_3] = a_{34}X_3, \quad [Y_1, Y_4] = a_{44}X_3, \\ &\qquad\qquad\qquad [Y_2, Y_3] = a_{34}X_4, \quad [Y_2, Y_4] = a_{44}X_4, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad [Y_3, Y_4] = -a_{43}X_3 - a_{44}X_4 \end{aligned}$$

and therefore the operators (2.6) do not define a group.

1.2. $a_{23} \neq 0, a_{24} = a_{25} = 0$. We suppose $a_{23} = 1, a_{33} = a_{43} = 0$.

1.2.1. $a_{34} = a_{35} = 0$. Then $a_{32} \neq 0$ and taking $a_{32} = 1, a_{22} = a_{42} = 0$ we have 1.1.1.

1.2.2. $a_{34} \neq 0, a_{35} = 0$. We assume $a_{34} = 1, a_{44} = 0$.

1.2.2.1. $a_{42} \neq 0, a_{45} = 0$. Putting $a_{42} = 1, a_{22} = a_{32} = 0$ we get H_4^1.

1.2.2.2. $a_{42} = 0, a_{45} \neq 0$. Now we choose $a_{45} = 1$. The operators $Y_1 = X_1, Y_2 = a_{22}X_2 + X_3, Y_3 = a_{32}X_2 + X_4, Y_4 = X_5$ define a group iff $a_{22} = a_{32} = 0$. We have the subgroup

$$H_4^{3'} = \{X_1, X_3, X_4, X_5\}. \tag{2.7}$$

1.2.2.3. $a_{42} \neq 0, a_{45} \neq 0$. The operators

$$Y_1 = X_1,\ Y_2 = a_{22}X_2 + X_3,\ Y_3 = a_{32}X_2 + X_4,\ Y_4 = a_{42}X_2 + a_{45}X_5 \tag{2.8}$$

define a group iff $a_{22} = 0,\ a_{32}(a_{42} - a_{45}) = 0$. There are the following cases:

1.2.2.3.1. $a_{32} = 0$. From (2.8) we get $Y_1 = X_1, Y_2 = X_3, Y_3 = X_4, Y_4 = a_{42}X_2 + a_{45}X_5$ and replacing $\dfrac{a_{42}}{a_{45}} = \alpha$ we find the subgroup

$$H_4^{3''} = \{X_1, X_3, X_4, \alpha X_2 + X_5 \mid \alpha \neq 0; \alpha \in \mathbf{R}\}. \tag{2.9}$$

Unifying (2.7) and (2.9) we have the subgroup

$$H_4^3 = \{X_1, X_3, X_4, \alpha X_2 + X_5 \mid \alpha \in \mathbf{R}\}.$$

1.2.2.3.2. $a_{42} = a_{45}$. We suppose $a_{42} = a_{45} = 1$ and therefore $Y_1 = X_1, Y_2 = X_3, Y_3 = a_{32}X_2 + X_4, Y_4 = X_2 + X_5$.

1.2.2.3.2.1. $a_{32} = 0$. Now we obtain H_4^3 with $\alpha = 1$.

1.2.2.3.2.2. $a_{32} \neq 0$. If we make the change

$$\bar{x} = x, \quad \bar{y} = -\frac{1}{a_{32}}x + y,$$

then we get the subgroup H_4^2.

1.2.3. $a_{35} \neq 0$. We choose $a_{35} = 1, a_{45} = 0$.

1.2.3.1. $a_{44} = 0$. Then $a_{42} \neq 0$ and putting $a_{42} = 1, a_{22} = a_{32} = 0$ we obtain 1.1.3.1.

1.2.3.2. $a_{44} \neq 0$. We assume $a_{44} = 1, a_{34} = 0$. The operators $Y_1 = X_1, Y_2 = a_{22}X_2 + X_3, Y_3 = a_{32}X_2 + X_5, Y_4 = a_{42}X_2 + X_4$ define a group iff $a_{22} = a_{32} = a_{42} = 0$ and we find the subgroup H_4^3 with $\alpha = 0$

1.3. $a_{24} \neq 0, a_{25} = 0$. Now we take $a_{24} = 1, a_{34} = a_{44} = 0$.

1.3.1. $a_{33} = a_{35} = 0$. Then $a_{32} \neq 0$ and choosing $a_{32} = 1, a_{22} = a_{42} = 0$ we have 1.1.2.

1.3.2. $a_{33} \neq 0, a_{35} = 0$. We put $a_{33} = 1, a_{23} = a_{43} = 0$ and we get 1.2.2.

1.3.3. $a_{35} \neq 0$. We take $a_{35} = 1, a_{45} = 0$.

1.3.3.1. $a_{43} = 0$. Therefore $a_{42} \neq 0$ and in this case the operators do not define a group.

1.3.3.2. $a_{43} \neq 0$. We suppose $a_{43} = 1, a_{23} = a_{33} = 0$ and we have 1.2.3.2.

1.4. $a_{25} \neq 0$. Now we choose $a_{25} = 1, a_{35} = a_{45} = 0$.

1.4.1. $a_{33} = a_{34} = 0$. Then $a_{32} \neq 0$ and replacing $a_{32} = 1, a_{22} = a_{42} = 0$ we obtain 1.1.3.

1.4.2. $a_{33} \neq 0, a_{34} = 0$. We put $a_{33} = 1, a_{23} = a_{43} = 0$ and we get 1.2.3.

1.4.3. $a_{34} \neq 0$. We suppose $a_{34} = 1, a_{24} = a_{44} = 0$ and we find 1.3.3.

2. $a_{12} \neq 0, a_{13} = a_{14} = a_{15} = 0$. We can choose $a_{12} = 1$, $a_{22} = a_{32} = a_{42} = 0$.

2.1. $a_{23} = a_{24} = a_{25} = 0$. Therefore $a_{21} \neq 0$ and taking $a_{21} = 1, a_{11} = a_{31} = a_{41} = 0$ we have 1.1.

2.2. $a_{23} \neq 0, a_{24} = a_{25} = 0$. We put $a_{23} = 1, a_{33} = a_{43} = 0$.

2.2.1. $a_{34} = a_{35} = 0$. Now $a_{31} \neq 0$ and replacing $a_{31} = 1, a_{11} = a_{21} = a_{41} = 0$ we get 1.1.1.

2.2.2. $a_{34} \neq 0, a_{35} = 0$. We suppose $a_{34} = 1, a_{44} = 0$.

2.2.2.1. $a_{45} = 0$. Then $a_{41} \neq 0$ and choosing $a_{41} = 1, a_{11} = a_{21} = a_{31} = 0$ we obtain 1.2.2.1.

2.2.2.2. $a_{45} \neq 0$. Let $a_{45} = 1$. The operators $Y_1 = a_{11}X_1 + X_2$, $Y_2 = a_{21}X_1 + X_3$, $Y_3 = a_{31}X_1 + X_4$, $Y_4 = a_{41}X_1 + X_5$ define a group iff $a_{21} = a_{31} = a_{41} = 0$. We make the change

$$\bar{x} = a_{11} + x, \quad \bar{y} = y$$

and obtain the subgroup

$$H_4^4 = \{X_2, X_3, X_4, X_5\}.$$

2.2.3. $a_{35} \neq 0$. We put $a_{35} = 1, a_{45} = 0$.

2.2.3.1. $a_{44} = 0$. Consequently, $a_{41} \neq 0$ and replacing $a_{41} = 1, a_{11} = a_{21} = a_{31} = 0$ we have 1.1.3.1.

2.2.3.2. $a_{44} \neq 0$. Now we take $a_{44} = 1, a_{34} = 0$ and we get 2.2.2.2.

2.3. $a_{24} \neq 0, a_{25} = 0$. We suppose $a_{24} = 1, a_{34} = a_{44} = 0$.

2.3.1. $a_{33} = a_{35} = 0$. Then $a_{31} \neq 0$ and putting $a_{31} = 1, a_{11} = a_{21} = a_{41} = 0$ we obtain 1.1.2.

2.3.2. $a_{33} \neq 0, a_{35} = 0$. We take $a_{33} = 1, a_{23} = a_{43} = 0$ and we have 2.2.2.

2.3.3. $a_{35} \neq 0$. Let $a_{35} = 1, a_{45} = 0$.

2.3.3.1. $a_{43} = 0$. Therefore $a_{41} \neq 0$ and using $a_{41} = 1, a_{11} = a_{21} = a_{31} = 0$ we find that the operators do not define a group.

2.3.3.2. $a_{43} \neq 0$. We choose $a_{43} = 1, a_{23} = a_{33} = 0$ and we get 2.2.2.2.

