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Principal Functions of the Non-Selfadjoint Operator Generated by System of Differential Equations [1]

Elgiz Bairomov * *and Esra Kir* **

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In this article, we have defined an operator L by system of differential equations of the first order in the space $L_2(0,\infty;\mathbf{C}_2)$. Discussing the principal functions of L, we have proved that the principal functions of L corresponding to the eigenvalues are in $L_2(0,\infty;\mathbf{C}_2)$ and the principal functions corresponding to the spectral singularities are in another Hilbert space that contains $L_2(0,\infty;\mathbf{C}_2)$.

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1. Introduction and preleminaries

The spectral analysis of the non-selfadjoint differential operators with continuous and point spectrum has been investigated by Naimark [9]. He has proved that some of the poles of the resolvent are not the eigenvalues of the operator and also shown that they are on the continuous spectrum. In [12] these points are called spectral singularities. So, in [9], it has been shown that spectral singularities have a special role in the spectral analysis of an operator. In [7], the properties of the principal functions corresponding to the spectral singularities are studied and the effect of the spectral singularities to the spectral expansion is investigated. Some problems of the spectral analysis of non-selfadjoint operators with spectral singularities are studied in [1] – [3], [5], [8].

Let us consider the non-selfadjoint operator L generated in $L_2(0,\infty;\mathbf{C}_2)$ by the system of differential equations

$$(1)\qquad \begin{aligned} &i\frac{d}{dx}\, u_1(x,\lambda)+q_1(x)\, u_2(x,\lambda)=\lambda\, u_1(x,\lambda)\\ &-i\frac{d}{dx}\, u_2(x,\lambda)+q_2(x)\, u_1(x,\lambda)=\lambda\, u_2(x,\lambda)\qquad x\in[0,\infty)\end{aligned}$$

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and the boundary condition

$$(2) \qquad u_2(0,\lambda) - h\, u_1(0,\lambda) = 0\,,$$

where q_i, $i = 1,2$ are complex valued functions, h is a complex number and

$$L_2(0,\infty;\mathbf{C}_2) := \left\{ f: \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \int_0^\infty \left\{ |f_1(x)|^2 + |f_2(x)|^2 \right\} dx < \infty \right\}.$$

By applying the transformation of

$$y_1(x,\lambda) = \frac{1}{2}\left[u_1(x,\lambda) + u_2(x,\lambda)\right],\ y_2(x,\lambda) = \frac{1}{2i}\left[u_2(x,\lambda) - u_1(x,\lambda)\right]$$

to the system (1) we can see that it has become the following

$$(3) \qquad B\frac{d}{dx}\, y(x,\lambda) + \Omega(x)\, y(x,\lambda) = \lambda\, y(x,\lambda)\,,$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ y(x,\lambda) = \begin{pmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{pmatrix},\ \Omega(x) = \begin{pmatrix} p(x)\ , & q(x) \\ q(x)\ , & -p(x) \end{pmatrix}$$

$$p(x) = \frac{1}{2}\left[q_1(x) + q_2(x)\right],\ q(x) = \frac{i}{2}\left[q_1(x) - q_2(x)\right]$$

The equation (3) is called the canonical Dirac system.

Assume that,

$$(4) \qquad |q_i(x)| \le c\, \exp(-\epsilon\sqrt{x}),\quad \epsilon > 0,\quad i = 1,2$$

where $c > 0$ is constant.

This paper is a continuation of [1] and we use the notation therein.

Let us consider the following vector solutions of the equation (1)

$$(5)\ e^{(1)}(x,\lambda) = \begin{pmatrix} e_1^{(1)}(x,\lambda) \\ e_2^{(1)}(x,\lambda) \end{pmatrix} = \begin{pmatrix} e^{-i\lambda x} + e^{-i\lambda x}\int_0^\infty H_{11}(x,x+t)e^{-i\lambda t}dt \\ e^{-i\lambda x}\int_0^\infty H_{21}(x,x+t)e^{-i\lambda t}dt \end{pmatrix},$$

$$(6) \quad e^{(2)}(x,\lambda) = \begin{pmatrix} e_1^{(2)}(x,\lambda) \\ e_2^{(2)}(x,\lambda) \end{pmatrix} = \begin{pmatrix} e^{i\lambda x}\int_0^\infty H_{12}(x,x+t)e^{i\lambda t}dt \\ e^{i\lambda x} + e^{i\lambda x}\int_0^\infty H_{22}(x,x+t)e^{i\lambda t}dt \end{pmatrix},$$

where the matrix function

$$H(x,t) = \begin{pmatrix} H_{11}(x,t) & , & H_{12}(x,t) \\ H_{21}(x,t) & , & H_{22}(x,t) \end{pmatrix}$$

has the role of a kernel of the operator transformation in the quantum scattering theory ([1] , [11]).

