Provided for non-commercial research and educational use.
Not for reproduction, distribution or commercial use.

Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

The attached copy is furnished for non-commercial research and education use only. Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.
Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on Mathematica Balkanica visit the website of the journal
http://www.mathbalkanica.info
or contact:
Mathematica Balkanica - Editorial Office;
Acad. G. Bonchev str., Bl. 25A, 1113 Sofia, Bulgaria
Phone: +359-2-979-6311, Fax: +359-2-870-7273,
E-mail: balmat@bas.bg

Mathematica
Balkanica

New Series Vol. 13, 1999, Fasc. 1-2

The Bang – Bang Principle for a Parabolic Optimal Control Problem with Hölderian Cost Functional [1]

I. V. Iordanov *, *Ts. Tsachev* **

Presented by P. Kenderov

The problem of controlling the temperature distribution of a heated body $\Omega \subset \mathbf{R}^n$ by the temperature φ of the medium surrounding Ω is considered. The aim is to choose φ (subject to certain constraints) in such a way that the temperature distribution of Ω at time $t = T$ be as close as possible (with respect to the Hölder $C^{0,\varepsilon}(\bar{\Omega})$ – norm $(0 < \varepsilon < 1)$) to a given function. The question of existence of optimal controls is studied and the bang–bang principle is proved.

AMS Sub. Classification: 49K20

Key Words: bang–bang principle, parabolic optimal control problem, Besov spaces, real interpolation

Introduction

A large class of constrained optimal control problems are characterized with the so called "bang–bang" property, i.e. the optimal control function attains its extreme values almost everywhere. These problems vary widely with respect to the type of the equation describing the process, the type of the control and its constraints, the type of the cost functional etc. The literature on this topic is abundant (cf., e.g. [B], [L], [Gl], [GS], [GW], [HW] and the references there).

This paper is devoted to the problem of controlling the temperature distribution of a heated body $\Omega \subset \mathbf{R}^n$ by the heat flow through its boundary $\partial\Omega$. It is motivated by the paper [GW], where the authors considered an optimal control problem whose cost functional was the deviation of the obtained temperature distribution $u(x, T)$ at fixed moment $T > 0$ from the desired one $z(x)$

[1] The first author I. Iordanov has been partially supported by the Ministry of Education and Science by Contract No. MM-612/96, and the second author Ts. Tsachev - partially supported by the Ministry of Education and Science by Contract No. MM-807/98

in the supremum norm (the norm in $C(\bar{\Omega})$ of $u(\cdot, T) - z(\cdot)$) – a fact emphasized by them as the main distinction between their paper and previous publications on the topic.

The function $u(x,t)$ is the solution of the following boundary value problem (BVP)

$$\frac{\partial u}{\partial t} - Au = 0 \quad \text{in} \quad \Omega \times (0,T)$$

$$\frac{\partial u}{\partial \nu} + \beta u = \varphi \quad \text{on} \quad \Sigma = \partial\Omega \times (0,T), \beta = \text{const} > 0$$

$$u(x,0) \qquad \text{on} \quad \Omega,$$

where $\dfrac{\partial u}{\partial \nu}$ is the conormal derivative on $\partial\Omega$ corresponding to the uniformly elliptic operator of second order

$$Au = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + a(x)u.$$

The set of admissible controls is

$$\{\varphi \in L_\infty(\Sigma) | \|\varphi\|_{L_\infty(\Sigma)} \le 1\}$$

(i.e. in this case the "bang–bang" principle means that the optimal control $\hat{\varphi}$ satisfies $|\hat{\varphi}| = 1$ a.e. Σ).

A crucial role in [GW] plays the Fourier method (of separating the variables) for solving the BVP and the analyticity of the coefficients. Entirely changing the technique used by K. Glashoff and N. Weck and following their idea, we extend the result of [GW] in the following directions:

— the coefficients of A depend on all variables x, t, so the Fourier method no longer applies;

— the cost functional is defined by a stronger norm, namely the one of an appropriate Besov space. In view of the Sobolev embedding theorem for every $\varepsilon \in (0,1)$ we can choose a Besov space in such a way that its norm be stronger than the Hölder $C^{0,\varepsilon}(\bar{\Omega})$–norm (see Remark 1.3 and Corollary 1.4).

Unfortunately we were not able to drop the analyticity assumption on the coefficients and for the time being we do not know whether it can be replaced by a weaker one. This problem has found a partial solution in [SW], where the coefficients are C^∞, but are not time-dependent. Of course, a property close to the "bang–bang" principle can easily be established under C^∞ (or even C^k) smoothness assumption on the coefficients (cf. Theorem 1.5).