2.4. $a_{25} \neq 0$. We suppose $a_{25} = 1, a_{35} = a_{45} = 0$.

2.4.1. $a_{33} = a_{34} = 0$. Then $a_{31} \neq 0$ and putting $a_{31} = 1, a_{11} = a_{21} = a_{41} = 0$ we obtain 1.1.3.

2.4.2. $a_{33} \neq 0, a_{34} = 0$. Replacing $a_{33} = 1, a_{23} = a_{43} = 0$, we get 2.2.3.

2.4.3. $a_{34} \neq 0$. We take $a_{34} = 1, a_{24} = a_{44} = 0$ and we have 2.3.3.

3. $a_{13} \neq 0, a_{14} = a_{15} = 0$. We assume $a_{13} = 1, a_{23} = a_{33} = a_{43} = 0$.

3.1. $a_{22} = a_{24} = a_{25} = 0$. Consequently $a_{21} \neq 0$ and choosing $a_{21} = 1, a_{11} = a_{31} = a_{41} = 0$ we get 1.2.

3.2. $a_{22} \neq 0, a_{24} = a_{25} = 0$. Now we take $a_{22} = 1, a_{12} = a_{32} = a_{42} = 0$ and we obtain 2.2.

3.3. $a_{24} \neq 0, a_{25} = 0$. We can write $a_{24} = 1, a_{34} = a_{44} = 0$.

3.3.1. $a_{32} = a_{35} = 0$. Then $a_{31} \neq 0$ and putting $a_{31} = 1, a_{11} = a_{21} = a_{41} = 0$ we have 1.2.2.

3.3.2. $a_{32} \neq 0, a_{35} = 0$. We choose $a_{32} = 1, a_{12} = a_{22} = a_{42} = 0$ and we get 2.2.2.

3.3.3. $a_{35} \neq 0$. We suppose $a_{35} = 1, a_{45} = 0$.

3.3.3.1. $a_{42} = 0$. Therefore $a_{41} \neq 0$ and taking $a_{41} = 1, a_{11} = a_{21} = a_{31} = 0$ we obtain that the operators $Y_1 = a_{12}X_2 + X_3, Y_2 = a_{22}X_2 + X_4, Y_3 = a_{32}X_2 + X_5, Y_4 = X_1$ define a group iff $a_{12} = 0, a_{22}(1 - a_{32}) = 0$.

3.3.3.1.1. $a_{22} = 0$. We have H_4^3.

3.3.3.1.2. $a_{32} = 1$. Now we find again H_4^3 with $\alpha = 1$.

3.3.3.2. $a_{42} \neq 0$. Replacing $a_{42} = 1, a_{12} = a_{22} = a_{32} = 0$ we reduce to 2.2.2.2.

3.4. $a_{25} \neq 0$. We assume $a_{25} = 1, a_{35} = a_{45} = 0$.

3.4.1. $a_{32} = a_{34} = 0$. Then $a_{31} \neq 0$ and taking $a_{31} = 1, a_{11} = a_{21} = a_{41} = 0$ we get 1.2.3.

3.4.2. $a_{32} \neq 0, a_{34} = 0$. Now we put $a_{32} = 1, a_{12} = a_{22} = a_{42} = 0$ and we obtain 2.2.3.

3.4.3. $a_{34} \neq 0$. Choosing $a_{34} = 1, a_{24} = a_{44} = 0$ we have 3.3.3.

4. $a_{14} \neq 0, a_{15} = 0$. We suppose $a_{14} = 1, a_{24} = a_{34} = a_{44} = 0$.

4.1. $a_{22} = a_{23} = a_{25} = 0$. Consequently $a_{21} \neq 0$ and replacing $a_{21} = 1, a_{11} = a_{31} = a_{41} = 0$ we get 1.3.

4.2. $a_{22} \neq 0, a_{23} = a_{25} = 0$. Now we choose $a_{22} = 1, a_{12} = a_{32} = a_{42} = 0$ and we obtain 2.3.

4.3. $a_{23} \neq 0, a_{25} = 0$. We put $a_{23} = 1, a_{13} = a_{33} = a_{43} = 0$ and reduce to 3.3.

4.4. $a_{25} \neq 0$. We assume $a_{25} = 1, a_{35} = a_{45} = 0$.

4.4.1. $a_{32} = a_{33} = 0$. Therefore $a_{31} \neq 0$ and taking $a_{31} = 1, a_{11} = a_{21} = a_{41} = 0$ we get 1.3.3.

4.4.2. $a_{32} \neq 0, a_{33} = 0$. Choosing $a_{32} = 1, a_{12} = a_{22} = a_{42} = 0$, we obtain 2.3.3.

4.4.3. $a_{33} \neq 0$. Now we put $a_{33} = 1, a_{13} = a_{23} = a_{43} = 0$ and we have 3.3.3.

5. $a_{15} \neq 0$. We suppose $a_{15} = 1, a_{25} = a_{35} = a_{45} = 0$.

5.1. $a_{22} = a_{23} = a_{24} = 0$. Then $a_{21} \neq 0$ and taking $a_{21} = 1, a_{11} = a_{31} = a_{41} = 0$ we reduce to 1.4.

5.2. $a_{22} \neq 0, a_{23} = a_{24} = 0$. We take $a_{22} = 1, a_{12} = a_{32} = a_{42} = 0$ and we have 2.4.

5.3. $a_{23} \neq 0, a_{24} = 0$. Puttting $a_{23} = 1, a_{13} = a_{33} = a_{43} = 0$ we get 3.4.

5.4. $a_{24} \neq 0$. Now we choose $a_{24} = 1, a_{14} = a_{34} = a_{44} = 0$ and we obtain 4.4.

Renumbering the last two subgroups we are in a position to state the following result:

Theorem 1. *The four-parametric subgroups of H_5 can be reduced to one of the subgroups*

$$H_4^1 = \{X_1, X_2, X_3, X_4\}, H_4^2 = \{X_1, X_2, X_3, X_5\},$$

$$H_4^3 = \{X_2, X_3, X_4, X_5\}, H_4^4 = \{X_1, X_3, X_4, \alpha X_2 + X_5 \mid \alpha \in \mathbf{R}\}.$$

Remark 1. The subgroups H_4^4 with $\alpha = 1$ and H_4^4 with $\alpha = 0$ are treated in [5] and the transformations of these subgroups are called similitudes of the first type and similitudes of the second type, respectively.

Remark 2. The subgroups H_4^1 and H_4^4 with $\alpha = 0, 1, -1$ are treated in [8]. The transformations of H_4^4 with $\alpha = 1$ are called in [7] equiform transformations.

3. Three-parametric subgroups of H_5

A three-parametric subgroup of H_5 can be defined by three linearly independent infinitesimal operators $Y_h, h = 1, 2, 3$ in the form (2.1), which satisfy (2.2) for $i \neq j$; $i, j, k = 1, 2, 3$. Consider the possible cases.

1. $a_{12} = a_{13} = a_{14} = a_{15} = 0$. Then $a_{11} \neq 0$ and we can assume that $a_{11} = 1, a_{21} = a_{31} = 0$.

1.1. $a_{23} = a_{24} = a_{25} = 0$. Therefore $a_{22} \neq 0$ and we can choose $a_{22} = 1$, $a_{32} = 0$. The operators $Y_1 = X_1$, $Y_2 = X_2$, $Y_3 = a_{33}X_3 + a_{34}X_4 + a_{35}X_5$

define a group iff $a_{34} = 0$. There are the following two subcases: (i) $a_{35} = 0$ and (ii) $a_{35} \neq 0$.

1.1.1. $a_{35} = 0$. Consequently $a_{33} \neq 0$ and we find the subgroup

$$H_3^1 = \{X_1, X_2, X_3\}.$$

1.1.2. $a_{35} \neq 0$. Now we make the substitution

$$\bar{x} = x, \quad \bar{y} = a_{33} + a_{35} y$$

and we obtain the subgroup

$$H_3^2 = \{X_1, X_2, X_5\}.$$

1.2. $a_{23} \neq 0$, $a_{24} = a_{25} = 0$. Now we can assume $a_{23} = 1, a_{33} = 0$. The operators $Y_1 = X_1$, $Y_2 = a_{22}X_2 + X_3$, $Y_3 = a_{32}X_2 + a_{34}X_4 + a_{35}X_5$ define a group iff $a_{22}a_{34} = 0, a_{22}a_{35} + \sigma a_{32} = 0, \sigma a_{34} = 0, \sigma a_{35} = 0$, where σ is a real number.

1.2.1. $a_{22} = 0$.

1.2.1.1. $a_{35} = 0$.

