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Some Remarks on Weighted Marcinkiewicz Inequalities

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Presented by V. Kiryakova

Dedicated to Prof. J. Szabados on his 60th birthday

We investigate the case when the necessary and sufficient conditions for the $L^p_u([-1,1])$ Marcinkiewicz inequalities, based on the zeros of the orthogonal polynomial $p_m(w)$, are not satisfied. Then, by suitable modification, we state new necessary and sufficient conditions for the obtained inequalities.

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1. Introduction and main results

Marcinkiewicz inequalities link up the L^p-weighted norm of a polynomial with a suitable quadrature sum.

Recently (see [14],[15],[2]) some Marcinkiewicz inequalities have been stated using the so-called generalized Ditzian-Totik weights, which are weight functions of the following type:

$$w(x) := \prod_{k=0}^{M} |x-\tau_k|^{\Gamma_k} \Phi_k(|x-\tau_k|^{\delta_k}), \quad |x| \le 1, \quad \Gamma_k > -1, \tag{1.1}$$

where

$$-1 = \tau_0 < \tau_1 < .. < \tau_{M-1} < \tau_M = 1,$$

$$\delta_k := \begin{cases} 1/2 & \text{if } k \in \{0, M\} \\ 1 & \text{otherwise} \end{cases},$$

and the functions Φ_k are either equal to 1, or they are concave moduli of continuity of the first order (i.e. Φ_k are semi-additive, nonnegative, continuous and nondecreasing functions on $[0,1]$ with $\Phi_k(0) = 0$ and $2\Phi_k(\frac{a+b}{2}) \geq \Phi_k(a) + \Phi_k(b), \forall a, b \in [0,1]$) verifying the following property:

$$\forall \epsilon > 0, \frac{\Phi_k(x)}{x^\epsilon} \text{ is a non-increasing function on [0,1]: } \lim_{x\to 0+} \frac{\Phi_k(x)}{x^\epsilon} = \infty.$$

If w is a generalized Ditzian-Totik weight, then we briefly write $w \in GDT$ and denote by $\{p_m(w)\}$ the corresponding system of orthonormal polynomials having positive leading coefficients.

For each $m \in \mathbb{N}$, let $\lambda_m(w,x) := \left[\sum_{k=0}^{m-1} p_k^2(w,x)\right]^{-1}$ be the $m-th$ Christoffel function corresponding to w and $x_k := x_{k,m}(w)$, $k = 1,..,m$, be the zeros of $p_m(w)$ (arranged in increasing order). Furthermore, we denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m and by L^p the space of all measurable functions such that $\|f\|_p := (\int_{-1}^1 |f|^p)^{\frac{1}{p}} < \infty$.

The following theorem is proved in [14].

Theorem. ([14, Theorem 2.7]) Let $1 < p < \infty$, $u, w \in GDT$, $u \in L^p$ and $\varphi(x) := \sqrt{1-x^2}$. Then the following statements are equivalent:

i) The weights u, w and φ are such that:

$$\frac{u}{\sqrt{w\varphi}} \in L^p \quad \text{and} \quad \frac{\sqrt{w\varphi}}{u} \in L^{p'}, \qquad p' := \frac{p}{p-1}. \tag{1.2}$$

ii) For all $m \in \mathbb{N}$ and $P \in \mathbb{P}_{m-1}$

$$\|Pu\|_p^p \leq C \sum_{i=1}^m \lambda_m(u^p, x_i)|P(x_i)|^p \tag{1.3}$$

holds, where C is a constant independent of m and P.

In the present paper we generalize this result showing how slight modifications on the system of knots in (1.3) influence the conditions on the weights u and w.

We explicitly note that the distribution of the knots x_i in (1.3) is of *arc cosine type* [21], i.e.

$$x_i = \cos\vartheta_i, \quad \vartheta_i - \vartheta_{i+1} \sim \frac{1}{m}, \quad i = 0, \dots, m, \tag{1.4}$$

where we set $x_0 = \cos\vartheta_0 \equiv -1$, $x_{m+1} = \cos\vartheta_{m+1} \equiv 1$. Remark that all the sets of knots we will consider in the sequel, have a similar arc cosine type distribution.

For the sake of simplicity, we restrict our consideration to the simplest case $u, w \in GDT$ having a unique (the same) zero or singularity inside the interval $[-1, 1]$, i.e.

$$(1.5)\quad w(x) := (1-x)^{\alpha}(1+x)^{\beta}|x-\tau|^{\varrho}\Phi_1(\sqrt{1-x})\Phi_2(\sqrt{1+x})\Phi_3(|x-\tau|),$$
$$(1.6)\quad u(x) := (1-x)^{\gamma}(1+x)^{\delta}|x-\tau|^{\sigma}\Psi_1(\sqrt{1-x})\Psi_2(\sqrt{1+x})\Psi_3(|x-\tau|).$$

Then all investigations can be immediately generalized to the more general Ditzian-Totik weights (1.1).

As a first problem, we study the construction of Marcinkiewicz inequalities of the type (1.3) involving also some preassigned points $\zeta_1, \ldots, \zeta_r$ in $[-1, 1]$.

This problem is not trivial, if we require that the number of the coefficients of the polynomial P is equal to the number of knots in the quadrature sum. The main difficulty is that the fixed points $\{\zeta_i\}$ can be "too close" to the knots $\{x_i\}$.

Using an idea from [3], [4], such problem can be overcome replacing the zeros closest to the fixed points by the latter ones. In this way, the new set of knots still has an arc cosine distribution (see [3]) and an inequality of the type (1.3) still holds.

More precisely, if we denote by $x_c(\zeta_j)$ the zero of $p_m(w)$ nearest to ζ_j, $j = 1, \ldots, r$ (i.e. $|x_c(\zeta_j) - \zeta_j| = \min_{1\le i\le m} |x_i - \zeta_j|$) and by $\mathcal{I} = \{i : x_i \neq x_c(\zeta_j), j = 1, \ldots, r\}$, then we can state the following result.

Theorem 1.1. *Let $1 < p < \infty$, $u, w \in GDT$ be given by (1.6) and (1.5) respectively, $u \in L^p$, $\varphi(x) = \sqrt{1-x^2}$ and $\zeta_1, \ldots, \zeta_r \in [-1, 1]$ fixed. Then for all $m \in \mathbb{N}$ sufficiently large and each $P \in \mathbb{P}_{m-1}$, there exists a positive constant C, independent of m and P, such that*

$$(1.7)\qquad \|Pu\|_p^p \le C\left[\sum_{i\in\mathcal{I}} \lambda_m(u^p, x_i)|P(x_i)|^p + \sum_{i=1}^{r} \lambda_m(u^p, \zeta_i)|P(\zeta_i)|^p\right]$$

holds if and only if

$$(1.8)\qquad \frac{u}{\sqrt{w\varphi}} \in L^p \quad \textit{and} \quad \frac{\sqrt{w\varphi}}{u} \in L^{p'}, \qquad p' := \frac{p}{p-1}.$$

We note that (1.2) and (1.8) give the same conditions for the weights u and w, hence the Marcinkiewicz inequalities (1.3) and (1.7) are equivalent.

Now let us examine the following question: what about the case when (1.2) is not satisfied? Is it still possible to state a Marcinkiewicz inequality on the zeros of $p_m(w)$ in $L^p_u = \{f : fu \in L^p\}$?

First we consider the case $\frac{u}{\sqrt{w\varphi}} \notin L^p$ but $\frac{\sqrt{w\varphi}}{u} \in L^{p'}$. In this case we succeed in obtaining a Marcinkiewicz inequality, involving the weight u and the zeros $x_{m,k}(w)$, provided some additional knots are used in the quadrature sum.

The idea of adding nodes to a system of zeros of orthogonal polynomials has been first introduced by Egerváry and Turán [6]. Subsequently these techniques have been extensively used by many authors in different contexts (see for instance [25], [24], [11], [13], and the references therein). Recently, this procedure has been used also in constructing Marcinkiewicz inequalities [15], [2].