In this paper we use an adaptation of the technique developed in [LM2], Ch. IV, but using interpolation spaces based on $L_p, p \neq 2$. Our approach was announced in [IT], where a weaker result was presented – the deviation in the cost functional is in the $H^{1/4}(\Omega)$–norm. The approach consists in using the results in the smooth case (cf. e.g. [G1], [G2]) and then doing transposition and subsequent interpolation. We mention that the choice of appropriate interpolation spaces is important part of the implementation of the just mentioned technique. Crucial for our treatment are our results in §2 for parabolic BVP with time–dependent operator and nonsmooth data. A number of silmilar results have been obtained recently. In [R] semilinear parabolic problems in $R^n \times [0,T]$ are considered using Besov spaces. In [KY], again in $R^n \times [0,T]$, even more complicated interpolation spaces (based on Morrey spaces instead on L_p spaces) are used for considering Navier–Stokes and semilinear heat equations with distributions as initial data. In [C] linear as well as nonlinear parabolic BVP with nonsmooth data are considered. However the obtained solutions do not possess the smoothness required for our analysis.

The paper is organized as follows:

In §1 the necessary notations are introduced and the main results are formulated. Some preliminary facts for parabolic BVP and interpolation spaces are collected in §2. There we recall some properties of the Besov spaces and following the ideas from [LM2] we prove an existence and uniqueness theorem for a parabolic BVP with nonsmooth data. The principal result needed in our subsequent analysis is Proposition 2.17 stating that the trace on Ω_T of the initial–BVP is in a suitable Besov space (rather then just in $C^0(\bar{\Omega})$ as proved in [GW]) and the corresponding dual result. The proofs of the main results are given in §3.

1. Basic notations and main results

Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary $\Gamma = \partial\Omega$, T be a positive real number, $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$. We denote $\Omega_t = \Omega \times \{t\}$ for $t \in [0,T]$.

Let L be the differential operator $L = \dfrac{\partial}{\partial t} - A$, where $Au = \sum_{i,j=1}^{n} \dfrac{\partial}{\partial x_j}$ $(a_{ij}(x,t)\dfrac{\partial u}{\partial x_i})$ $-a(x,t)u$. Here we assume that $a_{ij}(x,t) = a_{ji}(x,t)$ for $i,j = 1,2,\ldots,n$ and that the operator A is elliptic, i.e. that $\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq c.|\xi|^2$ (c =const > 0) holds true for every $\xi \in \mathbf{R}^n$ and for every $(x,t) \in Q$.

Let B be the differential operator defined by $Bu = \dfrac{\partial u}{\partial \nu} + \beta(x,t)u$, where $\beta(x,t)$ is a smooth positive function defined on Σ and $\dfrac{\partial}{\partial \nu} = \sum\limits_{i,j=1}^{n} a_{ij}(x,t)$. $\cos(\nu, x_i)\, \dfrac{\partial}{\partial x_j}$ is the conormal (with respect to Γ) derivative ($\nu = (\nu_1(x), \nu_2(x), \ldots, \nu_n(x))$ is the unit outer normal with respect to Ω in the points of Γ).

Remark 1.1. The coefficients of the operators A and B will be called data. In order to apply the theory developed in [LM2], [G1] and [G2] we shall assume that the data are C^∞, although C^k- regularity is sufficient. Actually only in the proof of Theorem 1.2 we can not dispense with analyticity assumption for the data and for $\Gamma = \partial\Omega$ as we mentioned in the introduction.

We state next our optimal control problem. Let $B_p^{\bar{p}}(\Omega)$, $p, \bar{p} > 1$, be the usual Besov space (see §2). Let $z(x) \in B_p^{\bar{p}}(\Omega)$. Let $u(x,t;\varphi)$ be the solution to the following BVP:

$$\begin{aligned} Lu &= 0 \quad \text{in} \quad Q \\ Bu &= \varphi \quad \text{on} \quad \Sigma \\ u(x,0) &= 0 \quad \text{on} \quad \Omega, \end{aligned}$$

where $\varphi \in U = \{\varphi \in L_\infty(\Sigma) \mid |\varphi| \le 1 \text{ a.e. on } \Sigma\}$. We consider the problems

$$(P) \qquad \text{minimize} \|u(\cdot, T; \varphi) - z(\cdot)\|_{B_p^{\bar{p}}(\Omega)} \text{ on } U$$

and

$$(P') \qquad \text{minimize} \|u(\cdot, T; \varphi) - z(\cdot)\|_{C^{0,\varepsilon}(\bar{\Omega})} \text{ on } U.$$