1.2.1.1.1. $a_{32} \neq 0$. Changing the variables in the form

$$\bar{x} = a_{32}x, \quad \bar{y} = -\frac{a_{34}}{a_{32}}x + y$$

we find the subgroup H_3^1.

1.2.1.1.2. $a_{32} = 0$. Then $a_{34} \neq 0$ and the operators define the subgroup

$$H_3^3 = \{X_1, X_3, X_4\}.$$

1.2.1.2. $a_{35} \neq 0$. Now we obtain the subgroup

$$H_3^4 = \{X_1, X_3, \alpha X_2 + \beta X_4 + X_5 \,|\, \alpha, \beta \in \mathbf{R}\}.$$

1.2.2. $a_{34} = 0$.

1.2.2.1. $a_{35} = 0$. Therefore $a_{32} \neq 0$ and we find again H_3^1.

1.2.2.2. $a_{35} \neq 0$. Then $\sigma = 0$ and from $a_{22}a_{35} = 0$ it follows that $a_{22} = 0$. Now we have H_3^4 with $\beta = 0$.

1.3. $a_{24} \neq 0, a_{25} = 0$. We suppose $a_{24} = 1, a_{34} = 0$ and the operators $Y_1 = X_1$, $Y_2 = a_{22}X_2 + a_{23}X_3 + X_4$, $Y_3 = a_{32}X_2 + a_{33}X_3 + a_{35}X_5$ define a group iff $a_{32} = 0, a_{33} \neq 0, a_{35} = 0$.

1.3.1. $a_{22} = 0$. We have H_3^3.

1.3.2. $a_{22} \neq 0$. Now we make the change

$$\bar{x} = x, \quad \bar{y} = -\frac{1}{a_{22}}x + y$$

and we obtain H_3^1.

1.4. $a_{25} \neq 0$. We choose $a_{25} = 1, a_{35} = 0$. The operators $Y_1 = X_1$, $Y_2 = a_{22}X_2 + a_{23}X_3 + a_{24}X_4 + X_5$, $Y_3 = a_{32}X_2 + a_{33}X_3 + a_{34}X_4$ define a group iff

$$\begin{array}{lll} \lambda a_{32} = 0, & \lambda a_{33} = a_{24}, & \lambda a_{34} = 0, \\ \mu a_{32} = 0, & \mu a_{33} = a_{34}, & \mu a_{34} = 0, \\ \nu a_{32} = 0, & \nu a_{33} = -a_{33}, & a_{22}a_{34} - a_{24}a_{32} = 2a_{34}, \end{array} \tag{3.1}$$

where λ, μ and ν are real numbers.

1.4.1. $a_{32} = a_{34} = 0$. Then $a_{33} \neq 0$ and we get H_3^4.

1.4.2. $|a_{32}| + |a_{34}| \neq 0$. Now the system (3.1) gives $a_{32} \neq 0, a_{24} = a_{33} = a_{34} = 0$ and changing the variables in the form

$$\bar{x} = x, \quad \bar{y} = a_{23} + y$$

we find H_3^2.

2. $a_{12} \neq 0, a_{13} = a_{14} = a_{15} = 0$. We suppose $a_{12} = 1, a_{22} = a_{32} = 0$.

2.1. $a_{23} = a_{24} = a_{25} = 0$. Therefore $a_{21} \neq 0$ and putting $a_{21} = 1, a_{11} = a_{31} = 0$ we reduce to 1.1.

2.2. $a_{23} \neq 0, a_{24} = a_{25} = 0$. We assume $a_{23} = 1, a_{33} = 0$. The operators $Y_1 = a_{11}X_1 + X_2$, $Y_2 = a_{21}X_1 + X_3$, $Y_3 = a_{31}X_1 + a_{34}X_4 + a_{35}X_5$ define a group iff

$$\begin{array}{l} \lambda a_{31} = -a_{21}, \ \lambda a_{34} = 0, \ \lambda a_{35} = 0, \ a_{21}(a_{21}a_{34} + a_{35}) = 0, \\ \mu a_{31} = -a_{11}a_{21}a_{34} - a_{31}, \ \mu a_{34} = a_{34}, \ \mu a_{35} = 0, \end{array} \tag{3.2}$$

where λ and μ are real numbers.

2.2.1. $a_{34} = a_{35} = 0$. We have $a_{31} \neq 0$ and taking $a_{31} = 1, a_{11} = a_{21} = 0$ we get H_3^1.

2.2.2. $a_{34} \neq 0$. Then from (3.2) we find $a_{21} = a_{31} = a_{35} = 0$. We make the substitution

$$\bar{x} = a_{11} + x, \quad \bar{y} = y \tag{3.3}$$

and we obtain the subgroup

$$H_3^5 = \{X_2, X_3, X_4\}.$$

2.2.3. $a_{35} \neq 0$. Now $a_{21} = a_{31} = a_{34} = 0$ and making the change (3.3) we get the subgroup

$$H_3^6 = \{X_2, X_3, X_5\}.$$

2.3. $a_{24} \neq 0, a_{25} = 0$. We suppose $a_{24} = 1, a_{34} = 0$. The operators $Y_1 = a_{11}X_1 + X_2$, $Y_2 = a_{21}X_1 + a_{23}X_3 + X_4$, $Y_3 = a_{31}X_1 + a_{33}X_3 + a_{35}X_5$ define a group iff

$$\begin{aligned} \lambda a_{31} &= -2a_{21}, \ \lambda a_{33} = a_{11} - a_{23}, \ \lambda a_{35} = 0, \\ \mu a_{31} &= -a_{31}, \ \mu a_{33} = 0, \ \mu a_{35} = 0, \\ \nu a_{31} &= -a_{21}a_{35}, \ \nu a_{33} = -a_{31}, \ \nu a_{35} = 0, \end{aligned}$$

where λ, μ and ν are real numbers.

2.3.1. $a_{35} \neq 0$. The last system gives $a_{21} = a_{31} = 0, a_{23} = a_{11}$ and replacing

$$\bar{x} = a_{11} + x, \quad \bar{y} = \frac{a_{33}}{a_{35}} + y$$

we find the subgroup

$$H_3^7 = \{X_2, X_4, X_5\}.$$

2.3.2. $a_{35} = 0$. Now we have $a_{21} = a_{31} = 0, a_{33} \neq 0$ and the change (3.3) brings to H_3^5.

2.4. $a_{25} \neq 0$. We put $a_{25} = 1, a_{35} = 0$. The operators $Y_1 = a_{11}X_1 + X_2$, $Y_2 = a_{21}X_1 + a_{23}X_3 + a_{24}X_4 + X_5$, $Y_3 = a_{31}X_1 + a_{33}X_3 + a_{34}X_4$ define a group iff

$$\begin{aligned} \lambda a_{31} &= -a_{21}, \ \lambda a_{33} = a_{11}a_{24}, \ \lambda a_{34} = a_{24}, \\ \mu a_{31} &= -a_{31}, \ \mu a_{33} = a_{11}a_{34}, \ \mu a_{34} = a_{34}, \\ \nu a_{31} &= 0, \ \nu a_{33} = a_{21}a_{34} - a_{24}a_{31} - a_{33}, \ \nu a_{34} = -a_{34}, \end{aligned} \tag{3.4}$$

where λ, μ and ν are real numbers.

2.4.1. $a_{34} = 0$. It follows from (3.4) that $a_{24} = 0$.

2.4.1.1. $a_{31} = 0$. Since $Y_3 \neq 0$ we have $a_{33} \neq 0$ and (3.4) gives $a_{21} = 0$. We choose $a_{33} = 1, a_{23} = 0$ and applying (3.3) we find H_3^6.

2.4.1.2. $a_{33} = 0$. Now $a_{31} \neq 0$ and we put $a_{31} = 1, a_{11} = a_{21} = 0$. Replacing

$$\bar{x} = x, \quad \bar{y} = a_{23} + y,$$

we obtain H_3^2.

2.4.2. $a_{34} \neq 0$. We find from (3.4) $a_{21} = a_{31} = 0, a_{33} = a_{11}a_{34}$ and making the change

$$\bar{x} = a_{11} + x, \quad \bar{y} = a_{23} + a_{24}x + y$$

we get H_2^7.

3. $a_{13} \neq 0, a_{14} = a_{15} = 0$. We suppose $a_{13} = 1, a_{23} = a_{33} = 0$.

3.1. $a_{22} = a_{24} = a_{25} = 0$. Then $a_{21} \neq 0$ and taking $a_{21} = 1, a_{31} = a_{41} = 0$ we have 1.2.

3.2. $a_{22} \neq 0, a_{24} = a_{25} = 0$. We choose $a_{22} = 1, a_{12} = a_{32} = 0$ and we reduce to 2.2.

