Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol. 13, 1999, Fasc. 1-2

On the Multi-Integral Iteration Methods for Solving a Non-Linear Differential Methods

Salah M. El-Sayed

Presented by P. Kenderov

This research is concerned with the multi-integral iteration methods for solving a non-linear differential equations of fifth order with boundary conditions. The existence and uniqueness of the solution generated by the Picard method are established. The bounds on sup norm for the derivative of a certain function f contained in considered differential equations, are computed. The rate of convergence of the iterative sequence of approximate solution is discussed and obtained. Some numerical examples are given.

AMS Subj. Classification: 65L10

Key Words: multi-integral method, non-linear fifth order boundary value problem, Picard method, convergence of iteration sequence

1. Introduction

This paper deals with a class of non-linear boundary value problems arising in the scientific and industrial application problems. The solution of the following non-linear fifth order differential equation (NFDE) is considered:

$$\frac{d^5u}{dx^5} + f(x,u) = e(x), \qquad 0 < x < 1 \tag{1.1}$$

under the boundary conditions

$$u(0) = u'(0) = u'''(0) = u(1) = u'(1) = 0, \tag{1.2}$$

where $f(x,y)$ is a continuous real valued function for every $x \in [0,1]$, $y \in \mathbf{R}$, differentiable for every y, and $e(x) \in \mathbf{L}^1[0,1]$.

Several authors considered methods for solving nonlinear differential equations [2-5]. Gupta [1] established and proved theorems for the existence and uniqueness of the solution to the deformation of elastic beam equation $u^{(4)} + f(x,u) = e(x), 0 < x < 1$. Sharma and Gupta [7] applied the multi-integral iterative methods to obtain numerical solutions of non-linear fourth order differential equation under several different boundary conditions. Theorems for the existence and uniqueness of the solution were established. Also, the rate of convergence for the sequence generated by the iteration method under the upper bound $\sup|f_u| < 96$, was obtained. In [6] Salah used the multi-integral for solving a non-linear singular two-point boundary value problem.

The main goal of this paper is to present Picard type implicit methods to compute solutions to non-linear equation (1.1) under boundary condition (1.2). In Section 2, Theorem 1 gives a multi-integral representation of the solution. Theorem 2 proves that the condition $\sup|f_u| < 318.78437$ is sufficient for the existence of a unique solution of our problem. In Section 3, the errors and the rate of convergence are discussed and obtained for some examples.

The following notations are used throughout the paper. The sequence of points generated by every iteration is denoted $\{u^k\}$. A superscripted function means the value of the function evaluated at a particular point. For example, $u^k \equiv u(x_k)$, $e^k \equiv e(x_k)$, $f^k \equiv f(u_k, x)$ and so on. Finally, all the norms used in this paper are ℓ_∞-norms.

2. Multi-integral method

We will establish the solution of problem (1.1) under boundary conditions (1.2). The following theorem describes the method used for this purpose.

Theorem 2.1. *If $u(x)$ is the solution of (1.1), (1.2), the real valued function $f(u,x)$ is continuous for every $x \in [0,1]$, $y \in \Re$, differentiable for every y ,and $e(x) \in L^1[0,1]$, then*

$$u(x) = \frac{x^2}{2}(1-x^2)\int_0^1\int_0^t\int_0^s\int_0^r r\,[e(r)-f(r,u)]\,dr\,ds\,dt\,du$$

$$(2.1)\qquad +\int_x^1 \frac{1}{u^5}\int_0^u\int_0^t\int_0^s\int_0^r r\,[e(r)-f(r,u)]\,dr\,ds\,dt\,du\,dv.$$

Proof. Since $u(x)$ is a solution of (1.1),(1.2), then we have

$$u^{(5)}(x) = e(x) - f(x,u),$$

multiplying both said by r and take integration four times, we get

$$\int_0^x\int_0^t\int_0^s\int_0^r r\,u^{(5)}(r)\,dr\,ds\,dt\,dx = \int_0^x\int_0^t\int_0^s\int_0^r r\,[e(r)-f(r,u)]\,dr\,ds\,dt\,dx,$$

i.e.