The main results of this paper are:

Theorem 1.2. ("bang–bang" principle) *Let $p > \max\{3, n+1\}$ and $\bar{p} \in (\frac{n}{p}, 1 - \frac{1}{p})$ hold. Let $z(\cdot) \notin \{u(\cdot, T; \varphi) \mid \varphi \in U\}$. Then if the data and $\Gamma = \partial\Omega$ are analytic, each solution $\hat{\varphi}$ of the problem (P) satisfies $|\hat{\varphi}(x,t)| = 1$ a.e. on Σ.*

Remark 1.3. The Sobolev embedding theorem implies that the $B_p^{\bar{p}}(\Omega)$– norm is stronger than the $C^{0,\varepsilon}(\bar{\Omega})$– norm, $\varepsilon = \bar{p} - \frac{n}{p}$. Hence, given $\varepsilon \in (0,1)$, we can choose a sufficiently large p and $\bar{p} \in (\frac{n}{p}, 1 - \frac{1}{p})$, such that the Besov norm in consideration is stronger than the $C^{0,\varepsilon}(\bar{\Omega})$– norm.

Corollary 1.4. *Let $\varepsilon \in (0,1)$ and $z(.) \notin \{u(., T; \varphi) | \varphi \in U\}$. Then if the data and $\Gamma = \partial\Omega$ are analytic, each solution $\hat{\varphi}$ of the problem (P') satisfies $|\hat{\varphi}(x,t)| = 1$ a.e. on Σ.*

Theorem 1.5. (restricted "bang–bang" principle) *Let $p > \max\{3, n+1\}$ and $\bar{p} \in (\frac{n}{p}, 1 - \frac{1}{p})$ hold. Let $z(\cdot) \notin \{u(\cdot, T; \varphi) \mid \varphi \in U\}$. Then if the data and $\Gamma = \partial\Omega$ are C^∞, for each solution $\hat{\varphi}$ of the problem (P) the set $\{(x,t) \in \Sigma \mid |\hat{\varphi}(x,t)| < 1\}$ has no internal points.*

§2. Preliminaries: Interpolation theory, Besov spaces and the mixed problem for a parabolic operator of second order

Following [G1] and [G2] we recall the definitions and some basic facts of the theory of the Besov spaces.

Let $\mathbf{N}$ be the set of all natural numbers. For $r, s \in \mathbf{N} \cup \{0\}$ and $p \in (1, +\infty)$, the Sobolev space $W_p^{r,s}(\mathbf{R}_x^n \times \mathbf{R}_t^1)$ is defined as the set of all functions from $L_p(\mathbf{R}^{n+1})$ such that

$$\frac{\partial^l u}{\partial x_j^l} \in L_p(\mathbf{R}^{n+1}), \quad 0 \le l \le r, j = 1, \ldots, n,$$

and

$$\frac{\partial^l u}{\partial t^l} \in L_p(\mathbf{R}^{n+1}), \quad 0 \le l \le s,$$

which is a Banach space with the norm

$$u \longmapsto \left(\sum_{l=0}^{s} \| \frac{\partial^l u}{\partial t^l} \|_{L_p}^p + \sum_{j=1}^{n} \sum_{l=1}^{r} \| \frac{\partial^l u}{\partial x_j^l} \|_{L_p}^p \right)^{\frac{1}{p}}.$$

This space can also be defined as the closure of $C_0^\infty(\mathbf{R}^{n+1})$ with respect to the same norm. The so called Besov spaces $B_p^{r,s}(\mathbf{R}^{n+1})$, $p \in (1, +\infty)$, $r > 0$, $s > 0$, form a continuous scale of Banach spaces (for their definition see, e.g., [G1], p.192, Definition 9.2. or [G2], p.334 and p.348). These spaces (which are different from $W_p^{r,s}(\mathbf{R}^{n+1})$ for $r, s \in \mathbf{N}$) are stable with respect to real interpolation (cf.,e.g. [BL], Ch.III, VI). In what follows, unless otherwise stated, we shall use the notation $(X, Y)_\theta$ instead of $(X, Y)_{\theta,q}$ for real interpolation between the spaces X and Y, always assuming that $q = p$.

We shall need only some of the spaces $B_p^{2s,s}(\mathbf{R}^{n+1})$, where $s \in [-1, 2]$. According to Theorem 9.1. on p.193 in [G1] we have

Proposition 2.1. *It is true that*

$$B_p^{(1-\theta)2r,(1-\theta)r}(\mathbf{R}^{n+1}) = (W_p^{2r,r}(\mathbf{R}^{n+1}), L_p(\mathbf{R}^{n+1}))_\theta$$

for $r \in \mathbf{N}$, $\theta \in (0,1)$ and $p \in [1, +\infty)$.