3.3. $a_{24} \neq 0, a_{25} = 0$. Now we put $a_{24} = 1, a_{34} = 0$. The operators $Y_1 = a_{11}X_1 + a_{12}X_2 + X_3$, $Y_2 = a_{21}X_1 + a_{22}X_2 + X_4$, $Y_3 = a_{31}X_1 + a_{32}X_2 + a_{35}X_5$ define a group iff

$$\begin{aligned}
&(a_{11} - a_{22})a_{11} = -\lambda a_{31}, \ (a_{11} + a_{22})a_{12} = -\lambda a_{32}, \ \lambda a_{35} = 0, \\
&a_{11}(a_{32} - a_{35}) - a_{12}a_{31} = \mu a_{31}, \ a_{12}a_{35} = -\mu a_{32}, \ \mu a_{35} = 0, \\
&(a_{11} - a_{22})a_{31} + 2a_{21}a_{32} - a_{21}a_{35} = \nu a_{31}, \\
&a_{12}a_{31} + (a_{32} - a_{35})a_{22} = \nu a_{32}, \ \nu a_{35} = 0,
\end{aligned} \tag{3.5}$$

where λ, μ and ν are real numbers.

3.3.1. $a_{35} \neq 0$. The system (3.5) gives $\lambda = \mu = \nu = 0$ and

$$\begin{aligned}
&(a_{11} - a_{22})a_{11} = 0, \ a_{12} = 0, \ (a_{32} - a_{35})a_{11} = 0, \\
&(a_{11} - a_{22})a_{31} + (2a_{32} - a_{35})a_{21} = 0, \ (a_{32} - a_{35})a_{22} = 0.
\end{aligned} \tag{3.6}$$

3.3.1.1. $a_{11} = 0$. Now (3.6) has the form

$$a_{12} = 0, \ a_{22}a_{31} - (2a_{32} - a_{35})a_{21} = 0, \ (a_{32} - a_{35})a_{22} = 0. \tag{3.7}$$

3.3.1.1.1. $a_{21} = a_{22} = 0$. The system (3.7) is satisfied and the corresponding operators are $Y_1 = X_3, Y_2 = X_4, Y_3 = a_{31}X_1 + a_{32}X_2 + a_{35}X_5$.

3.3.1.1.1.1. $a_{32} = 0$. We obtain the subgroup

$$H_3^8 = \{X_3, X_4, \alpha X_1 + X_5 \mid \alpha \in \mathbf{R}\}.$$

3.3.1.1.1.2. $a_{32} \neq 0$. We make the change

$$\bar{x} = \frac{a_{31}}{a_{35}} + \frac{a_{32}}{a_{35}}x, \quad \bar{y} = y$$

and we find the subgroup

$$H_3^9 = \{X_3, X_4, \alpha X_2 + X_5 \mid \alpha \neq 0; \quad \alpha \in \mathbf{R}\}.$$

3.3.1.1.2. $a_{21} \neq 0, a_{22} = 0$. We get from (3.7) $a_{12} = 0, a_{32} = \frac{1}{2}a_{35}$. Replacing

$$\bar{x} = a_{31} + \frac{1}{2}a_{35}x, \quad \bar{y} = y$$

we obtain the subgroup

$$H_3^{10} = \{X_3, X_2 + 2X_5, \alpha X_1 + X_4 \mid \alpha \neq 0; \quad \alpha \in \mathbf{R}\}.$$

3.3.1.1.3. $a_{21} = 0, a_{22} \neq 0$. Now (3.7) gives $a_{12} = a_{31} = 0, a_{32} = a_{35} \neq 0$ and using the change

$$\bar{x} = x, \quad \bar{y} = x - a_{22}y$$

we find H_3^6.

3.3.1.1.4. $a_{21} \neq 0, a_{22} \neq 0$. Then from (3.7) it follows $a_{12} = 0, a_{32} = a_{35}, a_{31} = \dfrac{a_{21}a_{35}}{a_{22}}$ and replacing

$$\bar{x} = a_{21} + a_{22}x, \quad \bar{y} = x - a_{22}y$$

we get again H_3^6.

3.3.1.2. $a_{11} \neq 0$. We find from (3.6) that $a_{11} = a_{22}, a_{12} = a_{21} = 0, a_{32} = a_{35}$ and applying the change

$$\bar{x} = x, \quad \bar{y} = -\frac{a_{31}}{a_{11}} - \frac{a_{35}}{a_{11}}x + a_{35}y$$

we obtain H_3^2.

3.3.2. $a_{35} = 0$. Therefore $|a_{31}| + |a_{32}| \neq 0$.

3.3.2.1. $a_{31} \neq 0, a_{32} = 0$. Putting $a_{31} = 1, a_{11} = a_{21} = 0$ we find from (3.5) $a_{12} = 0$.

3.3.2.1.1. $a_{22} \neq 0$. Changing the variables in the form

$$\bar{x} = x, \quad \bar{y} = -\frac{1}{a_{22}}x + y$$

we get the subgroup H_3^1.

3.3.2.1.2. $a_{22} = 0$. We have H_3^3.

3.3.2.2. $a_{31} = 0, a_{32} \neq 0$. From (3.5) it follows that $a_{11} = a_{21} = 0$ and we obtain H_3^5.

3.3.2.3. $a_{31} \neq 0, a_{32} \neq 0$. Now (3.5) has the form

$$a_{11}a_{22} = 0, \quad a_{12}a_{22} = 0, \quad a_{12} = \frac{a_{32}}{a_{31}}a_{11}, \quad a_{22} = \frac{a_{32}}{a_{31}}a_{21}. \tag{3.8}$$

3.3.2.3.1. $a_{22} = 0$. From (3.8) it follows that $a_{12} = \dfrac{a_{32}}{a_{31}}a_{11}, a_{21} = 0$ and using the change

$$\bar{x} = a_{31} + a_{32}x, \quad \bar{y} = y$$

we find the subgroup H_3^5.

3.3.2.3.2. $a_{22} \neq 0$. According to (3.8) we have $a_{11} = a_{12} = 0, a_{22} = \dfrac{a_{32}}{a_{31}}a_{21}$ and we get H_3^5.

3.4. $a_{25} \neq 0$. We take $a_{25} = 1, a_{35} = 0$ and the operators

$$\begin{aligned} Y_1 &= a_{11}X_1 + a_{12}X_2 + X_3, \\ Y_2 &= a_{21}X_1 + a_{22}X_2 + a_{24}X_4 + X_5, \\ Y_3 &= a_{31}X_1 + a_{32}X_2 + a_{34}X_4 \end{aligned}$$

define a group iff

$$\begin{aligned} & a_{11}(a_{11}a_{24} - a_{22} + 1) + a_{12}a_{22} = -\lambda a_{31}, \ a_{12}(a_{11}a_{24} + 1) = -\lambda a_{32}, \\ & a_{12}a_{24} = \lambda a_{34}, \ a_{11}(a_{32} - a_{11}a_{34}) - a_{12}a_{31} = \mu a_{31}, \\ & a_{11}a_{12}a_{34} = -\mu a_{32}, \ a_{12}a_{34} = \mu a_{34}, \\ & a_{21}a_{32} - a_{31}a_{22} - a_{11}(a_{21}a_{34} - a_{31}a_{24}) = \nu a_{31}, \\ & a_{12}(a_{21}a_{34} - a_{31}a_{24}) = \nu a_{32}, \ (a_{22} - 1)a_{34} - a_{32}a_{24} = \nu a_{34}, \end{aligned} \tag{3.9}$$

where λ, μ and ν are real numbers.

3.4.1. $a_{31} \neq 0, a_{32} = a_{34} = 0$. We can put $a_{31} = 1, a_{11} = a_{21} = 0$. From (3.9) it follows that $a_{12} = 0$ and we get the subgroup H_3^4.

3.4.2. $a_{31} = 0, a_{32} \neq 0, a_{34} = 0$. Now we choose $a_{32} = 1, a_{12} = a_{22} = 0$. The system (3.9) gives $a_{11} = a_{21} = a_{24} = 0$ and hence we have again H_3^6.