The solution $e^{(1)}(x,\lambda)$ is analytic with respect to λ in lower half-plane ($Im\lambda < 0$) and continuous up to the real axis and $e^{(2)}(x,\lambda)$ is analytic in upper half-plane and continuous up to the real axis ([11]).

It is obvious that under the condition (4), the following inequality hold ([1]) :

$$(7) \qquad |H_{ij}(x,t)| \leq c \exp\left\{-\epsilon\sqrt{\frac{x+t}{2}}\right\}, \quad \epsilon > 0, \quad i,\ j = 1,2$$

where $c > 0$ is a constant.

Let us define the following functions:

$$D_{+}(\lambda) = e_2^{(2)}(0,\lambda) - h\, e_1^{(2)}(0,\lambda)$$

$$D_{-}(\lambda) = e_2^{(1)}(0,\lambda) - h\, e_1^{(1)}(0,\lambda)\,.$$

Also, $\sigma_d(L)$ and $\sigma_{ss}(L)$ will denote the eigenvalues and the spectral singularities of the operator L, respectively.

Theorem 1.1. ([1]).

$a)\sigma_d(L) = \{\lambda : Im\lambda > 0, \quad D_{+}(\lambda) = 0\} \cup \{\lambda : Im\lambda < 0, \quad D_{-}(\lambda) = 0,\}$

$b)\sigma_{ss}(L) = \{\lambda : Im\lambda = 0, \quad D_{+}(\lambda) = 0\} \cup \{\lambda : Im\lambda = 0, \quad D_{-}(\lambda) = 0\}\,.$

Theorem 1.2. ([1]). *The operator L has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity, if condition (4) holds.*

Let

$$\varphi(x,\lambda) = \begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_2(x,\lambda) \end{pmatrix}$$

be the solution of the equation (1) subject to the intial conditions

$$\varphi_1(0,\lambda) = 1, \quad \varphi_2(0,\lambda) = h\,.$$

The solution

$$\varphi(x,\lambda) = \begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_2(x,\lambda) \end{pmatrix}$$

exists, is unique and an entire function of λ ([10]).

2. Principal functions

Let us express the system (1) as

$$\left[J\frac{d}{dx} + Q(x) - \lambda\right] u(x,\lambda) = 0\,, \tag{8}$$

where

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ Q(x) = \begin{pmatrix} 0 & q_1(x) \\ q_2(x) & 0 \end{pmatrix},\ u(x,\lambda) = \begin{pmatrix} u_1(x,\lambda) \\ u_2(x,\lambda) \end{pmatrix}.$$

We define

$$W\left[y^{(1)}(x,\lambda), y^{(2)}(x,\lambda)\right] = y_1^{(1)}(0,\lambda)\, y_2^{(2)}(0,\lambda) - y_2^{(1)}(0,\lambda)\, y_1^{(2)}(0,\lambda) \tag{9}$$

as the Wronskian of

$$y^{(1)}(x,\lambda) = \begin{pmatrix} y_1^{(1)}(x,\lambda) \\ y_2^{(1)}(x,\lambda) \end{pmatrix},\quad y^{(2)}(x,\lambda) = \begin{pmatrix} y_1^{(2)}(x,\lambda) \\ y_2^{(2)}(x,\lambda) \end{pmatrix}$$

which are the solutions of (8) (or of the system (1)). It is obvious from (9) that

$$W\left[\varphi(x,\lambda), e^{(2)}(x,\lambda)\right] = D_+(\lambda),\quad W\left[\varphi(x,\lambda), e^{(1)}(x,\lambda)\right] = D_-(\lambda). \tag{10}$$

In the following we use the notations

$$\mathbf{R} = (-\infty,\infty),\ \ \mathbf{C}_+ = \{\lambda : Im\lambda > 0\},\ \ \mathbf{C}_- = \{\lambda : Im\lambda < 0\}\ .$$