$$x\, u'(x) - 4\, u(x) + 4\, u(0) + 3x\, u'(0) + x^2\, u''(0) + \frac{1}{6}\, x^3\, u'''(0)$$

$$= \int_0^x \int_0^t \int_0^s \int_0^r r\,[e(r) - f(r,u)]\, dr\, ds\, dt\, dx. \tag{2.2}$$

Since $u(0) = u'(0) = u'''(0) = 0$, then (2.2) becomes

$$x\, u'(x) - 4\, u(x) + x^2\, u''(0)$$

$$= \int_0^t \int_0^s \int_0^r r\,[e(r) - f(r,u)]\, dr\, ds\, dt\, dx. \tag{2.3}$$

But $u(1) = 0, u'(1) = 0$ (given); substituting x=1 in (2.3), we have

$$u''(0) = \int_0^1 \int_0^t \int_0^s \int_0^r r\,[e(r) - f(r,u)]\, dr\, ds\, dt\, dx. \tag{2.4}$$

Therefore, from (2.3) and (2.4) we have

$$x\, u'(x) - 4\, u(x) = -x^2 \int_0^1 \int_0^t \int_0^s \int_0^r r\,[e(r) - f(r,u)]\, dr\, ds\, dt\, dx$$

$$+ \int_0^x \int_0^t \int_0^s \int_0^r r\,[e(r) - f(r,u)]\, dr\, ds\, dt\, dx.$$

This implies

$$\left(\frac{u(x)}{x^4}\right)' = \frac{-1}{x^3} \int_0^1 \int_0^t \int_0^s \int_0^r r\,[e(r) - f(r,u)]\, dr\, ds\, dt\, dx$$

$$+ \frac{1}{x^5} \int_0^x \int_0^t \int_0^s \int_0^r r\,[e(r) - f(r,u)]\, dr\, ds\, dt\, dx. \tag{2.5}$$

Integrate both sides of (2.5) from a to x (for $a > 0$), we get

$$\frac{u(x)}{x^4} - \frac{u(a)}{a^4} = \frac{1}{2}\left(\frac{1}{x^2} - \frac{1}{a^2}\right) \int_0^1 \int_0^t \int_0^s \int_0^r r\,[e(r) - f(r,u)]\, dr\, ds\, dt\, dx$$

$$+ \int_a^x \frac{1}{u^5} \int_0^u \int_0^t \int_0^s \int_0^r r\,[e(r) - f(r,u)]\, dr\, ds\, dt\, du\, dv. \tag{2.6}$$

Substituting $x = 1$ in the above equation ($u(1) = 0$ given), we get

$$\frac{u(a)}{a^4} = -\frac{1}{2}\left(1 - \frac{1}{a^2}\right) \int_0^1 \int_0^t \int_0^s \int_0^r r\,[e(r) - f(r,u)]\, dr\, ds\, dt\, dx$$

$$(2.7) \qquad -\int_a^1 \frac{1}{u^5} \int_0^u \int_0^t \int_0^s \int_0^r r\,[e(r) - f(r,u)]\,dr\,ds\,dt\,du\,dv.$$

From (2.6), (2.7), we have

$$u(x) = \frac{x^2}{2}(1-x^2) \int_0^1 \int_0^t \int_0^s \int_0^r r\,[e(r) - f(r,u)]\,dr\,ds\,dt\,du$$

$$-x^4 \int_x^1 \frac{1}{u^5} \int_0^u \int_0^t \int_0^s \int_0^r r\,[e(r) - f(r,u)]\,dr\,ds\,dt\,du\,dv.$$

This completes the proof of the theorem. ■

The following theorem proves that the iterative sequence $\{u^k\}$ of the solution of (1.1), (1.2) is a Cauchy sequence.