3.4.3. $a_{31} \neq 0, a_{32} \neq 0, a_{34} = 0$. According to (3.9) we have

$$a_{24} = 0, \qquad a_{11}a_{22} - a_{12}a_{21} = a_{11}, \tag{3.10}$$

$$a_{11}a_{32} - a_{12}a_{31} = 0, \qquad a_{21}a_{32} - a_{22}a_{31} = 0. \tag{3.11}$$

From (3.11) it follows that $a_{11}a_{22} - a_{12}a_{21} = 0$ and then the second equality of (3.10) gives $a_{11} = 0$. From this equality and (3.11) we get $a_{12} = 0, a_{21} = \rho a_{31}, a_{22} = \rho a_{32}$, where ρ is a real parameter. Now the operators have the form $Y_1 = X_3, Y_2 = \rho Y_3 + X_5, Y_3 = a_{31}X_1 + a_{32}X_2$ and we can change Y_2 by $\bar{Y}_2 = Y_2 - \rho Y_3$. We make the substitution

$$\bar{x} = a_{31} + a_{32}x, \quad \bar{y} = y$$

and we get the subgroup H_3^4 with $\alpha = \beta = 0$.

3.4.4. $a_{31} = a_{32} = 0, a_{34} \neq 0$. Putting $a_{34} = 1, a_{24} = 0$ from (3.9) we find $a_{11} = a_{12} = 0$.

3.4.4.1. $a_{22} = 0$. Now we have H_3^8.

3.4.4.2. $a_{22} \neq 0$. We make the change

$$\bar{x} = a_{21} + a_{22}x, \quad \bar{y} = y$$

and we get H_3^9.

3.4.5. $a_{31} \neq 0, a_{32} = 0, a_{34} \neq 0$. From (3.9) it follows that $a_{11} = a_{12} = 0, a_{22} = \frac{1}{2}$ and making the substitution

$$\bar{x} = a_{21} + \frac{1}{2}x, \quad \bar{y} = 2a_{24}x + y$$

we obtain the subgroup H_3^{10}.

3.4.6. $a_{31} = 0, a_{32} \neq 0, a_{34} \neq 0$. Now the system (3.9) has the form

$$\begin{aligned}
&a_{11}(a_{11}a_{24} - a_{22} + 1) + a_{12}a_{21} = 0,\\
&a_{12}((a_{11}a_{34} + a_{32})a_{24} + a_{34}) = 0,\\
&a_{11}(a_{32} - a_{11}a_{34}) = 0,\\
&a_{12}(a_{32} + a_{11}a_{34}) = 0,\\
&a_{21}(a_{32} - a_{11}a_{34}) = 0,\\
&a_{12}a_{21}a_{34} + \left(a_{22} - 1 - \frac{a_{24}a_{32}}{a_{34}}\right) a_{32} = 0.
\end{aligned} \tag{3.12}$$

3.4.6.1. $a_{12} = 0$. From (3.12) we deduce

$$\begin{aligned} &a_{11}(a_{11}a_{24} - a_{22} + 1) = 0, \quad a_{11}(a_{32} - a_{11}a_{34}) = 0, \\ &a_{21}(a_{32} - a_{11}a_{34}) = 0, \qquad (a_{22} - 1)a_{34} - a_{24}a_{32} = 0. \end{aligned} \tag{3.13}$$

3.4.6.1.1. $a_{11} = 0$. From (3.13) we get $a_{21} = 0, a_{21} - 1 = \rho a_{32}, a_{24} = \rho a_{34}$, where ρ is a real parameter. The operators are $Y_1 = X_3, Y_2 = X_2 + \rho Y_3 + X_5, Y_3 = a_{32}X_2 + a_{34}X_4$ and replacing Y_2 by $\bar{Y}_2 = Y_2 - \rho Y_3$ and changing the variables in the form

$$\bar{x} = x, \quad \bar{y} = -\frac{a_{34}}{a_{32}}x + y$$

we find the subgroup H_3^6.

3.4.6.1.2. $a_{11} \neq 0$. Now from (3.13) we get $a_{11} = \dfrac{a_{32}}{a_{34}}, (a_{22} - 1)a_{34} - a_{24}a_{32} = 0$ and using the change

$$\bar{x} = x, \quad \bar{y} = -\frac{a_{34}}{a_{32}}(a_{21} + x) + y$$

we obtain the subgroup H_3^2.

3.4.6.2. $a_{12} \neq 0$. In this case the system (3.9) is incompatible and the corresponding operators do not define a group.

3.4.7. $a_{31} \neq 0, a_{32} \neq 0, a_{34} \neq 0$. Then (3.9) becomes

$$\begin{aligned} &a_{11}(a_{11}a_{24} - a_{22} + 1) + a_{12}a_{21} = -\frac{a_{12}a_{24}a_{31}}{a_{34}}, \\ &(a_{11}a_{24} + 1)a_{12} = -\frac{a_{12}a_{24}a_{32}}{a_{34}}, \\ &a_{11}(a_{32} - a_{11}a_{34}) = 2a_{12}a_{31}, \\ &a_{12}(a_{32} + a_{11}a_{34}) = 0, \\ &a_{21}a_{32} - a_{22}a_{31} - a_{11}(a_{21}a_{34} - a_{24}a_{31}) = a_{31}(a_{22} - \frac{a_{24}a_{32}}{a_{34}} - 1), \\ &a_{12}(a_{21}a_{34} - a_{24}a_{31}) = -a_{32}\left(a_{22} - \frac{a_{24}a_{32}}{a_{34}} - 1\right). \end{aligned} \tag{3.14}$$

3.4.7.1. $a_{12}=0$. From (3.14) we obtain

$$a_{11}(a_{22}-a_{11}a_{24}-1)=0,$$
(3.15)
$$a_{11}(a_{32}-a_{11}a_{34})=0,$$

$$(a_{21}-1)a_{34}-a_{24}a_{32}=0,$$
(3.16)
$$a_{21}a_{32}-a_{22}a_{31}=a_{11}(a_{21}a_{34}-a_{24}a_{31}).$$

Since $a_{32}\neq 0, a_{34}\neq 0$ from (3.16) we have $a_{21}(a_{22}-a_{11}a_{24}-1)\neq 0$ and according to (3.15) we get $a_{11}=0$. The last relation and (3.16) give $a_{22}-1=\rho a_{32}$, $a_{24}=\rho a_{34}, a_{21}=\frac{1+\rho a_{32}}{a_{32}}a_{31}$, $\rho\in\mathbf{R}$, and the corresponding operators are $Y_1=X_3, Y_2=\frac{a_{31}}{a_{32}}X_1+X_2+X_5+\rho Y_3, Y_3=a_{31}X_1+a_{32}X_2+a_{34}X_4$. Replacing Y_2 by $\bar{Y}_2=Y_2-\rho Y_3$ and using the change

$$\bar{x}=a_{31}+a_{32}x,\quad \bar{y}=-\frac{a_{34}}{a_{32}}x+y,$$

we obtain the subgroup H_3^6.

3.4.7.2. $a_{12}\neq 0$. The system (3.9) is incompatible and the operators do not define a group.

4. $a_{14}\neq 0, a_{15}=0$. We asume that $a_{14}=1, a_{24}=a_{34}=0$.

4.1. $a_{22}=a_{23}=a_{25}=0$. Hence $a_{21}\neq 0$ and putting $a_{21}=1, a_{11}=a_{31}=0$ we reduce to 1.3.

4.2. $a_{22}\neq 0, a_{23}=a_{25}=0$. We choose $a_{22}=1, a_{12}=a_{32}=0$ and we get 2.3.

4.3. $a_{23}\neq 0, a_{25}=0$. We suppose $a_{23}=1, a_{13}=a_{33}=0$ and we obtain 3.3.

4.4. $a_{25}\neq 0$. We put $a_{25}=1, a_{35}=0$ and the operators $Y_1=a_{11}X_1+a_{12}X_2+a_{13}X_3+X_4, Y_2=a_{21}X_1+a_{22}X_2+a_{23}X_3+X_5, Y_3=a_{31}X_1+a_{32}X_2+a_{33}X_3$ define a group iff

$$\begin{aligned}
&a_{11}(2a_{22}-1)-a_{12}a_{21}=\lambda a_{31}, && a_{12}(a_{22}-1)=\lambda a_{32}, && a_{13}a_{22}-a_{21}=\lambda a_{33},\\
&2a_{11}a_{32}-a_{12}a_{31}=\mu a_{31}, && a_{12}a_{32}=\mu a_{32}, && -a_{31}+a_{13}a_{32}=\mu a_{33},\\
&a_{21}a_{32}-a_{22}a_{31}=\nu a_{31}, && \nu a_{32}=0, && a_{33}=-\nu a_{33},
\end{aligned}$$
(3.17)

where λ, μ and ν are real numbers.

It can be shown directly that for the cases

4.4.1. $a_{31} \neq 0, a_{32} = a_{33} = 0$;

4.4.2. $a_{31} = 0, a_{32} \neq 0, a_{33} \neq 0$;

4.4.3. $a_{31} \neq 0, a_{32} \neq 0, a_{33} \neq 0$;

the system (3.17) is incompatible and therefore the corresponding operators do not define a group.