Following the cited papers, we choose the additional knots near the singularities and/or the zeros of the weights i.e., fixed $\mu, \eta, \nu \in \mathbb{N}$, we define:

$$\begin{aligned} y_i &:= -1+\frac{i}{\mu+1}(1+x_1), & i=1,..,\mu,\\ z_i &:= x_m+\frac{i}{\eta+1}(1-x_m), & i=1,..,\eta,\\ \tau_i &:= \begin{cases} x_d+\frac{i}{\nu+1}(\tau-x_d) & \text{if } \tau-x_d\geq x_{d+1}-\tau \\ \tau+\frac{i}{\nu+1}(x_{d+1}-\tau) & \text{if } \tau-x_d\leq x_{d+1}-\tau \end{cases}, & i=1,..,\nu, \end{aligned}$$

where we have denoted by x_d and x_{d+1} the zeros of $p_m(w)$ which are closest to τ (for m sufficiently large, surely there exists $d \in \{1,..,m\}$ such that $x_d \leq \tau \leq x_{d+1}$).

We will use the following unified notation for the additional points:

$$t_i := \begin{cases} y_i & \text{if } i=1,..,\mu \\ \tau_{i-\mu} & \text{if } i=\mu+1,..,\mu+\nu \\ z_{i-\mu-\nu} & \text{if } i=\mu+\nu+1,..,\mu+\nu+\eta. \end{cases} \tag{1.9}$$

We explicitly note that the above points depend on m, but the numbers $\eta, \mu, \nu \in \mathbb{N}$ are independent of m.

Furthermore we remark that the system of knots $\{x_i\} \cup \{t_i\}$ has an arc cosine distribution and that other choices for the additional knots are possible on condition that such distribution is preserved (see [4]).

Set

$$v(x) := (1-x)^\eta (1+x)^\mu |x-\tau|^\nu, \qquad |x| \leq 1, \tag{1.10}$$

with η, μ, ν as in (1.9), we establish the following theorem.

Theorem 1.2. *Let $1 < p < \infty$, $w, u, v \in GDT$ be given by (1.5), (1.6), (1.10) respectively, $u \in L^p$ and $\varphi(x) := \sqrt{1-x^2}$. The following statements are equivalent:*

i) The weights w, u, v and φ satisfy:

$$\frac{uv}{\sqrt{w\varphi}} \in L^p \quad \text{and} \quad \frac{\sqrt{w\varphi}}{uv} \in L^{p'}, \qquad p' := \frac{p}{p-1}. \tag{1.11}$$

ii) For all $m \in \mathbb{N}$ (sufficiently large) and for each $P \in \mathbb{P}_{m+\eta+\mu+\nu-1}$

$$\|Pu\|_p^p \le C \left[\sum_{i=1}^{m} \lambda_m(u^p, x_i)|P(x_i)|^p + \sum_{i=1}^{\eta+\mu+\nu} \lambda_m(u^p, t_i)|P(t_i)|^p \right] \tag{1.12}$$

holds, where C is a constant independent of m and P and $\{t_i\}$ are given by (1.9).

Finally we consider the case $\frac{\sqrt{w\varphi}}{u} \notin L^{p'}$ but $\frac{u}{\sqrt{w\varphi}} \in L^p$.

We observe that this situation is peculiar with respect to the previous one, thus it is natural dropping, instead of adding, some suitable knots. The idea of dropping nodes has appeared for the first time in a paper by Horvát and Szabados [8].

We choose the knots to be dropped as follows. Taking v as in (1.10), i.e. $v(x) = (1-x)^\eta(1+x)^\mu|x-\tau|^\nu$, $\mu, \nu, \eta \in \mathbb{N}$, we remove μ zeros of $p_m(w)$ near -1, η zeros near 1 and ν zeros "around" τ. These last ones are chosen by turns on the left and the right of τ, starting from x_d, which is the zero closest to τ. Thus we state a Marcinkiewicz inequality on the following set of nodes

$$x_{\mu+1}, \ldots, x_{d-[\frac{\nu+1}{2}]}, x_{d+1+[\frac{\nu}{2}]}, \ldots, x_{m-\eta},$$

that, we remark, has still an arc cosine distribution.

Theorem 1.3. *Let $w, u, v \in GDT$ be given by (1.5), (1.6), (1.10) respectively and $\varphi(x) := \sqrt{1-x^2}$. If $1 < p < \infty$ and $u/v \in L^p$, the following statements are equivalent:*

i) The weights w, u, v and φ verify:

$$\frac{u}{v\sqrt{w\varphi}} \in L^p \quad \text{and} \quad \frac{v\sqrt{w\varphi}}{u} \in L^{p'}, \qquad p' := \frac{p}{p-1}. \tag{1.13}$$

ii) For all $m \in \mathbb{N}$ (sufficiently large) and for any $P \in \mathbb{P}_{m-(\eta+\mu+\nu)-1}$

$$\|Pu\|_p^p \le C \left[\sum_{i=\mu+1}^{d-[\frac{\nu+1}{2}]} \lambda_m(u^p, x_i)|P(x_i)|^p + \sum_{i=d+1+[\frac{\nu}{2}]}^{m-\eta} \lambda_m(u^p, x_i)|P(x_i)|^p \right] \tag{1.14}$$

holds, where d is the index of the zero x_d closest to τ and C is a constant independent of m and P.

Summing up Marcinkiewicz inequalities in L^p_u, involving the zeros of $p_m(w)$, can be constructed also when one of the two conditions in (1.2) is not verified, but the other one holds. In such cases we can look for three positive integers η, μ, ν such that the weight v, given by (1.10), satisfies (1.11) or (1.13). These integers "tell us" how many points we have to add or remove from the set of knots in (1.3).

We remark that it is not always possible to determine the numbers η, μ, ν since they must be integers. For example, taking $p = 2$, $u = 1$ and $w = \varphi$ the first condition in (1.2) is not fulfilled, but the second one holds.

On the other hand, in order that (1.11) is satisfied, the integers μ and η have to verify the condition

$$0 < \mu, \eta < 1, \quad \mu, \eta \in \mathbb{N},$$

and obviously this is impossible.

Finally, since v is bounded, if $\frac{u}{v\sqrt{w\varphi}} \in L^p$, then $\frac{u}{\sqrt{w\varphi}} \in L^p$, which implies that $\frac{uv}{\sqrt{w\varphi}} \in L^p$. On the other hand from $\frac{\sqrt{w\varphi}}{uv} \in L^{p'}$ it follows $\frac{\sqrt{w\varphi}}{u} \in L^{p'}$ and also $\frac{v\sqrt{w\varphi}}{u} \in L^{p'}$. Hence if one condition in (1.2) is relaxed, the other one has to be reinforced and then at least one of such conditions must be satisfied.

2. An application

Marcinkiewicz inequalities are important tools in several fields. One of these fields is the study of the boundedness of the Lagrange operator in some functional spaces (see for instance [23], [14], [15]).

In particular, the inequalities (1.7), (1.12) and (1.14) are equivalent to the boundedness of the corresponding Lagrange operators in some Besov subspaces of L^p_u. Here we illustrate such result just for the case of (1.7), since in the other cases it is possible to proceed in a similar way.

First of all, let us give some notations and preliminary results.

We denote by $C^0 = C^0[-1,1]$ the space of all continuous functions on $[-1,1]$, equipped with the norm $\|f\|_\infty = \sup_x |f(x)|$ and by C^0_{loc} the class of all locally continuous functions on $[-1,1]$ (i.e. continuous on each $[a,b] \subset [-1,1]$).