Theorem 2.2. *If the iteration process*

$$u^{k+1}(x) = \frac{x^2}{2}(1-x^2) \int_0^1 \int_0^t \int_0^s \int_0^r r\,[e(r) - f(r,u^k)]\,dr\,ds\,dt\,du$$

$$(2.8) \qquad -x^4 \int_x^1 \frac{1}{u^5} \int_0^u \int_0^t \int_0^s \int_0^r r\,[e(r) - f(r,u^k)]\,dr\,ds\,dt\,du\,dv,$$

$k = 0, 1, \cdots$, $\sup|f_u| < 318.78437$ *and* $\|u^k\| \le C_1$, *then the sequence* $\{u^k\}$ *convergent.*

Proof. To proof the theorem, it is enough to show that the sequence $\{u^k\}$ is a Cauchy sequence. Indeed, for arbitrary n, m which are sufficiently large, we have:

$$(u^n - u^m) = \frac{x^2}{2}(1-x^2) \int_0^1 \int_0^t \int_0^s \int_0^r r\,[f(r,u^{m-1}) - f(r,u^{n-1})]\,dr\,ds\,dt\,du$$

$$(2.9) \qquad -x^4 \int_x^1 \frac{1}{u^5} \int_0^u \int_0^t \int_0^s \int_0^r r\,[f(r,u^{m-1}) - f(r,u^{n-1})]\,dr\,ds\,dt\,du\,dv.$$

Taking the absolute values of both sides of equation (2.9) and using the mean value theorem, we get

$$|u^n - u^m| \le \sup|f_u| \left\{ \frac{x^2}{2}(1-x^2) \int_0^1 \int_0^t \int_0^s \int_0^r r\,dr\,ds\,dt\,du \right.$$

$$\left. + x^4 \int_x^1 \frac{1}{u^5} \int_0^u \int_0^t \int_0^s \int_0^r r\,dr\,ds\,dt\,du\,dv \} \|u^{n-1} - u^{m-1}\| \right\}$$

$$= \sup|f_u|\, \|u^{n-1} - u^{m-1}\| \left(\frac{x^2 + x^4 - 2x^5}{240} \right).$$

Now we put $p(x) = x^2 + x^4 - 2x^5$. Then for $\epsilon > 0$, we get

$$\|u^n - u^m\| \leq \sup|f_u|\, |u^{n-1} - u^{m-1}| \frac{p_{\max}}{240} \leq \frac{\sup|f_u|}{318.78437} \|u^{n-1} - u^{m-1}\|$$

$$\leq K^m\, |u^{n-m} - u0| \leq 2C_1 \frac{K^m}{1-K} \leq \epsilon, \tag{2.10}$$

where $p_{\max} = p(318.78437)$ and $K = \dfrac{\sup|f_u|}{318.78437} \in (0,1)$.

Then the sequence u^k is a Cauchy sequence.

This completes the proof of the theorem. ∎

Corollary 2.1. *The number m of iterations of our method with convergence tolerance ϵ is at least*

$$m = \frac{\ell n\epsilon - \ell n2 - \ell nC_1}{\ell n \sup|f_u| - 5.76452} + 1.$$

Proof. From inequality (2.10), we have $2\, K^m\, C_1 \leq \epsilon$. Directly we can proof the lemma by taking the ℓn on both sides. ∎

The existence and uniqueness of the solution of non-linear differential equation (1.1), (1.2) are established from the above two theorems under the condition $\sup|f_u| < 318.78437$.

3. Numerical results

We conclude the paper by reporting some numerical results, obtained from a set of test NFDE problems. These numerical results describe the performance of the algorithm. The tables indicate the convergence pattern of the iterative sequence of approximate solution. In all these examples Simpson's method is used to approximate the integrals. In the tables ε_u^k denotes $\|u - u^k\|$, ε^k denotes $\|u^k - u^{k-1}\|$ and R^k denotes $\varepsilon^{k-1}/\varepsilon^k$ (the rate of convergence). The algorithm has been tested on the following set of problems.

Example 1.

$$\frac{d^5u}{dx^5} + x^2 e^{-u^2}\, u = 360\,(2\,x - 1) + x^5(x-1)^3\, e^{-x^6(x-1)^6} \tag{3.1}$$

with boundary conditions

$$u(0) = u'(0) = u'''(0) = u(1) = u'(1) = 0. \tag{3.2}$$

The solution to (3.1), (3.2) is $u(x) = x^3(x-1)^3$.