The folloing theorem will be essential for our purposes.

Theorem 2.2. **([G1], p.195, Th. 9.3)** *If $p \in (1, +\infty)$, $\theta \in (0, 1)$ and s_0, s_1 are positive reals, then*

$$(B_p^{2s_0,s_0}(\mathbf{R}^{n+1}), B_p^{2s_1,s_1}(\mathbf{R}^{n+1}))_\theta = B_p^{2s_\theta,s_\theta}(\mathbf{R}^{n+1}),$$

where $s_\theta = (1-\theta)s_0 + \theta s_1$.

Following [G1], p.196, we give the following

Definition 2.3. For $s < 0$, $p \in (1, +\infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$,

$$B_p^{2s,s}(\mathbf{R}^{n+1}) = (B_{p'}^{-2s,-s}(\mathbf{R}^{n+1}))^*,$$

where "$*$" denotes the topological dual. Analogously,

$$W_p^{2s,s}(\mathbf{R}^{n+1}) = (W_{p'}^{-2s,-s}(\mathbf{R}^{n+1}))^* \text{ for } (-s) \in \mathbf{N} \cup \{0\}.$$

Let $\mathcal{S}(\mathbf{R}^{n+1})$ be the well known Schwartz space, $\mathcal{S}'(\mathbf{R}^{n+1})$ be its dual and $\mathcal{F}: \mathcal{S}' \to \mathcal{S}'$ denote the Fourier transform. Let the linear operator $\Lambda: \mathcal{S}' \to \mathcal{S}'$ be defined by

$$\Lambda f = \mathcal{F}^{-1}\{(1 + \tau^2 + \sum_{l=1}^{n} \xi_l^4)^{\frac{1}{2}} \mathcal{F} f\},$$

where τ and $\xi = (\xi_1, \ldots, \xi_n)$ are the dual variables of t and $x = (x_1, \ldots, x_n)$ respectively.

Lemma 2.4. *The operator* $\Lambda: W_p^{2s,s}(\mathbf{R}_x^n \times \mathbf{R}_t^1) \to W_p^{2s-2,s-1}(\mathbf{R}_x^n \times \mathbf{R}_t^1)$ *is an isomorphism for every* $p \in (1, +\infty)$ *and* $s = 0, 1$ *and* 2.

Proof. It is based on the Mihlin theorem for the L_p-multipliers (cf., e.g. the result of Stein given in [Tay], ch.XI, §1, theorem 1.5). Using it one can easily prove that the functions

$$\frac{\xi_j.\tau}{1+\xi_j^2+\tau^2}, \quad \frac{\xi_j.\xi_i^2}{1+\xi_j^2+\xi_i^4}, \quad \frac{\tau.\xi_j^2}{1+\tau^2+\xi_j^4}, \quad \frac{\xi_j^2.\xi_i^2}{1+\xi_j^4+\xi_i^4}$$

as well as $(1 + \tau^2 + \sum_{s=1}^{n} \xi_s^4)^{\frac{1}{2}}.(1 + i\tau + |\xi|^2)^{-1}$ are L_p-multipliers. It is easy to deduce from here, that $\Lambda: W_p^{4,2}(\mathbf{R}^{n+1}) \to W_p^{2,1}(\mathbf{R}^{n+1})$ is continuous. Using again the Mihlin theorem one can show that the functions $\xi_j^k(1 + \tau^2 + \sum_{s=1}^{n} \xi_s^4)^{-\frac{1}{2}}$

for $k = 0,1,2$ and $\tau.(1+\tau^2+\sum_{s=1}^{n}\xi_s^4)^{-\frac{1}{2}}$ also are L_p–multipliers and deduce from here that $\Lambda: W_p^{4,2}(\mathbf{R}^{n+1}) \to W_p^{2,1}(\mathbf{R}^{n+1})$ is onto. Hence, according to the Banach open mapping theorem $\Lambda: W_p^{4,2}(\mathbf{R}^{n+1}) \to W_p^{2,1}(\mathbf{R}^{n+1})$ is an isomorphism. Checking that $\Lambda: W_p^{2,1}(\mathbf{R}^{n+1}) \to L_p(\mathbf{R}^{n+1})$ is an isomorphism is even easier (see also [G1], p.194). Since Λ does not depend on p, we have that $\Lambda: W_{p'}^{2,1} \to L_{p'}$, $\frac{1}{p}+\frac{1}{p'}=1$, is also an isomorphism, and hence, its transposed operator ${}^t\Lambda: L_p \to W_p^{-2,-1}$ is an isomorphism. Because $\Lambda = {}^t\Lambda$ over the space $\mathcal{S}$, which is dense in the considered spaces, the lemma is proved. ■

Lemma 2.4 yields the following extension of Theorem 2.2.