Now fix u as in (1.6), but with $\sigma = 0$ and $\Psi_3 = 1$, i.e. let

$$u(x) := (1-x)^\gamma (1+x)^\delta \Psi_1(\sqrt{1-x})\Psi_2(\sqrt{1+x}). \tag{2.1}$$

Very important subspaces of L^p_u are the Besov-type spaces defined by means of the following seminorms

$$\|f\|_{u,p,q,r} := \begin{cases} \left(\int_0^1 \left[\frac{\Omega^k_\varphi(f,t)_{u,p}}{t^r} \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} & 1 \leq q < \infty \\ \sup\limits_{t>0} \frac{\Omega^k_\varphi(f,t)_{u,p}}{t^r} & q = \infty \end{cases} \qquad k > r$$

where $r > 0$ and $\Omega^k_\varphi(f,t)_{u,p}$ is the $k-th$ weighted modulus of continuity [5]:

$$\Omega^k_\varphi(f,t)_{u,p} := \sup_{0<h\leq t} \|(\Delta^k_{h\varphi} f)u\|_{L^p(I_{hk})}$$

with $\varphi(x) := \sqrt{1-x^2}$, $\Delta^k_{h\varphi} f(x) := \sum_{i=0}^k (-1)^i \binom{k}{i} f(x + \frac{kh}{2}\varphi(x) - ih\varphi(x))$, $k \in \mathbb{N}$ and $I_{hk} := [-1 + 2h^2k^2, 1 - 2h^2k^2]$.

If $1 < p < \infty$, $1 \leq q \leq \infty$ and $r > 0$, by using the above quasi-norm we define the Besov norm:

$$\|f\|_{B^p_{r,q}(u)} := \|fu\|_p + \|f\|_{u,p,q,r}$$

and the corresponding Besov space:

$$B^p_{r,q}(u) := \{f \in L^p_u : \|f\|_{B^p_{r,q}(u)} < \infty\}.$$

Finally for each $f \in C^0$ and $m \in \mathbb{N}$, given an arbitrary set of knots in $[-1,1]$, $S = \{s_j, j = 1, \ldots, m\}$, the Lagrange polynomial which interpolates f on the set of nodes S is defined as

$$L_m(S,f,x) = \sum_{j=1}^m l_{m,j}(S,x) f(s_j), \tag{2.2}$$

where $l_{m,j}$ are the so-called fundamental Lagrange polynomials given by

$$l_{m,j}(S,x) = \prod_{i=1, i\neq j}^m \frac{x - s_i}{s_j - s_i}. \tag{2.3}$$

In the case S is the set of the zeros of $\{p_m(w)\}$ we will simply write $L_m(w,f)$ and $l_{m,j}(w)$.

Here we consider the Lagrange operator $L_{m+\eta+\mu+\nu}(X,f)$ interpolating at the points $X = \{x_k\}_{k=1}^m \bigcup \{t_i\}_{i=1}^{\eta+\mu+\nu}$, where $\{x_k\}_{k=1}^m$ are the zeros of $p_m(w)$ and $\{t_i\}_{i=1}^{\eta+\mu+\nu}$ are the $\eta+\mu+\nu$ points introduced in (1.9).

By Theorem we can deduce the following result about the behaviour of the Lagrange operator $L_{m+\eta+\mu+\nu}(X)$ in L^p_u.

Theorem 2.1. *Let u, w, v be given by (2.1), (1.5) and (1.10) respectively. Then the following statements are equivalent:*

i) The weights w, u, v and $\varphi(x) := \sqrt{1-x^2}$ are such that

$$\frac{uv}{\sqrt{w\varphi}} \in L^p \quad \textit{and} \quad \frac{\sqrt{w\varphi}}{uv} \in L^{p'}, \tag{2.4}$$

where $1 < p < \infty$ and $p' := \frac{p}{p-1}$.

ii) For each $m \in \mathbb{N}$, $1 < p < \infty$ and $f \in C^0_{loc}$ satisfying $\int_0^1 \frac{\Omega^k_\varphi(f,t)_{u,p}}{t^{1+\frac{1}{p}}} dt < \infty$, there exists a constant C independent of m and f, such that

$$\|L_{m+\eta+\mu+\nu}(X,f)u\|_p \le C \left[\|fu\|_p + \frac{1}{m^{\frac{1}{p}}} \int_0^{\frac{1}{m}} \frac{\Omega^k_\varphi(f,t)_{u,p}}{t^{1+\frac{1}{p}}} dt \right], \tag{2.5}$$

or equivalently

$$\|(f - L_{m+\eta+\mu+\nu}(X,f))u\|_p \le \frac{C}{m^{\frac{1}{p}}} \int_0^{\frac{1}{m}} \frac{\Omega^k_\varphi(f,t)_{u,p}}{t^{1+\frac{1}{p}}} dt. \tag{2.6}$$

iii) For each $m \in \mathbb{N}$, $1 \le q \le \infty$, $1 < p < \infty$, $s > \frac{1}{p}$ and $f \in B^p_{s,q}(u) \bigcap C^0_{loc}$ it results

$$\|L_{m+\eta+\mu+\nu}(X,f)\|_{B^p_{s,q}(u)} \le C \|f\|_{B^p_{s,q}(u)}, \tag{2.7}$$

or equivalently for all $0 < r \le s$

$$\|f - L_{m+\eta+\mu+\nu}(X,f)\|_{B^p_{r,q}(u)} \le \frac{C}{m^{s-r}} \left(\int_0^{\frac{1}{m}} \left[\frac{\Omega^k_\varphi(f,t)_{u,p}}{t^{s+\frac{1}{q}}} \right]^q dt \right)^{\frac{1}{q}} \tag{2.8}$$

holds, where C is a constant independent of m and f.

In the case $\eta = \mu = \nu = 0$ (i.e. $v = 1$) such theorem has been stated in [14] for the classical Lagrange operator $L_m(w)$ while the case $\eta = \mu = 0, \nu > 0$ was studied in [2], where the authors stated only sufficient conditions.

Furthermore we observe that Theorem 2.1., as well as the Marcinkiewicz inequality (1.12), can be very useful when

$$\frac{u}{\sqrt{w\varphi}} \notin L^p \quad \text{and} \quad \frac{\sqrt{w\varphi}}{u} \in L^{p'}, \qquad p' := \frac{p}{p-1}.$$

3. Proofs.

Before giving the proofs, let us provide some notations and known results which will be useful in the sequel.

We denote by C a positive constant which may take different values in different formulas and we write $C \neq C(a, b, \ldots)$, if such a constant is independent of the parameters $a, b, \ldots$

If A, B are two positive quantities depending on some parameters, then $A \sim B$ means $C^{-1}B \leq A \leq CB$, C being independent of these parameters.

Moreover, we denote by AC_{loc} the set of all locally absolutely continuous functions on $[-1, 1]$, i.e. absolutely continuous on each $[a, b] \subset (-1, 1)$.

For each $w \in GDT$ given by (1.5), w_m denotes the function

$$\begin{aligned} w_m(x) \quad &:= \quad (\sqrt{1-x} + \frac{1}{m})^{2\alpha}(\sqrt{1+x} + \frac{1}{m})^{2\beta}(|x-\tau| + \frac{1}{m})^{\gamma} \times \\ &\times \quad \Phi_1(\sqrt{1-x} + \frac{1}{m})\Phi_2(\sqrt{1+x} + \frac{1}{m})\Phi_3(|x-\tau| + \frac{1}{m}) \end{aligned}$$

and $A_m(w)$ is the set

$$A_m(w) := \left[-1 + \frac{C_1}{m^2}, \tau - \frac{C_2}{m}\right] \cup \left[\tau + \frac{C_3}{m}, 1 - \frac{C_4}{m^2}\right], \tag{3.9}$$

where the constants C_i, $i = 1, 2, 3, 4$ (independent of m) will be suitably chosen according the circumstances.