We shall now treat the rest cases for a_{31}, a_{32} and a_{33} such that $|a_{31}| + |a_{32}| + |a_{33}| \neq 0$.

4.4.4. $a_{31} = 0, a_{32} \neq 0, a_{33} = 0$. We put $a_{32} = 1, a_{12} = a_{22} = 0$ and using (3.17) we find $a_{11} = a_{13} = a_{21} = 0$. We make the change

$$\bar{x} = x, \quad \bar{y} = a_{23} + y$$

and we obtain the subgroup H_3^7.

4.4.5. $a_{31} \neq 0, a_{32} \neq 0, a_{33} = 0$. Now the system (3.17) gives $a_{13}a_{22} - a_{21} = 0, a_{31} - a_{13}a_{32} = 0, a_{11}a_{32} - a_{12}a_{31} = 0, a_{21}a_{32} - a_{22}a_{31} = 0$ and consequently $a_{11} = \rho a_{31}, a_{12} = \rho a_{32}, a_{13} = \dfrac{a_{31}}{a_{32}}, a_{21} = \sigma a_{31}, a_{22} = \sigma a_{32}$, where ρ and σ are real numbers. Then the corresponding operators are $Y_1 = \rho Y_3 + \dfrac{a_{31}}{a_{32}} X_3 + X_4, Y_2 = \sigma Y_3 + a_{23}X_3 + X_5, Y_3 = a_{31}X_1 + a_{32}X_2$. We change the operators Y_1 and Y_2 by $\bar{Y_1} = Y_1 - \rho Y_3, \bar{Y_2} = Y_2 - \sigma Y_3$ and applying the substitution

$$\bar{x} = a_{31} + a_{32}x, \quad \bar{y} = a_{23} + y$$

we get the subgroup H_3^7.

4.4.6. $a_{31} = a_{32} = 0, a_{33} \neq 0$. We suppose that $a_{31} = 1, a_{13} = a_{23} = 0$ and from (3.17) it follows $a_{11}(2a_{22} - 1) - a_{12}a_{21} = 0, a_{12}(a_{22} - 1) = 0$.

4.4.6.1. $a_{11} = a_{12} = 0$.

4.4.6.1.1. $a_{22} = 0$. We have H_3^8.

4.4.6.1.2. $a_{22} \neq 0$. We make the change

$$\bar{x} = a_{21} + a_{22}x, \quad \bar{y} = y$$

and we obtain the subgroup H_3^9.

4.4.6.2. $a_{11} - a_{12}a_{21} = 0, a_{22} = 1$.

4.4.6.2.1. $a_{12} = 0$. By the change

$$\bar{x} = a_{21} + x, \quad \bar{y} = y$$

we get the subgroup H_3^9 with $\alpha = 1$.

4.4.6.2.2. $a_{12} \neq 0$. We make the substitution

$$\bar{x} = a_{21} + x, \quad \bar{y} = x - a_{12}y$$

and we find the subgroup H_3^6.

4.4.6.3. $a_{12} = 0, a_{22} = 1/2$.

4.4.6.3.1. $a_{11} = 0$. Changing the variables in the form

$$\bar{x} = a_{21} + \frac{1}{2}x, \quad \bar{y} = y \tag{3.18}$$

we obtain the subgroup H_3^9 with $\alpha = 1/2$.

4.4.6.3.2. $a_{11} \neq 0$. Applying (3.18) we get H_3^{10}.

4.4.7. $a_{31} \neq 0, a_{32} = 0, a_{33} \neq 0$. We get from (3.17)

$$a_{11} = \frac{a_{13}a_{31}}{a_{33}}, \quad a_{12} = \frac{a_{31}}{a_{33}}, \quad a_{22} = 1$$

and then the corresponding operators has the form

$$Y_1 = \frac{a_{13}}{a_{33}}Y_3 + \frac{a_{31}}{a_{33}}X_2 + X_4, Y_2 = a_{21}X_1 + X_2 + a_{23}X_3 + X_5, Y_3 = a_{31}X_1 + a_{33}X_3.$$

Obviously we can change Y_1 by $\bar{Y}_1 = a_{33}Y_1 - a_{13}Y_3 = a_{31}X_2 + a_{33}X_4$ and replacing

$$\bar{x} = a_{21} + x, \quad \bar{y} = a_{21}a_{33} - a_{23}a_{31} + a_{33}x - a_{31}y$$

we find the subgroup H_3^2.

5. $a_{15} \neq 0$. We assume that $a_{15} = 1, a_{25} = a_{35} = 0$.

5.1. $a_{22} = a_{23} = a_{24} = 0$. Then $a_{21} \neq 0$ and putting $a_{21} = 1, a_{11} = a_{31} = 0$ we have 1.4.

5.2. $a_{22} \neq 0, a_{23} = a_{24} = 0$. Now we choose $a_{22} = 1, a_{12} = a_{32} = 0$ and we get 2.4.

5.3. $a_{23} \neq 0, a_{24} = 0$. We take $a_{23} = 1, a_{13} = a_{33} = 0$ and we find 3.4.

5.4. $a_{24} \neq 0$. We assume $a_{24} = 1, a_{14} = a_{34} = 0$ and we obtain 4.4.

Now we summarize the foregoing results in the following

Theorem 2. *The three-parametric subgroups of H_5 can be reduced to one of the subgroups*

$$
\begin{aligned}
H_3^1 &= \{X_1, X_2, X_3\},\\
H_3^2 &= \{X_1, X_2, X_5\},\\
H_3^3 &= \{X_1, X_3, X_4\},\\
H_3^4 &= \{X_1, X_3, \alpha X_2 + \beta X_4 + X_5 \,|\, \alpha, \beta \in \mathbf{R}\},\\
H_3^5 &= \{X_2, X_3, X_4\},\\
H_3^6 &= \{X_2, X_3, X_5\},\\
H_3^7 &= \{X_2, X_4, X_5\},\\
H_3^8 &= \{X_3, X_4, \alpha X_1 + X_5 \,|\, \alpha \in \mathbf{R}\},\\
H_3^9 &= \{X_3, X_4, \alpha X_2 + X_5 \,|\, \alpha \neq 0, \alpha \in \mathbf{R}\},\\
H_3^{10} &= \{X_3, X_2 + 2X_5, \alpha X_1 + X_4 \,|\, \alpha \neq 0, \alpha \in \mathbf{R}\}.
\end{aligned}
$$

Remark 3. H_3^3 is the isometry group in Γ_2.

4. Two-parametric subgroups of H_5

A two-parametric subgroup of H_5 can be defined by two linearly independent infinitesimal operators $Y_h, h = 1, 2$, in the form (2.1), which satisfy (2.2) for $i \neq j$; $i, j, k = 1, 2$.

1. $a_{12} = a_{13} = a_{14} = a_{15} = 0$. Then $a_{11} \neq 0$ and we can put $a_{11} = 1, a_{21} = 0$.

1.1. $a_{23} = a_{24} = a_{25} = 0$. Consequently $a_{22} \neq 0$ and choosing $a_{21} = 1$ we obtain the subgroup

$$H_2^1 = \{X_1, X_2\}.$$

1.2. $a_{23} \neq 0, a_{24} = a_{25} = 0$. We take $a_{23} = 1$ and therefore $Y_1 = X_1, Y_2 = a_{22}X_2 + X_3$. From $[Y_1, Y_2] = a_{22}Y_1$ it follows that Y_1 and Y_2 define the subgroup

$$H_2^2 = \{X_1, \alpha X_2 + X_3 \,|\, \alpha \in \mathbf{R}\}.$$

1.3. $a_{24} \neq 0, a_{25} = 0$. We assume that $a_{24} = 1$. In this case the corresponding operators $Y_1 = X_1, Y_2 = a_{22}X_2 + a_{23}X_3 + X_4$ do not define a group.

1.4. $a_{25} \neq 0$. We choose $a_{25} = 1$. The operators $Y_1 = X_1, Y_2 = a_{22}X_2 + a_{23}X_3 + a_{24}X_4 + X_5$ define a group iff $a_{24} = 0$. We make the change

$$\bar{x} = x, \quad \bar{y} = a_{23} + y$$

and we get the subgroup

$$H_2^3 = \{X_1, \alpha X_2 + X_5 \,|\, \alpha \in \mathrm{R}\}.$$

2. $a_{12} \neq 0, a_{13} = a_{14} = a_{15} = 0$. We suppose $a_{12} = 1, a_{22} = 0$.