Let $\lambda_1^+, ..., \lambda_j^+$ and $\mu_1^-, ..., \mu_\ell^-$ denote the zeros of D_+ in $\mathbf{C}_+$ and D_- in $\mathbf{C}_-$ (i.e. the eigenvalues of L) with multiplicity $m_1^+, ..., m_j^+$ and $m_1^-, ..., m_\ell^-$, respectively. Similarly, let $\lambda_1, ..., \lambda_\alpha$ and $\mu_1, ..., \mu_\upsilon$ be the zeros of D_+ and D_- on the real axis (i.e. the spectral singularities of L) with multiplicities $m_1, ..., m_\alpha$ and $n_1, ..., n_\upsilon$, respectively. It is trivial that from (10)

$$\left\{\frac{\partial^n}{\partial\lambda^n} W\left[\varphi(x,\lambda), e^{(2)}(x,\lambda)\right]\right\}_{\lambda=\lambda_k^+} = \left\{\frac{d^n}{d\lambda^n} D_+(\lambda)\right\}_{\lambda=\lambda_k^+} = 0 \tag{11}$$

for $n = 0, 1, ..., m_k^+ - 1,\ \ i = 1, 2, ..., j$ and

$$(12)\qquad \left\{\frac{\partial^p}{\partial\lambda^p} W\left[\varphi(x,\lambda), e^{(1)}(x,\lambda)\right]\right\}_{\lambda=\mu_i^-} = \left\{\frac{d^p}{d\lambda^p} D_-(\lambda)\right\}_{\lambda=\mu_i^-} = 0$$

for $p = 0, 1, ..., m_i^- - 1,\ \ i = 1, 2, ..., \ell$. If $n = p = 0$ we get

$$(13)\qquad \varphi(x,\lambda_k^+) = a_0(\lambda_k^+)\, e^{(2)}(x,\lambda_k^+), \quad k = 1, 2, ..., j$$

and

$$(14)\qquad \varphi(x,\mu_i^-) = b_0(\mu_i^-)\, e^{(1)}(x,\mu_i^-), \quad i = 1, 2, ..., \ell\,.$$

So $a_0(\lambda_k^+) \neq 0$ and $b_0(\mu_i^-) \neq 0$.

Theorem 2.1. *The formulas*

$$(15)\qquad \left\{\frac{\partial^n}{\partial\lambda^n}\varphi(x,\lambda)\right\}_{\lambda=\lambda_k^+} = \sum_{i=0}^{n} \binom{n}{i} a_{n-i} \left\{\frac{\partial^i}{\partial\lambda^i} e^{(2)}(x,\lambda)\right\}_{\lambda=\lambda_k^+} ,$$

for $n = 0, 1, ..., m_k^+ - 1,\ \ k = 1, 2, ..., j$ *and*

$$(16)\qquad \left\{\frac{\partial^p}{\partial\lambda^p}\varphi(x,\lambda)\right\}_{\lambda=\mu_i^-} = \sum_{j=0}^{p} \binom{p}{j} b_{p-j} \left\{\frac{\partial^j}{\partial\lambda^j} e^{(1)}(x,\lambda)\right\}_{\lambda=\mu_i^-} ,$$

for $p = 0, 1, ..., m_i^- - 1,\ \ i = 1, 2, ..., \ell$ *hold, where the constants* $a_0, a_1, ..., a_n$ *and* $b_0, b_1, ..., b_p$ *depend on* λ_k^+ *and* μ_i^-*, respectively.*

Proof. Let us start with (15). We use mathematical induction. For $n = 0$ the proof is trivial from (13). Now let us prove (15) for $n = 1$. It is obvious from (11) that

$$(17)\qquad \left\{\frac{\partial}{\partial\lambda} W\left[\varphi(x,\lambda), e^{(2)}(x,\lambda)\right]\right\}_{\lambda=\lambda_k^+} = \left\{\frac{d}{d\lambda} D_+(\lambda)\right\}_{\lambda=\lambda_k^+} = 0\,.$$