Lemma 2.5. *Let $\theta, s_0, s_1 \in (0,1)$ and $p \in (1,+\infty)$. Then*

$$(B_p^{2s_0,s_0}(\mathbf{R}_x^n \times \mathbf{R}_t^1), B_p^{-2s_1,-s_1}(\mathbf{R}_x^n \times \mathbf{R}_t^1))_\theta = B_p^{2s_\theta,s_\theta}(\mathbf{R}_x^n \times \mathbf{R}_t^1),$$

where $s_\theta = (1-\theta)s_0 - \theta s_1$.

P r o o f. Because of Lemma 2.4, Proposition 2.1 and Definition 2.3 the mapping

$$\Lambda: B_p^{2s+2,s+1}(\mathbf{R}_x^n \times \mathbf{R}_t^1) \to B_p^{2s,s}(\mathbf{R}_x^n \times \mathbf{R}_t^1)$$

is an isomorphism for $s \in (-1,1)$. Hence

$$\Lambda^{-1}(B_p^{2s_0,s_0}, B_p^{-2s_1,-s_1})_\theta = (\Lambda^{-1}B_p^{2s_0,s_0}, \Lambda^{-1}B_p^{-2s_1,-s_1})_\theta =$$

$$= (B_p^{2+2s_0,1+s_0}, B_p^{2-2s_1,1-s_1})_\theta = B_p^{2+2s_\theta,1+s_\theta} = \Lambda^{-1}B_p^{2s_\theta,s_\theta},$$

where the equality before the last is derived from Theorem 2.2. ■

Remark 2.6. The symmetric Besov spaces (denoted by $B_p^s(\mathbf{R}^n)$) have analogous properties (cf.,e.g. [G1], p.196, Corollary 9.1 and its particular case $p_0 = p_1 = q_0 = q_1 = p \in (1,+\infty)$).

Definition 2.7. (cf.[G2], p.349, Def. 7.1). Let Ω be a domain in $\mathbf{R}_x^n$, I be an open interval in $\mathbf{R}_t^1$ and $Q = \Omega \times I$. For $s > 0$ the space $B_p^{2s,s}(Q)$ is defined as

$$\left\{v \in \mathcal{D}'(Q) | \exists u \in B_p^{2s,s}(\mathbf{R}^{n+1}), v = u|_Q\right\},$$

where $u|_Q$ denotes restriction on Q.

The space $B_p^{2s,s}(Q)$, $s > 0$, is a Banach space with respect to the norm

$$\|v\|_{B_p^{2s,s}(Q)} = \inf_{v=u|_Q} \|u\|_{B_p^{2s,s}(\mathbf{R}^{n+1})}.$$

It follows from here that the operator "restriction on Q":

$$r_Q: B_p^{2s,s}(\mathbf{R}^{n+1}) \to B_p^{2s,s}(Q)$$

is continuous.

Let $\overset{\circ}{B}{}_p^{2s,s}(Q)$, $s > 0$, be the closure of $C_0^\infty(Q)$ in the norm of $B_p^{2s,s}(Q)$.

Definition 2.8. For $s > 0$ we define

$$B_p^{-2s,-s}(Q) = (\overset{\circ}{B}{}_{p'}^{2s,s}(Q))^*, \frac{1}{p} + \frac{1}{p'} = 1,$$

where "$*$" denotes the topological dual.

Following [LM1] (cf. Ch.I, Lemma 12.2) next we obtain results analogous to Theorem 2.2 and Lemma 2.5.

Lemma 2.9. *There exists an extension operator P from Q to $\mathbf{R}^{n+1}$ with the properties:*

$$P \in \mathcal{L}(W_p^{4,2}(Q), W_p^{4,2}(\mathbf{R}^{n+1})),$$
$$P \in \mathcal{L}(W_p^{-2,-1}(Q), W_p^{-2,-1}(\mathbf{R}^{n+1})),$$
$$r_Q P u = u \text{ for each } u \in W_p^{-2,-1}(Q),$$

where r_Q is the restriction on Q and $\mathcal{L}(X, Y)$ denotes the set of all linear bounded operators from X to Y.