Comparing w and w_m it is simply to recognize that

$$w_m(x) \sim w(x), \qquad \forall x \in A_m(w). \tag{3.10}$$

About the Christoffel functions corresponding to w, the following estimate

$$\lambda_m(w, x) \sim \frac{w_m(x)\varphi_m(x)}{m} \tag{3.11}$$

holds [1],[12],[18]. Regarding the orthonormal polynomials with respect to w, we have [1], [21]:

$$|p_m(w, x)| \leq \frac{C}{\sqrt{w_m(x)\varphi_m(x)}}, \qquad \forall x \in [-1, 1], \tag{3.12}$$

$$|p_m(w, x)| \sim m^{\alpha+1/2}, \qquad \forall x \in \left[1 - \frac{C}{m^2}, 1\right], \tag{3.13}$$

$$|p_m(w, x)| \sim m^{\beta+1/2}, \qquad \forall x \in \left[-1, -1 + \frac{C}{m^2}\right], \tag{3.14}$$

$$|p_m(w, x)| \sim m^{\gamma/2}, \qquad \forall x \in \left[\tau - \frac{C}{m}, \tau + \frac{C}{m}\right], \tag{3.15}$$

and, by (1.4), the zeros of $p_m(w,x)$ verify [21]:

$$x_{m,k+1}(w) - x_{m,k}(w) \sim \frac{\sqrt{1-x_{m,k}^2(w)}}{m}. \tag{3.16}$$

Moreover for each $\sigma \in GDT$ and $1 < p < \infty$ we have [19] (see also [23]):

$$\left\| \frac{u\sigma}{\sqrt{w\varphi}} \right\|_p \leq C \limsup_m \|p_m(w)\sigma u\|_p . \tag{3.17}$$

For each $w \in GDT$, the following polynomial inequalities hold [12].

• If $m \in \mathbb{N}$, then there exists a polynomial $P \in \mathbb{P}_m$ such that

$$|P(x)| \sim w_m(x). \tag{3.18}$$

• If $m \in \mathbb{N}$, $0 < a < m$ and $E \subset [-1,1]$ is such that $meas(\arccos E) \leq \frac{a}{m}$, then $\forall P \in \mathbb{P}_m$

$$\|Pw\|_p \leq C\|Pw\|_{L^p([-1,1]-E)}, \qquad C \neq C(m,P) \tag{3.19}$$

holds.

• If $m \in \mathbb{N}$ and the points $-1 \leq z_1 < z_2 < .. < z_m \leq 1$ are such that $\forall k$, $z_k := \cos\theta_k$ with $|\theta_{k-1} - \theta_k| \sim m^{-1}$, then for each $l \in \mathbb{N}$ and for every $P \in \mathbb{P}_m$ it results

$$\left(\sum_{k=1}^m \lambda_m(w^p, z_k)|P(z_k)|^p \right)^{\frac{1}{p}} \leq C\|Pw\|_{L^p([z_1,z_m])}, \tag{3.20}$$

with $C \neq C(m,P)$.

Regarding the modulus of continuity $\Omega_\varphi^k(f,t)_{u,p}$ it results [5]

$$\Omega_\varphi^k(f,t)_{u,p} \sim \tilde{K}_{k,\varphi}(f,t^k)_{u,p}, \tag{3.21}$$

where $\tilde{K}_{k,\varphi}(f,t^k)_{u,p}$ is the following K-functional

$$\tilde{K}_{k,\varphi}(f,t^k)_{u,p} := \sup_{0<h\leq t} \quad \inf\{\|(f-g)u\|_{L^p[-1+Ch^2,1-Ch^2]}$$

$$+h^k\|g^{(k)}\varphi^k u\|_{L^p[-1+Ch^2,1-Ch^2]}, \quad g^{(k-1)} \in AC_{loc}\}. \tag{3.22}$$

For the boundedness of the Hilbert transform

$$H(f,x) := \lim_{\epsilon\to 0} \int_{|x-t|>\epsilon} \frac{f(t)}{t-x} dt,$$

in L^p_u with $u \in GDT$ and $1 < p < \infty$, we have the following result [20], [14]:

$$(3.23) \qquad u \in L^p, \quad u^{-1} \in L^{\frac{p}{p-1}} \iff \|H(f)u\|_p \le C\|fu\|_p, \quad C \ne C(f).$$

Finally the following lemma holds.

Lemma. [10] *If f is a continuous function in $[a, b]$ such that*

$$\int_0^{b-a} \frac{\omega(f,t)_{L^p[a,b]}}{t^{1+\frac{1}{p}}} dt < \infty, \quad \textit{with} \quad 1 < p < \infty,$$

then

$$(3.24) \qquad \sup_{x \in [a,b]} |f(x)| \le C \left[(b-a)^{-\frac{1}{p}} \|f\|_{L^p[a,b]} + \int_0^{b-a} \frac{\omega(f,t)_{L^p[a,b]}}{t^{1+\frac{1}{p}}} dt \right]$$

holds, where $C \ne C(f)$ and $\omega(f,t)_{L^p[a,b]}$ is the ordinary modulus of continuity, i.e. $\omega(f,t)_{L^p[a,b]} = \sup_{0<h\le t} \|f(\cdot + h) - f(\cdot)\|_{L^p[a,b]}$.

Proof of Theorem 1.1.

The crucial fact is the following. Since $x_c(\zeta_j)$ is the zero of $p_m(w)$ closest to ζ_j, $j = 1, \dots, r$, we have

$$(3.25) \qquad |x - \zeta_j| \sim |x - x_c(\zeta_j)|, \quad \forall x \in A_m(w), \quad j = 1, \dots, r.$$

Set $S = \{s_k\}_{k=1}^m := \{\zeta_j\}_{j=1}^r \cup \{x_i\}_{i\in\mathcal{I}}$, $\mathcal{I} = \{i : x_i \ne x_c(\zeta_j), \forall j = 1, \dots, r\}$, with the s_k arranged in increasing order. Then $\forall x \in A_m(w)$ and $k = 1, \dots, m$, the fundamental Lagrange polynomials $l_{m,k}(S, x)$ satisfy

$$(3.26) \qquad |l_{m,k}(S,x)| = \prod_{i=1, i\ne k}^{m} \frac{|x - s_i|}{|s_k - s_i|} \sim \prod_{i=1, i\ne k}^{m} \frac{|x - x_i|}{|x_k - x_i|} = |l_{m,k}(w, x)|.$$

Hence (1.7) $\Longrightarrow$ (1.8) can be proved exactly as in [14, Proof of Th.2.7].

For the proof of (1.8) $\Longrightarrow$ (1.7), we note that by (3.26) and (3.19) it follows

$$\|Pu\|_p^p \le C\|L_m(S,P)u\|^p_{L^p(A_m(w))} \le C \sum_{k=1}^m |P(s_k)|^p \|l_{m,k}(S)u\|^p_{L^p(A_m(w))}$$

$$\le C \sum_{k=1}^m |P(s_k)|^p \|l_{m,k}(w)u\|^p_{L^p(A_m(w))} \le C \sum_{k=1}^m |P(s_k)|^p \|l_{m,k}(w)u\|^p_p .$$

Hence, applying (1.3), (3.11) and (3.25), we obtain

$$\|Pu\|_p^p \le C\left[\sum_{i\in\mathcal{I}}\lambda_m(u^p,x_i)|P(x_i)|^p+\sum_{i=1}^{r}\lambda_m(u^p,x_c(\zeta_i))|P(\zeta_i)|^p\right]$$

$$\le C\left[\sum_{i\in\mathcal{I}}\lambda_m(u^p,x_i)|P(x_i)|^p+\sum_{i=1}^{r}\lambda_m(u^p,\zeta_i)|P(\zeta_i)|^p\right].$$

■

Proof of Theorem 1.2.
First we observe that setting

$$\pi(x):=\prod_{i=1}^{\eta+\mu+\nu}(x-t_i), \tag{3.27}$$

we obtain

$$|\pi(x)|\sim|v(x)|, \qquad \forall x\in A_m(w). \tag{3.28}$$

Hence by (3.19), for each $\tilde P\in \mathbb{P}_{m-1}$, we have

$$\|\tilde Puv\|_p\sim\|\tilde Puv\|_{L^p(A_m(w))}\sim\|\tilde P\pi u\|_{L^p(A_m(w))}\sim\|\tilde P\pi u\|_p$$

and by (3.11) and (3.10), we get

$$\lambda_m(u^pv^p,x_i)\sim\lambda_m(u^p,x_i)v_m^p(x_i)\sim\lambda_m(u^p,x_i)|\pi(x_i)|^p.$$

Then recalling that (1.11) is equivalent to [14]:

$$\|\tilde Puv\|_p^p\le C\sum_{i=1}^{m}\lambda_m(u^pv^p,x_i)|\tilde P(x_i)|^p,\quad \tilde P\in\mathbb{P}_{m-1},\quad C\ne C(m,\tilde P),$$

the conditions in (1.11) are equivalent to:

$$\|\tilde P\pi u\|_p^p\le C\sum_{i=1}^{m}\lambda_m(u^p,x_i)|(\tilde P\pi)(x_i)|^p,\quad \tilde P\in\mathbb{P}_{m-1},\quad C\ne C(m,\tilde P). \tag{3.29}$$

Thus, let us prove (1.12) $\Longleftrightarrow$ (3.29) instead of (1.12) $\Longleftrightarrow$ (1.11).