If $u(x,\lambda)$ is a solution of (8), then

$$(18)\qquad \left\{J\frac{d}{dx} + Q(x) - \lambda\right\}\frac{\partial}{\partial\lambda}u(x,\lambda) = u(x,\lambda)$$

holds. Using (18), we have

$$(19)\qquad \left\{J\frac{d}{dx} + Q(x) - \lambda_k^+\right\}\left\{\frac{\partial}{\partial\lambda}\varphi(x,\lambda)\right\}_{\lambda=\lambda_k^+} = \varphi(x,\lambda_k^+)$$

$$\left\{J\frac{d}{dx}+Q(x)-\lambda_k^+\right\}\left\{\frac{\partial}{\partial\lambda}e^{(2)}(x,\lambda)\right\}_{\lambda=\lambda_k^+}=e^{(2)}(x,\lambda_k^+)\,. \tag{20}$$

From (13), (19) and (20) we get

$$\left\{J\frac{d}{dx}+Q(x)-\lambda_k^+\right\}f_1(x,\lambda_k^+)=0,$$

where

$$f_1(x,\lambda_k^+)=\left\{\frac{\partial}{\partial\lambda}\varphi(x,\lambda)\right\}_{\lambda=\lambda_k^+}-a_0(\lambda_k^+)\left\{\frac{\partial}{\partial\lambda}e^{(2)}(x,\lambda)\right\}_{\lambda=\lambda_k^+}.$$

By (17) we obtain

$$W\left[f_1(x,\lambda_k^+),\,e^{(2)}(x,\lambda_k^+)\right]=\left\{\frac{\partial}{\partial\lambda}W\left[\varphi(x,\lambda),\,e^{(2)}(x,\lambda)\right]\right\}_{\lambda=\lambda_k^+}=0.$$

Hence there exist a constant $a_1(\lambda_k^+)$ such that

$$f_1(x,\lambda_k^+)=a_1(\lambda_k^+)e^{(2)}(x,\lambda_k^+),$$

or

$$\left\{\frac{\partial}{\partial\lambda}\varphi(x,\lambda)\right\}_{\lambda=\lambda_k^+}=a_1(\lambda_k^+)e^{(2)}(x,\lambda_k^+)+a_0(\lambda_k^+)\left\{\frac{\partial}{\partial\lambda}e^{(2)}(x,\lambda)\right\}_{\lambda=\lambda_k^+}.$$

Thus (15) holds for $n=1$.

Let us assume that for $2\le n_0\le m_k^+-2$ the equality (15) holds; i.e.

$$\left\{\frac{\partial^{n_0}}{\partial\lambda^{n_0}}\varphi(x,\lambda)\right\}_{\lambda=\lambda_k^+}=\sum_{i=0}^{n_0}\binom{n_0}{i}a_{n_0-i}(\lambda_k^+)\left\{\frac{\partial^i}{\partial\lambda^i}e^{(2)}(x,\lambda)\right\}_{\lambda=\lambda_k^+}. \tag{21}$$

Now we prove that (15) holds for n_0+1, too. If $u(x,\lambda)$ is a solution of (8), then $\frac{\partial^n}{\partial\lambda^n}u(x,\lambda)$ satisfies

$$\left\{J\frac{d}{dx}+Q(x)-\lambda\right\}\frac{\partial^n}{\partial\lambda^n}u(x,\lambda)=n\frac{\partial^{n-1}}{\partial\lambda^{n-1}}u(x,\lambda)\,. \tag{22}$$

Writting (22) for $\varphi(x,\lambda_k^+)$ and $e^{(2)}(x,\lambda_k^+)$, and then using (21), we find

$$\left\{J\frac{d}{dx}+Q(x)-\lambda_k^+\right\}f_{n_0+1}(x,\lambda_k^+)=0\,,$$

where

$$f_{n_0+1}(x,\lambda_k^+) = \left\{\frac{\partial^{n_0+1}}{\partial\lambda^{n_0+1}}\varphi(x,\lambda)\right\}_{\lambda=\lambda_k^+}$$

$$-\sum_{i=0}^{n_0+1}\binom{n_0+1}{i} a_{n_0+1-i}(\lambda_k^+)\left\{\frac{\partial^i}{\partial\lambda^i}e^{(2)}(x,\lambda)\right\}_{\lambda=\lambda_k^+} .$$

From (11) we have

$$W\left[f_{n_0+1}(x,\lambda_k^+), e^{(2)}(x,\lambda_k^+)\right] = \left\{\frac{\partial^{n_0+1}}{\partial\lambda^{n_0+1}}W\left[\varphi(x,\lambda), e^{(2)}(x,\lambda)\right]\right\}_{\lambda=\lambda_k^+} = 0.$$