2.1. $a_{23} = a_{24} = a_{25} = 0$. Then $a_{21} \neq 0$ and taking $a_{21} = 1, a_{11} = 0$ we obtain H_2^1.

2.2. $a_{23} \neq 0, a_{24} = a_{25} = 0$. We put $a_{23} = 1$. The operators $Y_1 = a_{11}X_1 + X_2, Y_2 = a_{21}X_1 + X_3$ define a group iff $a_{21} = 0$. We make the change

$$\bar{x} = a_{11} + x, \quad \bar{y} = y \tag{4.1}$$

and we get the subgroup

$$H_2^4 = \{X_2, X_3\}.$$

2.3. $a_{24} \neq 0, a_{25} = 0$. We assume $a_{24} = 1$. The operators $Y_1 = a_{11}X_1 + X_2, Y_2 = a_{21}X_1 + a_{23}X_3 + X_4$ define a group iff $a_{21} = 0, a_{23} = a_{11}$. Applying (4.1) we get the subgroup

$$H_2^5 = \{X_2, X_4\}.$$

2.4. $a_{25} \neq 0$. We take $a_{25} = 1$ and the operators $Y_1 = a_{11}X_1 + X_2, Y_2 = a_{21}X_1 + a_{23}X_3 + a_{24}X_4 + X_5$ define a group iff $a_{21} = a_{23} = a_{24} = 0$. We have the subgroup

$$H_2^6 = \{X_2, X_5\}.$$

3. $a_{13} \neq 0, a_{14} = a_{15} = 0$. We suppose $a_{13} = 1, a_{23} = 0$.

3.1. $a_{22} = a_{24} = a_{25} = 0$. Now $a_{21} \neq 0$ and putting $a_{21} = 1, a_{11} = 0$ we obtain 1.2.

3.2. $a_{22} \neq 0, a_{24} = a_{25} = 0$. We choose $a_{22} = 1, a_{12} = 0$ and we get 2.2.

3.3. $a_{24} \neq 0, a_{25} = 0$. We put $a_{24} = 1$. The operators $Y_1 = a_{11}X_1 + a_{12}X_2 + X_3, Y_2 = a_{21}X_1 + a_{22}X_2 + X_4$ define a group iff $a_{11}(a_{11} - a_{22}) + 2a_{12}a_{21} = 0, a_{12}(a_{11} + a_{22}) = 0$.

3.3.1. $a_{11} = a_{12} = 0$.

3.3.1.1. $a_{22} = 0$. If $a_{21} \neq 0$, then we have the subgroup

$$H_2^7 = \{X_3, \alpha X_1 + X_4 \mid \alpha \neq 0, \alpha \in \mathrm{R}\},$$

and if $a_{21} = 0$ - the subgroup

$$H_2^{8'} = \{X_3, X_4\}.$$

3.3.1.2. $a_{22} \neq 0$. Using the change

$$\bar{x} = \frac{a_{21}}{a_{22}} + x, \quad \bar{y} = -\frac{1}{a_{22}}x + y \tag{4.2}$$

we find the subgroup H_2^4.

3.3.2. $a_{11} = a_{22}, a_{12} = 0$.

3.3.2.1. $a_{11} = 0$. We have 3.3.1.1.

3.3.2.2. $a_{11} \neq 0$. Now we apply (4.2) and if $a_{21} = 0$ (resp. $a_{21} \neq 0$), then we get H_2^1 (resp. H_2^2 with $\alpha \neq 0$).

3.3.3. $a_{11} = -a_{22}, a_{11}^2 + a_{12}a_{21} = 0$.

3.3.3.1. $a_{11} = 0$. We have 3.3.1.1.

3.3.3.2. $a_{12} \neq 0$. We find $a_{21} = -\dfrac{a_{11}^2}{a_{12}}$ and replacing

$$\bar{x} = a_{11} + a_{12}x, \quad \bar{y} = y$$

we obtain the subgroup

$$H_2^{8''} = \{X_4, \alpha X_2 + X_3 \mid \alpha \neq 0, \alpha \in \mathrm{R}\}.$$

Unifying the last two subgroups we have the subgroup

$$H_2^8 = \{X_4, \alpha X_2 + X_3 \mid \alpha \in \mathrm{R}\}.$$

3.4. $a_{25} \neq 0$. We put $a_{25} = 1$ and the operators $Y_1 = a_{11}X_1 + a_{12}X_2 + X_3, Y_2 = a_{21}X_1 + a_{22}X_2 + a_{24}X_4 + X_5$ define a group iff

$$\begin{aligned} a_{11}(a_{11}a_{24} - a_{22} + 1) + a_{12}a_{21} &= 0, \\ a_{12}(a_{11}a_{24} + 1) &= 0, \\ a_{12}a_{24} &= 0. \end{aligned} \tag{4.3}$$

3.4.1. $a_{11} = a_{12} = 0$.

3.4.1.1. $a_{22} = 0$. After the change

$$\bar{x} = x, \quad \bar{y} = a_{21}a_{24} + a_{24}x + y$$

we get the subgroup

$$H_2^9 = \{X_3, \alpha X_1 + X_5 \mid \alpha \in \mathrm{R}\}.$$

3.4.1.2. $a_{22} \neq 0$. Replacing

$$\bar{x} = a_{21} + a_{22}x, \quad \bar{y} = a_{24}x + y$$

we find the subgroup

$$H_2^{10} = \{X_3, \alpha X_2 + \beta X_4 + X_5 \mid \alpha \neq 0, \alpha \in \mathrm{R}\}.$$

3.4.2. $a_{11}a_{24} - a_{22} + 1 = 0, a_{12} = 0$.

3.4.2.1. $a_{11} = 0$. We have 3.4.1.2.

3.4.2.2. $a_{11} \neq 0$.

3.4.2.2.1. $a_{22} = 0$. By the change

$$\bar{x} = x, \quad \bar{y} = -\frac{a_{21}}{a_{11}} - \frac{1}{a_{11}}x + y$$

we obtain the subgroup H_2^3 with $\alpha = 0$.

3.4.2.2.2. $a_{22} \neq 0$. Using the change

$$\bar{x} = \frac{1}{a_{11}}\left(\frac{a_{21}}{a_{22}} + x\right), \quad \bar{y} = -\frac{a_{21}}{a_{11}} - \frac{1}{a_{11}}x + y,$$

we get the subgroup H_2^3 with $\alpha \neq 0$.

We omit the rest of the cases for a_{ij} from (4.3) because they are reduced to preceding ones.

4. $a_{14} \neq 0, a_{15} = 0$. Then we assume that $a_{14} = 1, a_{24} = 0$.

4.1. $a_{22} = a_{23} = a_{25} = 0$. Now $a_{21} \neq 0$ and putting $a_{21} = 1, a_{11} = 0$ we obtain 1.3.

4.2. $a_{22} \neq 0, a_{23} = a_{25} = 0$. We choose $a_{22} = 1, a_{12} = 0$ and we get 2.3.

4.3. $a_{23} \neq 0, a_{25} = 0$. We put $a_{23} = 1, a_{13} = 0$ and we get 3.3.

4.4. $a_{25} \neq 0$. We suppose $a_{25} = 1$ and the operators $Y_1 = a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + X_4, Y_2 = a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + X_5$ define a group iff

$$\begin{aligned} a_{11}(2a_{22} - 1) - a_{12}a_{21} &= 0, \\ a_{12}(a_{22} - 1) &= 0, \\ a_{13}a_{22} &= a_{21}. \end{aligned}$$

4.4.1. $a_{11} = a_{12} = 0, \quad a_{21} = a_{13}a_{22}$.

4.4.1.1. $a_{22} = 0$. We have $a_{21} = 0$ and replacing

$$\bar{x} = a_{13} + x, \quad \bar{y} = a_{23} + y \tag{4.4}$$

we find the subgroup

$$H_2^{11'} = \{X_4, X_5\}.$$

4.4.1.2. $a_{22} \neq 0$. Now by the change (4.4) we get the subgroup

$$H_2^{11''} = \{X_4, \alpha X_2 + X_5 \mid \alpha \neq 0, \alpha \in \mathrm{R}\}.$$

Unifying the last two cases we have the subgroup

$$H_2^{11} = \{X_4, \alpha X_2 + X_5 \mid \alpha \in \mathrm{R}\}.$$

4.4.2. $a_{12} = 0$, $a_{21} = \frac{1}{2}a_{13}$, $a_{22} = \frac{1}{2}$.

4.4.2.1. $a_{11} = 0$. By the change (4.4) we get the subgroup H_2^{11} with $\alpha = \frac{1}{2}$.