Since $\forall\tilde P\in\mathbb{P}_{m-1}$, $\tilde P\pi\in\mathbb{P}_{m+\eta+\mu+\nu-1}$, then (1.12) $\Longrightarrow$ (3.29) trivially.

In order to prove the other implication, let $P\in\mathbb{P}_{m+\eta+\mu+\nu-1}$ be arbitrarily fixed. Dividing P by π we can write:

$$P(x)=Q(x)\pi(x)+R(x),\qquad Q\in\mathbb{P}_{m-1},\quad R\in\mathbb{P}_{\eta+\mu+\nu-1}$$

and consequently,

$$\|Pu\|_p^p \le C(\|Q\pi u\|_p^p + \|Ru\|_p^p). \tag{3.30}$$

For the first addendum, we can apply (3.29) with $\tilde{P} = Q$, obtaining

$$\|Q\pi u\|_p^p \le C \sum_{i=1}^{m} \lambda_m(u^p, x_i)|(Q\pi)(x_i)|^p. \tag{3.31}$$

For the remainder term R, since $R(t_i) = P(t_i), i = 1,..,\eta+\mu+\nu$, recalling the definition of the Lagrange polynomial given in (2.2),

$$R = L_{m+\eta+\mu+\nu}(S, R) = \pi L_m\left(w, \frac{R}{\pi}\right) + p_m(w) L_{\eta+\mu+\nu}\left(T, \frac{P}{p_m(w)}\right)$$

holds, where $S = \{x_i\}_{i=1}^m \cup \{t_i\}_{i=1}^{\eta+\mu+\nu}$ and $T = \{t_i\}_{i=1}^{\eta+\mu+\nu}$.

Hence using (3.29) with $\tilde{P} = L_m(w, \frac{R}{\pi})$, we obtain

$$\begin{aligned} \|Ru\|_p^p &\le C\left(\left\|L_m\left(w, \frac{R}{\pi}\right)\pi u\right\|_p^p + \left\|p_m(w)L_{\eta+\mu+\nu}\left(T, \frac{P}{p_m(w)}\right)u\right\|_p^p\right) \\ &\le C\sum_{i=1}^{m} \lambda_m(u^p, x_i)|R(x_i)|^p + C\left\|p_m(w)L_{\eta+\mu+\nu}\left(T, \frac{P}{p_m(w)}\right)u\right\|_p^p. \end{aligned}$$

Then the last result, (3.31) and (3.30) give

$$\begin{aligned} \|Pu\|_p^p &\le C\sum_{i=1}^{m} \lambda_m(u^p, x_i)(|(Q\pi)(x_i)|^p + |R(x_i)|^p) + \\ &+ C\left\|p_m(w)L_{\eta+\mu+\nu}\left(T, \frac{P}{p_m(w)}\right)u\right\|_p^p \le \\ &\le C\sum_{i=1}^{m} \lambda_m(u^p, x_i)|P(x_i)|^p + C\left\|p_m(w)L_{\eta+\mu+\nu}\left(T, \frac{P}{p_m(w)}\right)u\right\|_p^p. \end{aligned}$$

Thus in order to end the proof we have to show just that:

$$J := \left\|p_m(w)L_{\eta+\mu+\nu}\left(T, \frac{P}{p_m(w)}\right)u\right\|_p \le C\left(\sum_{i=1}^{\eta+\mu+\nu} \lambda_m(u^p, t_i)|P(t_i)|^p\right)^{\frac{1}{p}}. \tag{3.32}$$

Well by (3.19) we have

$$J \le C\left\|p_m(w)L_{\eta+\mu+\nu}\left(T, \frac{P}{p_m(w)}\right)u\right\|_{L^p(A_m(w))}$$

$$= C \sup_{\|g\|_{p'}=1} \int_{A_m(w)} p_m(w,x) \left[\sum_{i=1}^{\eta+\mu+\nu} \frac{\pi(x)}{\pi'(t_i)(x-t_i)} \frac{P(t_i)}{p_m(w,t_i)} \right] u(x)g(x)dx$$

$$= C \sup_{\|g\|_{p'}=1} \sum_{i=1}^{\eta+\mu+\nu} \frac{P(t_i)}{p_m(w,t_i)} \frac{1}{\pi'(t_i)} \int_{A_m(w)} \frac{\pi(x)p_m(w,x)}{x-t_i} u(x)g(x)dx$$

$$= C \sup_{\|g\|_{p'}=1} \sum_{i=1}^{\eta+\mu+\nu} \frac{P(t_i)}{p_m(w,t_i)} \frac{1}{\pi'(t_i)} q_g(t_i),$$

where q_g is the following polynomial:

$$q_g(t) := \int_{A_m(w)} \frac{\pi(x)p_m(w,x)\Theta(x) - \pi(t)p_m(w,t)\Theta(t)}{x-t} \frac{u(x)g(x)}{\Theta(x)} dx \tag{3.33}$$

with $\Theta \in \mathbb{P}_m$ arbitrarily chosen for the moments.

Now taking into account the arc cosine distribution of the knots and (3.13)-(3.15), it is simple to verify that

$$\frac{1}{\pi'(t_i)} \sim \frac{\varphi_m(t_i)}{m v_m(t_i)}, \qquad \frac{1}{p_m(w,t_i)} \sim \sqrt{(w_m\varphi_m)(t_i)}.$$

So by using these equivalences and the Hölder inequality, we get

$$\sum_{i=1}^{\eta+\mu+\nu} \frac{P(t_i)}{p_m(w,t_i)} \frac{1}{\pi'(t_i)} q_g(t_i) \le C \sum_{i=1}^{\eta+\mu+\nu} \frac{u_m(t_i)\varphi_m(t_i)}{m} P(t_i) \frac{\sqrt{(w_m\varphi_m)(t_i)}}{(u_m v_m)(t_i)} q_g(t_i)$$

$$\le C \left(\sum_{i=1}^{\eta+\mu+\nu} \frac{(u_m^p\varphi_m)(t_i)}{m} |P(t_i)|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{\eta+\mu+\nu} \frac{\varphi_m(t_i)}{m} \frac{\sqrt{(w_m\varphi_m)^{p'}(t_i)}}{(u_m v_m)^{p'}(t_i)} |q_g(t_i)|^{p'} \right)^{\frac{1}{p'}}$$

$$\le C \left(\sum_{i=1}^{\eta+\mu+\nu} \lambda_m(u^p, t_i) |P(t_i)|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{\eta+\mu+\nu} \lambda_m \left(\frac{\sqrt{(w_m\varphi_m)^{p'}}}{(u_m v_m)^{p'}}, t_i \right) |q_g(t_i)|^{p'} \right)^{\frac{1}{p'}}$$

having used (3.11) in the last inequality.

Then in order to obtain (3.32), we have to prove that for all g such that $\|g\|_{p'} = 1$

$$\left(\sum_{i=1}^{\eta+\mu+\nu} \lambda_m \left(\left[\frac{\sqrt{(w_m\varphi_m)}}{(u_m v_m)} \right]^{p'}, t_i \right) |q_g(t_i)|^{p'} \right)^{\frac{1}{p'}} \le C \tag{3.34}$$

holds.