Hence there exists a constant $a_{n_0+1}(\lambda_k^+)$ such that

$$f_{n_0+1}(x,\lambda_k^+) = a_{n_0+1}(\lambda_k^+)e^{(2)}(x,\lambda_k^+) .$$

This shows that (15) holds for $n = n_0 + 1$. In a similar way we can prove that (16) also holds. ■

Theorem 2.2.

$$(23)\ \left\{\frac{\partial^n}{\partial\lambda^n}\varphi(.,\lambda)\right\}_{\lambda=\lambda_k^+} \in L_2(0,\infty;\mathbf{C_2}),\ n = 0,1,...,m_k^+ - 1,\ k = 1,2,...,j$$

$$(24)\ \left\{\frac{\partial^p}{\partial\lambda^p}\varphi(.,\lambda)\right\}_{\lambda=\mu_i^-} \in L_2(0,\infty;\mathbf{C_2}),\ n = 0,1,...,m_i^- - 1,\ i = 1,2,...,\ell.$$

Proof. From (6) we have

$$\left\{\frac{\partial^n}{\partial\lambda^n}e^{(2)}(x,\lambda)\right\}_{\lambda=\lambda_k^+} = \begin{pmatrix} \left\{\frac{\partial^n}{\partial\lambda^n}e_1^{(2)}(x,\lambda)\right\}_{\lambda=\lambda_k^+} \\ \left\{\frac{\partial^n}{\partial\lambda^n}e_2^{(2)}(x,\lambda)\right\}_{\lambda=\lambda_k^+} \end{pmatrix}$$

$$(25)= \begin{pmatrix} \int_0^\infty [i(x+t)]^n H_{12}(x,x+t)\exp\left[i\lambda_k^+(x+t)\right]dt \\ (ix)^n\exp(i\lambda_k^+x) + \int_0^\infty [i(x+t)]^n H_{22}(x,x+t)\exp\left[i\lambda_k^+(x+t)\right]dt \end{pmatrix}$$

for $n = 0, 1, ..., m_k^+ - 1$, $k = 1, 2, ..., j$. Using (7) and (25) we obtain

$$\left|\left\{\frac{\partial^n}{\partial\lambda^n} e_1^{(2)}(x,\lambda)\right\}_{\lambda=\lambda_k^+}\right| = \left|\int_0^\infty [i(x+t)]^n H_{12}(x, x+t) \exp\left[i\lambda_k^+(x+t)\right] dt\right|$$

$$\leq \int_0^\infty (x+t)^n \left|H_{12}(x, x+t)\right| \exp\left\{-(x+t) Im\lambda_k^+\right\} dt$$

$$\leq c\int_0^\infty (x+t)^n \exp\left\{-(x+t) Im\lambda_k^+\right\} \exp\left\{-\epsilon\sqrt{\frac{2x+t}{2}}\right\} dt$$

$$\leq c\exp(-\epsilon\sqrt{x}) \int_0^\infty (x+t)^n \exp\left\{-(x+t) Im\lambda_k^+\right\} dt$$

$$\leq c\exp(-\epsilon\sqrt{x}) \int_0^\infty t^n \exp\left\{-t\, Im\lambda_k^+\right\} dt\,. \tag{26}$$

Since $Im\lambda_k^+ > 0$, then we have

$$\int_0^\infty t^n \exp\left\{-t\, Im\lambda_k^+\right\} dt < \infty\,. \tag{27}$$

From (26) and (27) we get

$$\left|\left\{\frac{\partial^n}{\partial\lambda^n} e_1^{(2)}(x,\lambda)\right\}_{\lambda=\lambda_k^+}\right| < c_1 \exp(-\epsilon\sqrt{x}) \tag{28}$$

for $n = 0, 1, ..., m_k^+ - 1$, $k = 1, 2, ..., j$, where $c_1 > 0$ is a constant. In a similar way we can show that

$$\left|\left\{\frac{\partial^n}{\partial\lambda^n} e_2^{(2)}(x,\lambda)\right\}_{\lambda=\lambda_k^+}\right| \leq x^n \exp(-x\, Im\lambda_k^+) + c_2 \exp(-\epsilon\sqrt{x}) \tag{29}$$

for $n = 0, 1, ..., m_k^+ - 1$, $k = 1, 2, ..., j$, where $c_2 > 0$ is a constant.

