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On a Binary Problem with Prime Numbers

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Let $1 < c < 17/16$, and N a sufficiently large integer. In this paper we prove that, almost all $n \in (N, 2N]$ can be represented as $n = [p_1{}^c] + [p_2{}^c]$, where $p_1, p_2 \leq N^{\frac{1}{c}}$ are prime numbers and $[x]$ denotes the integer part of x. Our method also yields an asymptotic formula for the number of representations of these n.

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1. Introduction

In [2] Tolev and the author proved that every sufficiently large integer N can be represented as $N = [p_1{}^c] + [p_2{}^c] + [p_3{}^c]$, where p_1, p_2, p_3 are prime numbers, c is a real number near to one, $1 < c < 17/16$, and $[x]$ denotes the integer part of x. The method also yields the following asymptotic formula for the weighted number of representations:

$$\sum_{[p_1{}^c]+[p_2{}^c]+[p_3{}^c]=N} \log p_1 \log p_2 \log p_3$$

$$(1) \qquad = \frac{\Gamma(1/c+1)^3}{\Gamma(3/c)} N^{\frac{3}{c}-1} + O(N^{\frac{3}{c}-1}\exp(-L^{\frac{1}{3}-\varepsilon})),$$

where Γ is the Euler gamma function, $L = \log N$ and $\varepsilon > 0$ is arbitrarily small. The study of this equation was motivated by Tolev's work [3] on a problem of Piatetski-Shapiro.

In this paper we show how one could obtain an analogous result for the corresponding binary problem

$$(2) \qquad [p_1{}^c] + [p_2{}^c] = n, \quad c > 1.$$

More precisely, let

$$R(n) = \sum_{[p_1{}^c]+[p_2{}^c]=n} \log p_1 \log p_2,$$

where $N/2 < n \leq N$, N is a sufficiently large integer and the summation is over all primes $p_1, p_2 \leq N^{\frac{1}{c}}$ satisfying (2). Our result is as follows:

Theorem 1. *Let $1 < c < 17/16$, $A > 0$ and $0 < \varepsilon < 1/3$ be arbitrary constants. Then*

$$\sum_{N/2<n\leq N} |R(n) - \frac{\Gamma(1/c+1)^2}{\Gamma(2/c)} n^{\frac{2}{c}-1}|^2 \ll N^{\frac{4}{c}-1} \exp(-AL^{1/3-\varepsilon}). \tag{3}$$

Clearly, Theorem 1 implies the following

Corollary 1. *Let $1 < c < 17/16$, $A, B > 0$ and $0 < \varepsilon < 1/3$ be arbitrary constants. Then for all $n \in (N/2, N]$ but $O(N \exp(-BL^{1/3-\varepsilon}))$ exceptions, the equation (2) is solvable with primes $p_1, p_2 \leq N^{\frac{1}{c}}$ and we have*

$$R(n) = \frac{\Gamma(1/c+1)^2}{\Gamma(2/c)} n^{\frac{2}{c}-1} + O(N^{\frac{2}{c}-1} \exp(-AL^{1/3-\varepsilon})).$$

The constants in the $\ll$ and O-symbols depend on A, c and ε.

After this manuscript was completed, it was made known to the author by D.I. Tolev that A. Kumchev and T. Nedeva [1] have recently improved the results of [2] and [3]. Mainly, they proved that (1) holds for $1 < c < 12/11$. The author is very grateful to D.I. Tolev for this and some helpful comments.

2. Notation and some formulas

Our notations are standard in number theory. Moreover,
c is a real number such that $1 < c < 17/16$;
η is a positive number such that $\eta < 0,001$;
N is a sufficiently large integer;
n, m are integers; p denotes a prime number;
$[x]$ is the integer part of the real number x;

$||x|| = \min_{n\in\mathbf{Z}}|x-n|$; $e(x) = \exp(2\pi i x)$;
$L = \log N$; $Q = N^{1/c}$; $\omega = Q^{1-c-\eta}$; $E = \exp(-AL^{1/3-\varepsilon})$; $P = L^{\frac{4}{3}} E^{-\frac{2}{3}}$

$$S(x) = \sum_{p \le Q} \log p \; e(x[p^c]); \qquad G(x) = \frac{1}{c} \sum_{m \le N} m^{1/c-1} e(xm);$$
$$R_1 = \int_{-\omega}^{\omega} S(x)^2 e(-nx)dx; \qquad R_2 = \int_{\omega}^{1-\omega} S(x)^2 e(-nx)dx;$$
$$H = \int_{-1/2}^{1/2} G(x)^2 e(-nx)dx; \qquad H_1 = \int_{-\omega}^{\omega} G(x)^2 e(-nx)dx.$$