By applying (3.20), (3.19) and taking into account the definition of q_g, we have

$$\left(\sum_{i=1}^{\eta+\mu+\nu} \lambda_m\left(\left[\frac{\sqrt{(w_m\varphi_m)}}{(u_m v_m)}\right]^{p'}, t_i\right)|q_g(t_i)|^{p'}\right)^{\frac{1}{p'}} \leq C\left\|q_g\frac{\sqrt{w\varphi}}{uv}\right\|_{p'}$$

$$\leq C\left\|H(\pi p_m(w)gu)\frac{\sqrt{w\varphi}}{uv}\right\|_{L^{p'}(A_m(w))} + C\left\|H\left(\frac{gu}{\Theta}\right)\pi p_m(w)\Theta\frac{\sqrt{w\varphi}}{uv}\right\|_{L^{p'}(A_m(w))}$$

Finally recalling (3.28) and (3.10), choosing $\Theta \in \mathbb{P}_m$ such that $\Theta \sim \frac{\sqrt{w_m\varphi_m}}{v_m}$ (see (3.18)) and using (3.12) and (3.23), we get

$$\begin{aligned}\left\|H(\pi p_m(w)gu)\frac{\sqrt{w\varphi}}{uv}\right\|_{L^{p'}(A_m(w))} &\leq C\|\pi p_m(w)g\frac{\sqrt{w\varphi}}{v}\|_{L^{p'}(A_m(w))}\\ &\leq C\|g\|_{L^{p'}(A_m(w))} \leq C\|g\|_{p'} = C\end{aligned}$$

and similarly

$$\begin{aligned}\left\|H\left(\frac{gu}{\Theta}\right)\pi p_m(w)\Theta\frac{\sqrt{w\varphi}}{uv}\right\|_{L^{p'}(A_m(w))} &\leq C\left\|H\left(\frac{gu}{\Theta}\right)\frac{\sqrt{w\varphi}}{uv}\right\|_{L^{p'}(A_m(w))}\\ &\leq C\|g\|_{L^{p'}(A_m(w))} \leq C\|g\|_{p'} = C,\end{aligned}$$

i.e. (3.34) holds. ■

P r o o f o f T h e o r e m 1.3.
First we set

$$\begin{aligned}X &= \left\{x_{\mu+1},\ldots,x_{d-[\frac{\nu+1}{2}]},x_{d+1+[\frac{\nu}{2}]},\ldots,x_{m-\eta}\right\},\\ K &= \{k \in \mathbb{N} : x_k \in X\}\end{aligned}$$

and

$$(3.35)\quad \pi(x) = \prod_{i=1}^{\mu}(x-x_i)\prod_{j=-[\frac{\nu-1}{2}]}^{[\frac{\nu}{2}]}(x-x_{d+j})\prod_{k=0}^{\eta-1}(x-x_{m-k}) = \prod_{i=1,i\notin K}^{m}(x-x_i).$$

¿From the definition (3.9) of $A_m(w)$ and for m sufficiently large, it follows

$$(3.36)\qquad |\pi(x)| \sim v(x), \qquad x \in A_m(w).$$

That said, let us prove that (1.13) $\Longrightarrow$ (1.14).

Let $m \in \mathbb{N}$ sufficiently large. We note that by (1.13) it follows (1.2) with u replaced by u/v. Hence applying (1.3), for all $\widetilde{P} \in \mathbb{P}_{m-1}$ we have

$$\left\|\widetilde{P}\frac{u}{v}\right\|_p^p \leq C\sum_{k=1}^{m} \lambda_m\left(\frac{u^p}{v^p}, x_k\right)|\widetilde{P}(x_k)|^p,$$

where C is independent of m and $\widetilde{P}$. Now let $P \in \mathbb{P}_{m-(\nu+\mu+\eta)-1}$ arbitrarily fixed and set $\widetilde{P} = P\pi \in \mathbb{P}_{m-1}$. From the definition (3.35) of π and the above inequality

$$\left\|P\pi\frac{u}{v}\right\|_p^p \leq C\left[\sum_{k=\mu+1}^{d-\left[\frac{\nu+1}{2}\right]} \lambda_m\left(\frac{u^p}{v^p}, x_k\right)|P(x_k)\pi(x_k)|^p\right.$$

$$\left. + \sum_{k=d+1+\left[\frac{\nu}{2}\right]}^{m-\eta} \lambda_m\left(\frac{u^p}{v^p}, x_k\right)|P(x_k)\pi(x_k)|^p\right]$$

follows.

About the left-hand side, by (3.19) and (3.36) we deduce

$$\left\|P\pi\frac{u}{v}\right\|_p^p \geq \left\|P\pi\frac{u}{v}\right\|_{L^p(A_m(w))}^p \geq C\,\|Pu\|_{L^p(A_m(w))}^p \geq C\|Pu\|_p^p.$$

Furthermore about the right-hand side, by (3.11) and (3.36)

$$\lambda_m\left(\frac{u^p}{v^p}, x_k\right) \leq C\frac{\lambda_m(u^p, x_k)}{|\pi(x_k)|^p}, \quad \forall k \in K$$

follows. So we can conclude

$$\|Pu\|_p^p \leq C\left\|P\pi\frac{u}{v}\right\|_p^p \leq C\left[\sum_{k=\mu+1}^{d-\left[\frac{\nu+1}{2}\right]} \lambda_m(u^p, x_k)|P(x_k)|^p\right.$$

$$\left. + \sum_{k=d+1+\left[\frac{\nu}{2}\right]}^{m-\eta} \lambda_m(u^p, x_k)|P(x_k)|^p\right],$$

i.e. (1.14) holds.

Now let us prove that (1.14) $\Longrightarrow$ (1.13). First we show that $\frac{u}{v\sqrt{w\varphi}} \in L^p$. Writing (1.14) with $P = L_{m-(\mu+\nu+\eta)}(X, f)$, $f \in C^0$, we obtain

$$\|L_{m-(\mu+\nu+\eta)}(X, f)u\|_p \leq C \left[\sum_{i=\mu+1}^{d-\left[\frac{\nu+1}{2}\right]} \lambda_m(u^p, x_i)|f(x_i)|^p \right.$$

$$\left. + \sum_{i=d+1+\left[\frac{\nu}{2}\right]}^{m-\eta} \lambda_m(u^p, x_i)|f(x_i)|^p \right]^{\frac{1}{p}} \leq C\|f\|_\infty,$$

i.e. $\sup_m \|L_{m-(\mu+\nu+\eta)}(X)\|_{C^0 \longrightarrow L^p_u} < \infty$.

On the other hand we have $\|\frac{p_m(w)}{\pi}u\|_p \leq C\|L_{m-(\mu+\nu+\eta)}(X)\|_{C^0 \longrightarrow L^p_u}$, with C independent of m [17].

Hence $\sup_m \left\|p_m(w)\frac{u}{v}\right\|_p < \infty$, and then the first condition in (1.13) immediately follows by (3.17).

Now we prove that $\frac{v\sqrt{w\varphi}}{u} \in L^{p'}$.

Since the fundamental Lagrange polynomials corresponding to X satisfy

$$l_{m-(\mu+\nu+\eta),k}(X, x) = \frac{\pi(x_k)}{\pi(x)} l_{m,k}(w, x), \quad k \in K,$$

and it is well known that (see for instance [22])

$$|l_{m,i}(w, x)| \sim \left| \lambda_m(w, x_i) p_{m-1}(w, x_i) \frac{p_m(w, x)}{x - x_i} \right|, \quad i = 1, \ldots, m,$$

we deduce $\forall k \in K$

$$|l_{m-(\mu+\nu+\eta),k}(X, x)| \sim \left| \lambda_m(w, x_k) p_{m-1}(w, x_k) \pi(x_k) \frac{p_m(w, x)}{\pi(x)(x - x_k)} \right|$$

is true.