By (25), (28) and (29) we obtain

$$\left\|\left\{\frac{\partial^n}{\partial\lambda^n} e^{(2)}(.,\lambda)\right\}_{\lambda=\lambda_k^+}\right\|^2_{L_2(0,\infty;C_2)} = \int_0^\infty \left|\left\{\frac{\partial^n}{\partial\lambda^n} e^{(2)}(x,\lambda)\right\}_{\lambda=\lambda_k^+}\right|^2 dx$$

$$= \int_0^\infty \left\{ \left| \left\{ \frac{\partial^n}{\partial \lambda^n} e_1^{(2)}(x,\lambda) \right\}_{\lambda=\lambda_k^+} \right|^2 + \left| \left\{ \frac{\partial^n}{\partial \lambda^n} e_2^{(2)}(x,\lambda) \right\}_{\lambda=\lambda_k^+} \right|^2 \right\} dx < \infty,$$

or

$$\left\{ \frac{\partial^n}{\partial \lambda^n} e^{(2)}(.,\lambda) \right\}_{\lambda=\lambda_k^+} \in L_2(0,\infty;\mathbf{C}_2) \tag{30}$$

for $n = 0, 1, ..., m_k^+ - 1$, $k = 1, 2, ..., j$. Using (15) and (30) we arrive to (23). (24) may be derived analogously. ■

$$\varphi(x,\lambda_k^+),\ \left\{ \frac{\partial}{\partial \lambda} \varphi(x,\lambda) \right\}_{\lambda=\lambda_k^+}, ..., \left\{ \frac{\partial^{m_k^+ - 1}}{\partial \lambda^{m_k^+ - 1}} \varphi(x,\lambda) \right\}_{\lambda=\lambda_k^+}$$

and

$$\varphi(x,\mu_i^-),\ \left\{ \frac{\partial}{\partial \lambda} \varphi(x,\lambda) \right\}_{\lambda=\mu_i^-}, ..., \left\{ \frac{\partial^{m_i^- - 1}}{\partial \lambda^{m_i^- - 1}} \varphi(x,\lambda) \right\}_{\lambda=\mu_i^-}$$

are called the principal functions corresponding to the eigenvalues $\lambda = \lambda_k^+$, $k = 1, 2, ..., j$ and $\lambda = \mu_i^-$, $i = 1, 2, ..., \ell$ of L, respectively. In the above, $\varphi(x,\lambda_k^+)$ and $\varphi(x,\mu_i^-)$ are eigenfunctions;

$$\left\{ \frac{\partial}{\partial \lambda} \varphi(x,\lambda) \right\}_{\lambda=\lambda_k^+}, ..., \left\{ \frac{\partial^{m_k^+ - 1}}{\partial \lambda^{m_k^+ - 1}} \varphi(x,\lambda) \right\}_{\lambda=\lambda_k^+}$$

are the associated functions of $\varphi(x,\lambda_k^+)$ and

$$\left\{ \frac{\partial}{\partial \lambda} \varphi(x,\lambda) \right\}_{\lambda=\mu_i^-}, ..., \left\{ \frac{\partial^{m_i^- - 1}}{\partial \lambda^{m_i^- - 1}} \varphi(x,\lambda) \right\}_{\lambda=\mu_i^-}$$

are the associated functions of $\varphi(x,\mu_i^-)$ ([6]).

If $\lambda_1, ..., \lambda_\alpha$ and $\mu_1, ..., \mu_v$ are spectral singularities of L (i.e. the real zeros of D_+ and D_-), then we obtain

$$\left\{ \frac{\partial^n}{\partial \lambda^n} W\left[\varphi(x,\lambda), e^{(2)}(x,\lambda)\right] \right\}_{\lambda=\lambda_k} = \left\{ \frac{d^n}{d\lambda^n} D_+(\lambda) \right\}_{\lambda=\lambda_k} = 0 \tag{31}$$

for $n = 0, 1, ..., m_k - 1$, $k = 1, 2, ..., \alpha$,

$$\left\{ \frac{\partial^p}{\partial \lambda^p} W\left[\varphi(x,\lambda), e^{(1)}(x,\lambda)\right] \right\}_{\lambda=\mu_i} = \left\{ \frac{d^p}{d\lambda^p} D_-(\lambda) \right\}_{\lambda=\mu_i} = 0 \tag{32}$$

for $p = 0, 1, ..., n_i - 1$, $i = 1, 2, ..., v$. Using (31) and (32) similarly to the proof of Theorem 2.1, we have the following remark.