We recall the following formulas from [2] (see also [4], Ch. 2, for (4), (6) and (8)).

$$H = \frac{\Gamma(1/c+1)^2}{\Gamma(2/c)} n^{\frac{2}{c}-1} + O(n^{\frac{1}{c}-1}), \tag{4}$$

$$\max_{\omega \le x \le 1-\omega} |\, S(x) \,| \ll Q^{\frac{11+2c}{14}} \log^5 Q, \tag{5}$$

$$H_1 - H \ll \int_{\omega \le |x| \le 1/2} |\, G(x) \,|^2 \, dx \ll Q^{2-c-\nu}, \quad \text{for some } \nu > 0, \tag{6}$$

$$\int_{-\omega}^{\omega} |\, S(x) \,|^2 \, dx \ll Q^{2-c} \log^4 Q, \tag{7}$$

$$\int_{-1/2}^{1/2} |\, G(x) \,|^2 \, dx \ll Q^{2-c}, \tag{8}$$

$$\max_{|x| \le \omega} |\, S(x) - G(x) \,| \ll QE. \tag{9}$$

3. Proof of Theorem 1

It follows from (4) that

$$\begin{aligned} \sum_{N/2<n\le N} |R(n) - \frac{\Gamma(1/c+1)^2}{\Gamma(2/c)} n^{\frac{2}{c}-1}|^2 \;&\ll\; \sum_{N/2<n\le N} |R_1 - H|^2 + NQ^{2-2c} \\ &+\; \sum_{N/2<n\le N} |R_2|^2. \end{aligned} \tag{10}$$

By Bessel's inequality, Parseval's identity, the prime number theorem and (5) we have

$$\begin{aligned} \sum_{N/2<n\le N} |R_2|^2 \;&\le\; \int_{\omega}^{1-\omega} |S(x)|^4 dx \le (\max_{\omega \le x \le 1-\omega} |\, S(x) \,|)^2 \int_0^1 |S(x)|^2 dx \\ &\ll\; Q^{\frac{11+2c}{7}+1} L^6 \ll N^{\frac{4}{c}-1} E^{\frac{1}{3}}. \end{aligned} \tag{11}$$

The inequality (6) implies that

$$(12) \qquad \sum_{N/2<n\le N} |R_1 - H|^2 \ll \sum_{N/2<n\le N} |R_1 - H_1|^2 + NQ^{4-2c-2\nu}.$$

Hence, from (10), (11) and (12), it follows that (3) is implied by

$$(13) \qquad \sum_{N/2<n\le N} |R_1 - H_1|^2 \ll N^{\frac{4}{c}-1}E^{\frac{1}{3}}.$$

We write

$$|R_1 - H_1|^2 = \int_{-\omega}^{\omega} (\overline{S(x)}^2 - \overline{G(x)}^2)\left(\int_{-\omega}^{\omega} (S(y)^2 - G(y)^2)e(n(x-y))dy\right)dx.$$

Then, using the Cauchy-Schwarz inequality and the well-known estimate $\sum_{a<n\le b} e(nx) \ll \min(b-a, 1/||x||)$, from (7) and (8) we obtain

$$\sum_{N/2<n\le N} |R_1 - H_1|^2$$

$$(14) \qquad \ll Q^{3-\frac{3}{2}c}L^6\left(\sup_{|x|\le\omega}\int_{-\omega}^{\omega} |S(y)-G(y)|^2 \min\left(N, \frac{1}{||x-y||}\right)^2 dy\right)^{\frac{1}{2}}.$$

Now we note that, uniformly with respect to $|x| \le \omega$, we have

$$\int_{-\omega}^{\omega} |S(y)-G(y)|^2 \min\left(N, \frac{1}{||x-y||}\right)^2 dy$$

$$\ll N^2\int_{[x-\frac{P}{N},x+\frac{P}{N}]\cap[-\omega,\omega]} |S(y)-G(y)|^2 dy + \frac{N^2}{P^2}\int_{-\omega}^{\omega} |S(y)-G(y)|^2 dy.$$

We estimate the first integral by using (9) and the second one by using (7) and (8). Therefore, we get

$$\begin{aligned}\int_{-\omega}^{\omega} |S(y)-G(y)|^2 \min\left(N, \frac{1}{||x-y||}\right)^2 dy &\ll N^2Q^{2-c}(PE^2 + P^{-2}L^4)\\ &\ll N^2Q^{2-c}L^{\frac{4}{3}}E^{\frac{4}{3}}.\end{aligned}$$

We substitute this formula in (14) and then we obtain (13). Theorem 1 is proved.

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