Now recalling that [21] (see also [14])

$$|p_{m-1}(w, x_k)| \geq C\sqrt{\frac{\varphi(x_k)}{w_m(x_k)}}, \quad C \neq C(m, k),$$

by (3.11) and (3.36) we obtain

$$|l_{m-(\mu+\nu+\eta),k}(X, x)| \geq C\sqrt{(w_m\varphi)(x_k)}v(x_k)\frac{\varphi(x_k)}{m} \left| \frac{p_m(w, x)}{\pi(x)(x - x_k)} \right|.$$

Hence by (3.19) and (3.36) it follows

$$\sqrt{(w_m\varphi)(x_k)}v(x_k)\frac{\varphi(x_k)}{m}\left\|\frac{p_m(w)}{(\cdot - x_k)}\frac{u}{v}\right\|_p \leq C\|l_{m-(\mu+\nu+\eta),k}(X)u\|_p.$$

On the other hand applying (1.14) with $P = l_{m-(\mu+\nu+\eta),k}(X)$, by (3.11) we have

$$\|l_{m-(\mu+\nu+\eta),k}(X)u\|_p \leq C\lambda_m^{\frac{1}{p}}(u^p, x_k) \leq C\frac{u_m(x_k)\varphi^{\frac{1}{p}}(x_k)}{m^{\frac{1}{p}}}.$$

Therefore, for each $k \in K$ we have

$$\sqrt{(w_m\varphi)(x_k)}v(x_k)\frac{\varphi(x_k)}{m}\left\|\frac{p_m(w)}{(\cdot - x_k)}\frac{u}{v}\right\|_p \leq C\frac{u_m(x_k)\varphi^{\frac{1}{p}}(x_k)}{m^{\frac{1}{p}}},$$

i.e.

$$\left\|\frac{p_m(w)}{(\cdot - x_k)}\frac{u}{v}\right\|_p \frac{\sqrt{(w_m\varphi)(x_k)}}{u_m(x_k)}v(x_k)\left(\frac{\varphi(x_k)}{m}\right)^{\frac{1}{p'}} \leq C.$$

Then using this last estimate and recalling (3.17), we can conclude the proof as in [14] (for the details see in that paper the proof of Theorem 2.7). ■

P r o o f o f T h e o r e m 2.1.
Let us prove that (2.4) $\Longrightarrow$ (2.5).
By Theorem 1.2 we have:

$$\|L_{m+\eta+\mu+\nu}(X,f)u\|_p^p \leq C\left[\sum_{i=1}^{m}\lambda_m(u^p, x_i)|f(x_i)|^p + \sum_{i=1}^{\eta+\mu+\nu}\lambda_m(u^p, t_i)|f(t_i)|^p\right].$$

Let us examine the addenda on the right-hand side.

Since in our proof we obtain the same results for the zeros $\{x_i\}$ as well as for the additional knots $\{t_i\}$, denote such points by an unique notation $\{\xi_i\} := \{x_i\} \cup \{t_i\}$ and, as usual, consider $\{\xi_i\}$ arranged in increasing order, i.e. set:

$$\xi_i := \begin{cases} y_i & \text{if } i = 1,\dots,\mu \\ x_{i-\mu} & \text{if } i = \mu+1,\dots,\mu+d \\ \tau_{i-(\mu+d)} & \text{if } i = \mu+d+1,\dots,\mu+d+\nu \\ x_{i-(\mu+\nu)} & \text{if } i = \mu+d+\nu+1,\dots,\mu+\nu+m \\ z_{i-(\mu+\nu+m)} & \text{if } i = \mu+\nu+m+1,\dots,\mu+\nu+m+\eta. \end{cases} \tag{3.37}$$

With this notation the previous inequality can be rewritten as follows:

$$\|L_{m+\eta+\mu+\nu}(X,f)u\|_p^p \leq C\sum_{i=1}^{\eta+\mu+\nu+m}\lambda_m(u^p, \xi_i)|f(\xi_i)|^p. \tag{3.38}$$

By applying (3.24) we have for $i = 1, 2, \ldots, \mu + \nu + \eta + m - 1$:

$$|f(\xi_i)| \leq \sup_{x \in [\xi_i, \xi_{i+1}]} |f(x)| \leq C \left[\frac{\|f\|_{L^p[\xi_i,\xi_{i+1}]}}{(\xi_{i+1} - \xi_i)^{\frac{1}{p}}} + \int_0^{\xi_{i+1}-\xi_i} \frac{\omega(f,t)_{L^p[\xi_i,\xi_{i+1}]}}{t^{1+\frac{1}{p}}} dt \right],$$

while in the case $i = \mu + \nu + \eta + m$ we estimate $|f(\xi_i)|$ like $|f(\xi_{i-1})|$, since we have

$$|f(\xi_i)| \leq \sup_{x \in [\xi_{i-1}, \xi_i]} |f(x)|.$$

Now since the nodes $\{\xi_i\}$ have an arc cosine distribution, it is simple to verify that

$$\xi_{i+1} - \xi_i \sim \frac{\varphi(\xi_i)}{m}, \qquad i = 1, 2, .., \mu + \nu + \eta + m - 1.$$

Then we have

$$|f(\xi_i)| \leq C \left[\left(\frac{m}{\varphi(\xi_i)} \right)^{\frac{1}{p}} \|f\|_{L^p[\xi_i,\xi_{i+1}]} + \int_0^{\frac{\varphi(\xi_i)}{m}} \frac{\omega(f,t)_{L^p[\xi_i,\xi_{i+1}]}}{t^{1+\frac{1}{p}}} dt \right]$$

$$= C \left[\left(\frac{m}{\varphi(\xi_i)} \right)^{\frac{1}{p}} \|f\|_{L^p[\xi_i,\xi_{i+1}]} + \frac{1}{\varphi(\xi_i)^{\frac{1}{p}}} \int_0^{\frac{1}{m}} \frac{\omega(f,t\varphi(\xi_i))_{L^p[\xi_i,\xi_{i+1}]}}{t^{1+\frac{1}{p}}} dt \right].$$

Consequently, by (3.11) it follows:

$$\lambda_m(u^p, \xi_i)|f(\xi_i)|^p \leq C \frac{\varphi(\xi_i) u^p(\xi_i)}{m} |f(\xi_i)|^p$$

$$\leq C \left[u^p(\xi_i) \|f\|^p_{L^p[\xi_i,\xi_{i+1}]} + \frac{1}{m} \left(\int_0^{\frac{1}{m}} \frac{u(\xi_i)\omega(f,t\varphi(\xi_i))_{L^p[\xi_i,\xi_{i+1}]}}{t^{1+\frac{1}{p}}} dt \right)^p \right].$$

But applying the well-known properties of the ordinary modulus of continuity $\omega(f,x)_p$ we deduce that $\forall g \in AC_{loc}$:

$$\omega(f,t\varphi(\xi_i))_{L^p[\xi_i,\xi_{i+1}]} \leq \omega(f-g,t\varphi(\xi_i))_{L^p[\xi_i,\xi_{i+1}]} + \omega(g,t\varphi(\xi_i))_{L^p[\xi_i,\xi_{i+1}]}$$

$$\leq 2\|f-g\|_{L^p[\xi_i,\xi_{i+1}]} + t\varphi(\xi_i)\|g'\|_{L^p[\xi_i,\xi_{i+1}]}.$$

Hence we have

$$\lambda_m(u^p, \xi_i)|f(\xi_i)|^p \leq C u^p(\xi_i) \|f\|^p_{L^p[\xi_i,\xi_{i+1}]}$$

$$+ \frac{C}{m} \left(\int_0^{\frac{1}{m}} \frac{u(\xi_i)\|f-g\|_{L^p[\xi_i,\xi_{i+1}]} + tu(\xi_i)\varphi(\xi_i)\|g'\|_{L^p[\xi_i,\xi_{i+1}]}}{t^{1+\frac{1}{p}}} dt \right)^p.$$

Now taking into account the arc cosine distribution of $\{\xi_i\}$, we get

$$u(x) \sim u(\xi_i), \quad \varphi(x) \sim \varphi(\xi_i), \qquad \forall x \in [\xi_i, \xi_{i+1}]$$

and thus we obtain

$$\lambda_m(u^p, \xi_i)|f(\xi_i)|^p \le C\|fu\|^p_{L^p[\xi_i,\xi_{i+1}]}$$

$$+\frac{C}{m}\left(\int_0^{\frac{1}{m}} \frac{\|(f-g)u\|_{L^p[\xi_i,\xi_{i+1}]} + t\|g'\varphi u\|_{L^p[\xi_i,\xi_{i+1}]}}{t^{1+\frac{1}{p}}}dt\right)^p.$$

Finally, using this inequality in (3.38) and applying Minkowski inequality (see [7], p.148) and (3.21), we obtain the thesis (for the details we refer the reader to [14, Proof of Th.3.1]).