S k e t c h o f P r o o f. Using the partition of unity, the problem is reduced to two different cases: extension from halfspace (which is treated in [LM1], Ch.I, Lemma 12.2) and extension from a corner between two hyperplanes (cf., e.g.,[LM2], Ch.IV, §2, p.15).

Thus, in view of the definition of $B^{2s,s}(Q)$, $s \in (-1, 2)$, we obtain the necessary retraction and coretraction which do not depend on s. Using them, a standard result from the theory of interpolation spaces ([BL], §6.4) and the already proved interpolation results for $B^{2s,s}(\mathbf{R}^{n+1})$, we obtain the following theorem.

Theorem 2.10. *Let $s_0, s_1 \in (-1, 2)$ and $\theta \in (0, 1)$. Then*

$$(B_p^{2s_0,s_0}(Q), B_p^{2s_1,s_1}(Q))_\theta = B_p^{2s_\theta,s_\theta}(Q),$$

where $s_\theta = (1 - \theta)s_0 + \theta s_1$.

Remark 2.11. Analogous result is valid for the symmetric Besov spaces $B_p^s(\Omega)$ (cf.[LM3]). The spaces $B_p^{r,s}(\Sigma)$, including those with negative r and s, are defined similarly to the respective spaces over Q. The difference here is, that, since Σ is an n–dimensional compact manifold, local cards has to be used (cf.[G2], p.349). A result analogous to Theorem 2.10 is true for the spaces $B_p^{r,s}(\Sigma)$. It is obtained by analogous to Lemma 2.9 result (cf. [LM2], Ch.IV, Lemma 14.2, p.79).

The above described scale of spaces will be essential in the proof of an existence theorem for the mixed problem for the parabolic operator from §1 with nonregular initial and boundary data. We follow the scheme developed in [LM2], Ch.IV.

We shall use the following

Theorem 2.12. ([G2], Th. 7.3, p.350). *Let n_x denote the unit outer normal to Q in the points of Σ and the mapping $u \mapsto \{f_k\}, \{g_j\}$ be defined by*

$$f_k = \left.\frac{\partial^k u}{\partial n_x^k}\right|_{\Sigma}, g_j = \left.\frac{\partial^j u}{\partial t^j}\right|_{\Omega_t} \qquad (t = 0 \text{ or } T).$$

Then it maps continuously the space $B_p^{2s,s}(Q)$, $s \in (0,2]$, (respectively $W_p^{2s,s}(Q)$ for $s = 1,2$), on a subspace of

$$\prod_{k<2s-\frac{1}{p}} B_p^{\alpha_k,\beta_k}(\Sigma) \times \prod_{j<s-\frac{1}{p}} B_p^{p_j}(\Omega_t),$$

where $\dfrac{\alpha_k}{2s} = \dfrac{\beta_k}{s} = \dfrac{2s-k-\frac{1}{p}}{2s}$, $p_j = 2(s-j-\frac{1}{p})$, *which subspace is defined by the respective compatibility conditions:*

(i) $\dfrac{\partial^j f_k}{\partial t^j} = \dfrac{\partial^k g_j}{\partial n_x^k}$ *on* $\partial\Omega_t$, $t = 0, T$, *if* $\dfrac{k}{2} + j < s - \dfrac{3}{2p}$;

(ii) integral compatibility conditions, if $\dfrac{k}{2} + j = s - \dfrac{3}{2p}$ *(see the following remark).*

Remark 2.13. We shall need the traces for $k = 0, 1$ and $j = 0$. Therefore even for $k = 1$ and $s = 1$ (the maximal value we shall use) under the condition

$$p > 3$$

only (i) has to be satisfied. In the principal case we shall be interested in (namely $s = \dfrac{1}{2}(1 + \dfrac{1}{p})$, see below), neither (i) nor (ii) has to be satisfied.

Theorem 9.13 on p.379 in [G2] for $\sigma = 0$ gives us the following basic result, which corresponds to (13.8) in [LM2], Ch.IV.