4.4.2.2. $a_{11} \neq 0$. Applying (4.4) we obtain the subgroup

$$H_2^{12} = \{X_2 + 2X_5, \alpha X_1 + X_4 \,|\, \alpha \neq 0,\ \alpha \in \mathrm{R}\}.$$

4.4.3. $a_{11} = a_{12}a_{13}$, $a_{21} = a_{13}$, $a_{22} = 1$.

4.4.3.1. $a_{12} = 0$. We have $a_{11} = 0$ and making the substitution (4.4) we find the subgroup H_2^{11} with $\alpha = 1$.

4.4.3.2. $a_{12} \neq 0$. Replacing

$$\bar{x} = a_{13} + x, \quad \bar{y} = a_{12}a_{23} - a_{13} - x + a_{12}y$$

we obtain the subgroup H_2^6.

5. $a_{15} \neq 0$. We put $a_{15} = 1, a_{25} = 0$.

5.1. $a_{22} = a_{23} = a_{24} = 0$. Then $a_{21} \neq 0$ and taking $a_{21} = 1, a_{11} = 0$ we obtain 1.4.

5.2. $a_{22} \neq 0, a_{23} = a_{24} = 0$. We choose $a_{22} = 1, a_{12} = 0$ and we obtain 2.4.

5.3. $a_{23} \neq 0, a_{24} = 0$. We suppose $a_{23} = 1, a_{13} = 0$ and we have 3.4.

5.4. $a_{24} \neq 0$. Now we put $a_{24} = 1, a_{13} = 0$ and we get 4.4.

Thus the following theorem is stated:

Theorem 3. *The two-parametric subgroups of H_5 can be reduced to one of the subgroups*

$$\begin{aligned}
H_2^1 &= \{X_1, X_2\}, \\
H_2^2 &= \{X_1, \alpha X_2 + X_3 \,|\, \alpha \in R\}, \\
H_2^3 &= \{X_1, \alpha X_2 + X_5 \,|\, \alpha \in R\}, \\
H_2^4 &= \{X_2, X_3\}, \\
H_2^5 &= \{X_2, X_4\}, \\
H_2^6 &= \{X_2, X_5\}, \\
H_2^7 &= \{X_3, \alpha X_1 + X_4 \,|\, \alpha \neq 0,\ \alpha \in R\}, \\
H_2^8 &= \{X_4, \alpha X_2 + X_3 \,|\, \alpha \in R\}, \\
H_2^9 &= \{X_3, \alpha X_1 + X_5 \,|\, \alpha \in R\}, \\
H_2^{10} &= \{X_3, \alpha X_2 + \beta X_4 + X_5 \,|\, \alpha \neq 0,\ \alpha \in R\}, \\
H_2^{11} &= \{X_4, \alpha X_2 + X_5 \,|\, \alpha \in R\}, \\
H_2^{12} &= \{X_2 + 2X_5, \alpha X_1 + X_4 \,|\, \alpha \neq 0,\ \alpha \in R\}.
\end{aligned}$$

5. One-parametric subgroups of H_5

An one-parametric subgroup of H_5 can be defined by an operator of the form

$$Y = \sum_{i=1}^{5} b_i X_i,$$

where $b_i \in \mathbf{R}$. We distinguish the following cases:

1. $b_2 = b_3 = b_4 = b_5 = 0$. Consequently $b_1 \neq 0$ and putting $b_1 = 1$ we obtain the subgroup

$$H_1^1 = \{X_1\}.$$

2. $b_2 \neq 0, b_3 = b_4 = b_5 = 0$. Now we suppose $b_2 = 1$. We make the change

$$\bar{x} = b_1 + x, \quad \bar{y} = y$$

and we get the subgroup

$$H_1^2 = \{X_2\}.$$

3. $b_3 \neq 0, b_4 = b_5 = 0$. We assume $b_3 = 1$.

3.1. $b_1 = b_2 = 0$. In this case we have the subgroup

$$H_1^3 = \{X_3\}.$$

3.2. $b_1 \neq 0, b_2 = 0$. Replacing

$$\bar{x} = \frac{1}{b_1}x, \quad \bar{y} = -\frac{1}{b_1}x + y$$

we get again the subgroup H_1^1.

3.3. $b_2 \neq 0$. By the change

$$\bar{x} = b_1 + b_2 x, \quad \bar{y} = y$$

we obtain the subgroup

$$H_1^4 = \{\alpha X_2 + X_3 \,|\, \alpha \neq 0, \alpha \in \mathrm{R}\}.$$

4. $b_4 \neq 0, b_5 = 0$. We choose $b_4 = 1$.

4.1. $b_1 = b_2 = 0$. Applying

$$\bar{x} = b_3 + x, \quad \bar{y} = y \tag{5.1}$$

we find the subgroup

$$H_1^5 = \{X_4\}.$$

4.2. $b_1 \neq 0, b_2 = 0$. Replacing (5.1) we get the subgroup

$$H_1^6 = \{\alpha X_1 + X_4 \mid \alpha \neq 0, \alpha \in \mathbb{R}\}.$$

4.3. $b_1 = 0, b_2 \neq 0$.

4.3.1. $b_3 \neq 0$. We make the substitution

$$\bar{x} = -\frac{1}{b_3}x, \quad \bar{y} = -\frac{1}{b_2}x + y$$

and we obtain the subgroup H_1^4.

4.3.2. $b_3 = 0$. Using the change

$$\bar{x} = x, \quad \bar{y} = -\frac{1}{b_2}x + y$$

we find the subgroup H_1^2.

4.4. $b_1 \neq 0, b_2 \neq 0$.

4.4.1. $b_1 - b_2 b_3 = 0$. By the change

$$\bar{x} = \frac{b_1}{b_2} + x, \quad \bar{y} = -\frac{1}{b_2}x + y \tag{5.2}$$

we get again the subgroup H_1^2.

4.4.2. $b_1 - b_2 b_3 \neq 0$. Replacing (5.2) we obtain the subgroup H_1^4.

5. $b_5 \neq 0$. We put $b_5 = 1$.

5.1. $b_1 = b_2 = 0$. After the substitution

(5.3) $$\bar{x} = x, \quad \bar{y} = b_3 + b_4 x + y$$

we find the subgroup

$$H_1^7 = \{X_5\}.$$

5.2. $b_1 \neq 0, b_2 = 0$. Now we make the change

$$\bar{x} = \frac{1}{b_1} x, \quad \bar{y} = b_1 b_4 + b_3 + b_4 x + y$$

and we get the subgroup

$$H_1^8 = \{X_1 + X_5\}.$$

5.3. $b_1 = 0, b_2 \neq 0$.

5.3.1. $b_4 = 0$. Replacing

$$\bar{x} = x, \quad \bar{y} = b_3 + y$$

we obtain the subgroup

$$H_1^9 = \{\alpha X_2 + X_5, \mid \alpha \neq 0, \ \alpha \in \mathrm{R}\}.$$

5.3.2. $b_4 \neq 0$. Using (5.3) we find the subgroup

$$H_1^{10} = \{\alpha X_2 + \beta X_4 + X_5, \mid \alpha\beta \neq 0, \ \alpha, \beta \in \mathrm{R}\}.$$

5.4. $b_1 \neq 0, b_2 \neq 0$.

5.4.1. $b_4 = 0$. We make the change

$$\bar{x} = b_1 + b_2 x, \quad \bar{y} = b_3 + y$$

and we get the subgroup H_1^9.

5.4.2. $b_4 \neq 0$. By the substitution

$$\bar{x} = b_1 + b_2 x, \ \bar{y} = b_3 + b_4 x + y$$

we obtain the subgroup H_1^{10}.

Re-numbering some groups we have:

Theorem 4. *The one-parametric subgroups of H_5 can be reduced to one of the subgroups*

$$H_1^1 = \{X_1\}, \qquad H_1^2 = \{X_2\},$$
$$H_1^3 = \{X_3\}, \qquad H_1^4 = \{X_4\},$$
$$H_1^5 = \{X_5\}, \qquad H_1^6 = \{\alpha X_1 + X_4, \mid \alpha \neq 0, \alpha \in R\},$$
$$H_1^7 = \{X_1 + X_5\}, \qquad H_1^8 = \{\alpha X_2 + X_3, \mid \alpha \neq 0, \alpha \in R\},$$
$$H_1^9 = \{\alpha X_2 + X_5, \mid \alpha \neq 0, \alpha \in R\} \quad H_1^{10} = \{\alpha X_2 + \beta X_4 + X_5, \mid \alpha\beta \neq 0, \alpha, \beta \in R$$

Remark 4. Geometrical interpretations of one-parametrical subgroups of H_5 are given in [2] and [7].

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