Remark 2.3. The formulas

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(x,\lambda)\right\}_{\lambda=\lambda_k} = \sum_{i=0}^{n}\binom{n}{i} d_{n-i}(\lambda_k)\left\{\frac{\partial^i}{\partial\lambda^i}e^{(2)}(x,\lambda)\right\}_{\lambda=\lambda_k} \tag{33}$$

for $n = 0, 1, ..., m_k - 1$, $k = 1, 2, ..., \alpha$, and

$$\left\{\frac{\partial^p}{\partial\lambda^p}\varphi(x,\lambda)\right\}_{\lambda=\mu_i} = \sum_{j=0}^{p}\binom{p}{j} g_{p-j}(\mu_j)\left\{\frac{\partial^j}{\partial\lambda^j}e^{(1)}(x,\lambda)\right\}_{\lambda=\mu_i} \tag{34}$$

for $p = 0, 1, ..., n_i - 1$, $i = 1, 2, ..., v$ hold.

Lemma 2.4.

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(.,\lambda)\right\}_{\lambda=\lambda_k} \notin L_2(0,\infty;\mathbf{C_2}),\ n = 0, 1, ..., m_k - 1,\ \ k = 1, 2, ..., \alpha,$$

$$\left\{\frac{\partial^p}{\partial\lambda^p}\varphi(.,\lambda)\right\}_{\lambda=\mu_i} \notin L_2(0,\infty;\mathbf{C_2}),\ p = 0, 1, ..., n_i - 1,\ \ i = 1, 2, ..., v.$$

Proof. May be easily obtained from (33), (34) and

$$\left\{\frac{\partial^n}{\partial\lambda^n}e^{(2)}(x,\lambda)\right\}_{\lambda=\lambda_k} = \begin{pmatrix} \int_x^\infty (it)^n H_{12}(x,t)\exp(i\lambda_k t)dt \\ (ix)^n\exp(i\lambda_k x) + \int_x^\infty (it)^n H_{22}(x,t)\exp(i\lambda_k t)dt \end{pmatrix}, \tag{35}$$

$$\left\{\frac{\partial^p}{\partial\lambda^p}e^{(1)}(x,\lambda)\right\}_{\lambda=\mu_i} = \begin{pmatrix} (-ix)^p\exp(-i\mu_i x) + \int_x^\infty (-it)^p H_{11}(x,t)\exp(-i\mu_i t)dt \\ \int_x^\infty (-it)^p H_{21}(x,t)\exp(-i\mu_i t)dt \end{pmatrix} \tag{36}$$

Now let us introduce the Hilbert spaces:

$$H(0,\infty;\mathbf{C_2},m) = \left\{f : f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},\ \int_0^\infty (1+x)^{2m}\left\{|f_1(x)|^2 + |f_2(x)|^2\right\}dx < \infty\right\}$$

$$H(0,\infty;\mathbf{C_2},-m)=\left\{g:g=\begin{pmatrix}g_1\\g_2\end{pmatrix},\ \int\limits_0^\infty(1+x)^{-2m}\left\{|g_1(x)|^2+|g_2(x)|^2\right\}dx<\infty\right\}$$

with

$$\|f\|^2_{H(0,\infty;\mathbf{C_2},m)}=\int\limits_0^\infty(1+x)^{2m}\left\{|f_1(x)|^2+|f_2(x)|^2\right\}dx,$$

$$\|g\|^2_{H(0,\infty;\mathbf{C_2},-m)}=\int\limits_0^\infty(1+x)^{-2m}\left\{|g_1(x)|^2+|g_2(x)|^2\right\}dx,$$

respectively. It is evident that

$$H(0,\infty;\mathbf{C_2},0)=L_2(0,\infty;\mathbf{C_2}),$$

$$H(0,\infty;\mathbf{C_2},m)\subset_{\neq}L_2(0,\infty;\mathbf{C_2})\subset_{\neq}H(0,\infty;\mathbf{C_2},-m),\quad m=1,2,...$$

Let $H'(0,\infty;\mathbf{C_2},m)$ denote the dual of $H(0,\infty;\mathbf{C_2},m)$, which is isomorphic to $H(0,\infty;\mathbf{C_2},-m)$ ([4]). So for every functional $F\in H'(0,\infty;\mathbf{C_2},m)$ there is a function f^* belonging to $H(0,\infty;\mathbf{C_2},-m)$ such that

$$F(f)=\int\limits_0^\infty f(x)\,f^*(x)\,dx$$

for all $f\in H(0,\infty;\mathbf{C_2},m)$.