Theorem 2.14. *The mapping*

$$u \mapsto \{f, \varphi, \psi\}$$

defined by

$$\begin{aligned} Lu &= f \quad in \quad Q \\ Bu &= \varphi \quad on \quad \Sigma \\ u &= \psi \quad on \quad \Omega_0 \end{aligned}$$

is an isomorphism between $W_p^{2,1}(Q)$ *and a subspace of*

$$L_p(Q) \times B_p^{\alpha,\beta}(\Sigma) \times B_p^{p_0}(\Omega),$$

where $\alpha = 2\beta = 1 - \frac{1}{p}$ *and* $p_0 = 2(1 - \frac{1}{p})$, *which subspace is defined by the compatibility condition*

$$\varphi(x,0) = (B\psi)(x,0), x \in \partial\Omega_0.$$

Bearing in mind the conditions of Theorem 9.13 on p.379 in [G2], we point out that under the condition $p > 3$ (imposed in Remark 2.13) the number $0 + 1 - \frac{1}{p} - \frac{1}{2}(1 + \frac{1}{p}) = \frac{1}{2}(1 - \frac{3}{p})$ is not integer. As far as the compatibility condition is concerned, it is only one, since the inequality

$$\frac{h}{1} + \frac{1}{2} < 1 - \frac{3}{2p}$$

has unique nonnegative integer solution (namely $h = 0$); it is easily seen that the condition is exactly the one given in Theorem 2.14.

In order to obtain result, corresponding to (13.9) in [LM2], Ch.IV, we shall use the so called adjoint isomorphism (cf.[LM2], Ch.IV, §7). Let $g \in L_{p'}(Q)$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, and v be the solution of the problem

$$\begin{aligned} -\frac{\partial v}{\partial t} - Av &= g \quad \text{in} \quad Q \\ Bv &= 0 \quad \text{on} \quad \Sigma \\ v(x,T) &= 0 \quad \text{on} \quad \Omega. \end{aligned}$$

According to Theorem 2.14 we have $v \in W_{p'}^{2,1}(Q)$. The space

$$X(Q) = \left\{ v \in W_{p'}^{2,1}(Q) | \ B\, v|_{\Sigma} = 0, v|_{\Omega_T} = 0 \right\}$$

equipped with the norm

$$\|v\|_{X(Q)} = \|v\|_{W_{p'}^{2,1}(Q)} + \|L^* v\|_{L_{p'}(Q)}$$

where $L^* = -\dfrac{\partial}{\partial t} - A)$ is a Banach space. Since $L^* \colon X(Q) \longmapsto L_{p'}(Q)$ is continuous ($\|L^* v\|_{L_{p'}(Q)} \leq \|v\|_{X(Q)}$) and is onto (existence theorem), according to the Banach open mapping theorem it is an isomorphism between $X(Q)$ and $L_{p'}(Q)$.

Using the above described adjoint isomorphism by transposition (cf. [LM2], Ch.IV, §8.1, p.44) we obtain the following proposition.

Proposition 2.15. *If $\Phi \in X^*(Q)$, the topological dual of $X(Q)$, and $\langle\cdot,\cdot\rangle$ denotes the duality between $L_p(Q)$ and $L_{p'}(Q)$, then there exists a unique element $u \in L_p(Q)$ such that*

$$\langle u, -\frac{\partial v}{\partial t} - Av\rangle = \Phi(v) \text{ for each } v \in X(Q).$$

Next we choose (formally) $\Phi(v) = f(v) + \varphi(Bv) + \psi(v(x,T))$, where $f \in (W_{p'}^{2,1}(Q))^*$, $\varphi \in (B_{p'}^{\bar\alpha,\bar\beta}(\Sigma))^*$ and $\psi \in (B_{p'}^{\bar p_0}(\Omega))^*$, where $\bar\alpha = 2\bar\beta = 1 + \frac{1}{p}, \bar p_0 = \frac{2}{p}$ and "$*$" denotes the topological dual.

Remark 2.16. We point out that f, φ and ψ may not be distributions, but the function $u \in L_p(Q)$ still can be interpreted as a solution to the boundary value problem from Theorem 2.14 (cf. [LM2], pp.45-46). The case $f \equiv 0$ will be sufficient for our purposes and in what follows we shall confine ourselevs to it. The initial and the boundary condition could be interpreted by generalized traces, as it is done in [LM2], ch.IV, §10, but we shall not need it. The solution of the BVP from Theorem 2.14 will be understood in the sense of Proposition 2.15.

In §3 two existence results (with zero initial and zero boundary condition respectively) under different regularity assumptions will be needed. The first case is with $f \equiv 0$ and $\psi \equiv 0$. Let G be the linear operator mapping the set of the boundary data φ into the set of solutions to the problem

$$\begin{aligned} \frac{\partial u}{\partial t} - Au &= 0 \quad \text{in} \quad Q \\ Bu &= \varphi \quad \text{on} \quad \Sigma \\ u &= 0 \quad \text{on} \quad \Omega_0. \end{aligned} \tag{2.1}$$