Table (3.1): Error analysis for (3.1), (3.2)

$Iter$	ε_u^k	ε^k	R^k
1	1.21347E-02	4.02383E-02	—
2	1.19513E-02	3.97884E-04	1.01131E+02
3	1.10029E-02	6.30085E-06	6.31477E+01
4	1.10028E-02	1.93863E-07	3.25016E+01

Example 2.

$$\frac{d^5u}{dx^5} + x\,e^{-u}u = C_3^8\,x^3 - 2C_2^7\,x^2 + 720\,x + \left(x^9 - 2x^8 + x^7\right)\,e^{-x^6(1-x)^2} \tag{3.3}$$

with boundary conditions

$$u(0) = u'(0) = u'''(0) = u(1) = u'(1) = 0. \tag{3.4}$$

The solution to (3.3), (3.4) is $u(x) = x^6(1-x)^2$.

Table (3.2): Error analysis for (3.3), (3.4)

$Iter$	ε_u^k	ε^k	R^k
1	1.23341E-04	4.65788E-02	—
2	1.89411E-04	5.50176E-04	8.46616E+01
3	1.65025E-04	2.44946E-05	2.24611E+01
4	1.44018E-04	8.96630E-07	2.73185E+01
5	1.44018E-04	3.86749E-08	2.31838E+01

Example 3.

$$\frac{d^5u}{dx^5} + x^2\,e^{-u^3}u = 240\,(3\,x-1) + \left(x^6 - 2x^7 + x^6\right)\,e^{-x^{12}(1-x)^6} \tag{3.5}$$

with boundary conditions

$$u(0) = u'(0) = u'''(0) = u(1) = u'(1) = 0. \tag{3.6}$$

The solution to (3.3), (3.4) is $u(x) = x^4(1-x)^2$.

Table (3.3): Error analysis for (3.5), (3.6)

$Iter$	ε_u^k	ε^k	R^k
1	1.76528E-03	3.64581E-03	—
2	1.86320E-03	5.09152E-04	7.16055E+00
3	1.46946E-03	8.97415E-06	5.67354E+01
4	1.46811E-03	1.80045E-07	4.98439E+01
5	1.46811E-03	6.32516E-08	2.84649E+01

In the above three tables column 1 represents the number of iteration, column 2 represents the comparison between the iterative solution and the exact solution and column 4 represents the rate of convergence of the iterative solution. We observe that in column 2 the accuracy achieved is not very high. But this accuracy is still better than the finite difference method [7]. The accuracy in the approximate solution can be improved significantly by using smaller grid size in Simpson's method or by using more accurate approximations for integrals. Which cannot be said about finite difference or finite element. Our methods on the other hand can be applied to NFDE with various boundary value problems. Also, they can be applied to all the natural, scientific and industrial non-linear boundary value problems.

References

[1] C.P. G u p t a, Existence and uniqueness theorems for the bending of an elastic beam equation, *Applicable Analysis*. To appear.

[2] M.K. J a i n, S.R.K. T y e n g e r, J.S.V. S a l d a n a h a, Numerical integration of a fourth order ordinary differential equation, *J. Engrg. Math.*, **11** (1977), 373-380.

[3] E.L. R e i s s, A.J. C a l l e g a r i, D.S. A h l u w a l i a, *Ordinary Differential Equations with Applications*, Holt, Rinehart and Winston, New York (1976).

[4] R i a z A. U s m a n i, M.J. M a r s d e n, Numerical solutions of some ordinary differential equations, *J. Engrg. Math.*, **9** (1975), 1-10.

[5] R i a z A. U s m a n i, A uniqueness the theorem for a boundary-value problem, *Mathematical Society*, **77** (1979), 329-335.

[6] S. M. E l - S a y e d, Multi-integral methods for nonlinear boundary-value problems, *Intern. J. Computer Math.*. To appear.

[7] R.K. S h a r m a, C.P. G u p t a, Iterative solutions to non-linear fourth order differential equations through multi-integral methods, *Intern. J. Computer Math.* **28** (1989), 219-226.

Dept. of Mathematics, Fac. of Science
Benha University, Benha 13518, EGYPT
e-mail: mselsayed@frcu.eun.eg

Received: 3.06.1997