For the proof of (2.5) $\Longrightarrow$ (2.7), as well as for the proofs of (2.5) $\Longleftrightarrow$ (2.6) and (2.7) $\Longleftrightarrow$ (2.8), once again we refer the reader to [14].

In conclusion, let us prove (2.8) $\Longrightarrow$ (2.4).

First of all, let us observe that (2.8) implies that

$$\|L_{m+\eta+\mu+\nu}(X,f)u\|_p \le \|fu\|_p + \|[f - L_{m+\eta+\mu+\nu}(X,f)]u\|_{B^p_{r,q}(u)}$$

$$\le C\|f\|_\infty + C\left(\int_0^{\frac{1}{m}}\left[\frac{\Omega^k_\varphi(f,t)_{u,p}}{t^{r+\frac{1}{q}}}\right]^q dt\right)^{\frac{1}{q}}.$$

Hence taking the $\limsup_{m\to\infty}$, we get

$$\|L_{m+\eta+\mu+\nu}(X,f)u\|_p \le C\|f\|_\infty, \qquad C \ne C(m,f). \tag{3.39}$$

Then taking into account that [17]

$$\|p_m(w)uv\|_p \le C \sup_{\|f\|_\infty=1} \|L_{m+\eta+\mu+\nu}(X,f)u\|_p, \qquad C \ne C(m),$$

by (3.39) and (3.17) we obtain $\frac{uv}{\sqrt{w\varphi}} \in L^p$.

Thus it remains to prove the second condition in (2.4). To this end, let us assume $\frac{\sqrt{w\varphi}}{uv} \notin L^{p'}$ and show such assumption leads to a contradiction.

Since $\frac{\sqrt{w\varphi}}{uv} \notin L^{p'}$ at least one of the following functions does not belong to $L^{p'}$:

$$(1-x)^{\alpha/2+1/4-\gamma-\eta}\sqrt{\Phi_1(\sqrt{1-x})}[\Psi_1(\sqrt{1-x})]^{-1},$$

$$(1+x)^{\beta/2+1/4-\delta-\mu}\sqrt{\Phi_2(\sqrt{1+x})}[\Psi_2(\sqrt{1+x})]^{-1},$$

$$|x-\tau|^{\rho/2-\nu}\sqrt{\Phi_3(|x-\tau|)}.$$

Well, let us suppose that

$$|x-\tau|^{\rho/2-\nu}\sqrt{\Phi_3(|x-\tau|)}\notin L^{p'}, \tag{3.40}$$

the proof works analogously in the other cases.

Since the function Φ_3 has subalgebraic growth, (3.40) implies

$$\rho/2-\nu+\frac{1}{p'}\le 0. \tag{3.41}$$

Now let x_d be the zero of $p_m(w)$ closest to τ and let us define the following function

$$f_d(x):=\begin{cases}\dfrac{x-x_{d-1}}{x_d-x_{d-1}} & x\in[x_{d-1},x_d]\\ \dfrac{x-\tau_1}{x_d-\tau_1} & x\in[x_d,\tau_1]\\ 0 & \text{elsewhere}\end{cases},$$

where we recall τ_1 is the first additional knot near τ.

Then let us consider a function $F_{d,j}$ such that $F_{d,j}^{(j)}=f_d$, $F_{d,j}(x_d)=1$, $F_{d,j}(x)=0\quad\forall x\notin]x_{d-1},\tau_1,[$ and $\sup_{x\in[-1,1]}|F_{d,j}(x)|\le C$.

By applying (2.8) with $f:=F_{d,j}$ we have

$$\begin{aligned}\|L_{m+\eta+\mu+\nu}(X,F_{d,j})u\|_p &\le \|F_{d,j}u\|_p+\|[F_{d,j}-L_{m+\eta+\mu+\nu}(X,F_{d,j})]u\|_{B^p_{r,q}(u)}\\ &\le \|F_{d,j}u\|_p+\frac{C}{m^{s-r}}\left(\int_0^{\frac{1}{m}}\left[\frac{\Omega_\varphi^k(F_{d,j},t)_{u,p}}{t^{s+\frac{1}{q}}}\right]^q\right)^{\frac{1}{q}}.\end{aligned}$$

Furthermore by (3.21) and (3.22) we deduce $\Omega_\varphi^k(F_{d,j},t)_{u,p}\le Ct^k\|F_{d,j}^{(k)}\varphi^k u\|_p$, hence taking the integers j and k such that $k=j+1$ and $j\ge s$, by standard calculation, we obtain

$$\begin{aligned}&\|L_{m+\eta+\mu+\nu}(X,F_{d,j})u\|_p\\ \le\ &\|F_{d,j}u\|_p+\frac{C}{m^{s-r}}\|f_d'\varphi^{j+1}u\|_p\left(\int_0^{\frac{1}{m}}t^{q(j+1-s-1/q)}dt\right)^{\frac{1}{q}}\\ \le\ &\|F_{d,j}u\|_p+\frac{C}{m^{j+1-r}}\|f_d'\varphi^{j+1}u\|_p\\ \le\ &\|F_{d,j}u\|_p+\frac{C}{m}\|f_d'\varphi u\|_p\le C\lambda_m^{\frac{1}{p}}(u^p,x_d).\end{aligned}$$

Now let us consider the fundamental Lagrange polynomial corresponding to x_d:

$$l_{m+\eta+\mu+\nu,d}(X,x) := \frac{\pi(x)p_m(w,x)}{\pi(x_d)p'_m(w,x_d)(x-x_d)},$$

where $\pi(x)$ is given by (3.27).

Obviously, $l_{m+\eta+\mu+\nu,d}(X,x) = L_{m+\eta+\mu+\nu}(X,F_{d,j},x)$, hence by the above estimate and (3.11)

$$\|l_{m+\eta+\mu+\nu,d}(X)u\|_p \le C\lambda_m^{\frac{1}{p}}(u^p,x_d) \sim \frac{u_m(x_d)\varphi_m^{\frac{1}{p}}(x_d)}{m^{\frac{1}{p}}} \tag{3.42}$$

14holds.

On the other hand, it is well known that

$$\left|\frac{p_m(w,x)}{p'_m(w,x_d)(x-x_d)}\right| \sim \left|\lambda_m(w,x_d)p_{m-1}(w,x_d)\frac{p_m(w,x)}{x-x_d}\right|.$$

Thus, taking into account that $|p_{m-1}(w,x_d)| \ge C\sqrt{\frac{\varphi_m(x_d)}{w_m(x_d)}}$ with $C \ne C(m,d)$ (see [21]), by the definition of $l_{m+\eta+\mu+\nu,d}(X)$ and by (3.11) and (3.28), we deduce

$$\|l_{m+\eta+\mu+\nu,d}(X)u\|_p \ge \frac{\sqrt{w_m(x_d)\varphi_m^3(x_d)}}{mv_m(x_d)}\left\|\frac{p_m(w)}{\cdot - x_d}uv\right\|_p. \tag{3.43}$$

In conclusion by (3.42) and (3.43) we obtain

$$\left\|\frac{p_m(w)}{\cdot - x_d}uv\right\|_p \frac{\sqrt{w_m(x_d)\varphi_m(x_d)}}{u_m(x_d)v_m(x_d)}\left(\frac{\varphi(x_d)}{m}\right)^{\frac{1}{p'}} \le C \ne C(m)$$

and the thesis can be obtained by this estimate reasoning ab absurdo as in [14] (see the proof of Theorem 2.7). ■

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