Theorem 2.5.

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(.,\lambda)\right\}_{\lambda=\lambda_k}\in H(0,\infty;\mathbf{C_2},-(n+1)),\ n=0,1,...,m_k-1,\ \ k=1,2,...,\alpha,\tag{37}$$

$$\left\{\frac{\partial^p}{\partial\lambda^p}\varphi(.,\lambda)\right\}_{\lambda=\mu_i}\in H(0,\infty;\mathbf{C_2},-(p+1)),\quad p=0,1,...,n_i-1,\quad i=1,2,...,v.\tag{38}$$

P r o o f. From (35) we have

$$\left|\left\{\frac{\partial^n}{\partial\lambda^n}e_1^{(2)}(x,\lambda)\right\}_{\lambda=\lambda_k}\right|\le\int\limits_x^\infty t^n\,|H_{12}(x,t)|\,dt\tag{39}$$

$$\left| \left\{ \frac{\partial^n}{\partial \lambda^n} e_2^{(2)}(x, \lambda) \right\}_{\lambda=\lambda_k} \right| \leq x^n + \int_x^\infty t^n \left| H_{22}(x, t) \right| dt \tag{40}$$

for $n = 0, 1, ..., m_k - 1$, $k = 1, 2, ..., \alpha$.

By the definition of the space $H(0, \infty; \mathbf{C}_2, -(n+1))$ and using (39), (40) we arrive to (37). In a similar way we can show that (38) also holds. ■

$$\varphi(x, \lambda_k), \left\{ \frac{\partial}{\partial \lambda} \varphi(x, \lambda) \right\}_{\lambda=\lambda_k}, ..., \left\{ \frac{\partial^{m_k - 1}}{\partial \lambda^{m_k - 1}} \varphi(x, \lambda) \right\}_{\lambda=\lambda_k}$$

and

$$\varphi(x, \mu_i), \left\{ \frac{\partial}{\partial \lambda} \varphi(x, \lambda) \right\}_{\lambda=\mu_i}, ..., \left\{ \frac{\partial^{n_i - 1}}{\partial \lambda^{n_i - 1}} \varphi(x, \lambda) \right\}_{\lambda=\mu_i}$$

are called the principal functions corresponding to the spectral singularities $\lambda = \lambda_k$, $k = 1, 2, ..., \alpha$ and $\lambda = \mu_i$, $i = 1, 2, ..., v$ of L, respectively.

Let us choose m_0 so that

$$m_0 = \max\{m_1, ..., m_\alpha, n_1, ..., n_v\}.$$

If we take

$$H_{m_0} = H(0, \infty; \mathbf{C}_2, m_0 + 1), \quad H_{-m_0} = H(0, \infty; \mathbf{C}_2, -(m_0 + 1)),$$

then

$$H_{m_0} \subset_{\neq} L_2(0, \infty; \mathbf{C}_2) \subset_{neq} H_{-m_0} .,$$

From Theorem 2.5 we have the following remark.

Remark 2.6.

$$\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(., \lambda) \right\}_{\lambda=\lambda_k} \in H_{-m_0}, \quad n = 0, 1, ..., m_k - 1, \ k = 1, 2, ..., \alpha,$$

$$\left\{ \frac{\partial^p}{\partial \lambda^p} \varphi(., \lambda) \right\}_{\lambda=\mu_i} \in H_{-m_0}, \quad p = 0, 1, ..., n_i - 1, \ i = 1, 2, ..., v.$$

Spectral expansion in term of the principal functions of L will be examined in a different work.

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* *Ankara University, Dept. of Mathematics*
06100 Beşevler Ankara - TURKEY

** *Gazi University, Dept. of Mathematics*
06500 Teknikokullar Ankara - TURKEY