According to Theorem 2.14 and Proposition 2.15 if $B_p^{\alpha,\beta}(\Sigma)\Big|_{\text{R.C.}}$ denotes the subspace of $B_p^{\alpha,\beta}(\Sigma)$, defined by the compatibility condition $\varphi(x,0) = 0$ on $\partial\Omega$,

$$G\colon B_p^{\alpha,\beta}(\Sigma)\Big|_{\text{R.C.}} \longrightarrow W_p^{2,1}(Q)$$

and

$$G\colon (B_{p'}^{\bar\alpha,\bar\beta}(\Sigma))^* \longrightarrow L_p(Q)$$

is a linear bounded operator, where $\alpha = 2\beta = 1 - \frac{1}{p}$, $\bar\alpha = 2\bar\beta = 1 + \frac{1}{p}$ and "$*$" denotes the topological dual. By interpolation we obtain that G is a linear bounded operator from

$$\left(B_p^{\alpha,\beta}(\Sigma)\Big|_{\text{R.C.}}, (B_{p'}^{\bar\alpha,\bar\beta}(\Sigma))^*\right)_\theta \tag{2.2}$$

to $(W_p^{2,1}(Q), L_p(Q))_\theta = B_p^{(1-\theta).2,1-\theta}(Q)$. In addition $u = G(\varphi)$ is a solution to (2.1) in the sense of proposition 2.15, namely

$$\langle u, -\frac{\partial v}{\partial t} - Av\rangle = \varphi(Bv) \text{ for each } v \in X(Q).$$

Now we look for θ, sich that $L_p(\Sigma)$ be included in the space (2.2). This will be satisfied if the space $(\mathring{B}_p^{\alpha,\beta}(\Sigma), (B_{p'}^{\bar{\alpha},\beta}(\Sigma))^*)_{\theta,p} = (B_{p'}^{-\alpha,-\beta}(\Sigma), B_{p'}^{\bar{\alpha},\beta}(\Sigma))^*_{\theta,p'}$ contains $L_p(\Sigma)$, which is equivalent to $(B_{p'}^{-\alpha,-\beta}(\Sigma), B_{p'}^{\bar{\alpha},\beta}(\Sigma))_{\theta,p'} \subset L_{p'}(\Sigma)$. Theorem 2.10 and Remark 2.11 yield that the last inclusion is true if $(1-\theta)(-\alpha) + \theta.\alpha' > 0$, i.e.

(2.3) $$\frac{1}{2}(1 - \frac{1}{p}) < \theta(< 1).$$

The trace on Ω_T of the respective solution to (2.1) belongs to $B_p^{\bar{p}}(\Omega)$, $\bar{p} = 2(1 - \theta - \frac{1}{p})$ (Theorem 2.12). According to the embedding theorems, this trace will be a Hölder function if $\bar{p} > \frac{n}{p}$, i.e. if

(2.4) $$1 - \frac{n+2}{2p} > \theta(> 0).$$

Conditions (2.3) and (2.4) are compatible if

(2.5) $$p > n + 1$$

Remark 2.13 and (2.5) can be unified in the following condition for p:

(2.6) $$p > \max\{3, n+1\}.$$

We proceed with the last result we shall need in §3. If u is the solution to the BVP

(2.7) $$\begin{aligned} \frac{\partial u}{\partial t} - Au &= 0 \quad \text{in} \quad Q \\ Bu &= 0 \quad \text{on} \quad \Sigma \\ u &= \psi \quad \text{on} \quad \Omega_0, \end{aligned}$$

where $\psi \in (B_p^{\bar{p}}(\Omega))^*$, $\bar{p} = 2(\frac{1}{p'} - \theta)$, $\theta \in (\frac{1}{2p'}, 1 - \frac{n+2}{2p})$, $\frac{1}{p} + \frac{1}{p'} = 1$, $p > \max\{3, n+1\}$, then the following condition is satisfied

(2.8) the trace $u|_\Sigma$ belongs to $L_{p'}(\Sigma)$.

(For $p > n+1$ and $\theta \in (\frac{1}{2p'}, 1 - \frac{n+2}{2p})$ we have $\frac{n}{p} < \frac{1}{p'}$ and $\bar{p} = 2(\frac{1}{p'} - \theta) \in (\frac{n}{p}, \frac{1}{p'})$).

Let $\bar{G}$ be the linear operator mapping the set of the initial data ψ into the set of solutions to Problem (2.7). According to Theorem 2.14 and Proposition 2.15,

$$\bar{G}\colon B^{q_0}_{p'}(\Omega)\Big|_{\text{R.C.}} \longrightarrow W^{2,1}_{p'}(Q), q_0 = \frac{2}{p}$$

and

$$\bar{G}\colon (B^{\bar{q}_0}_{p}(\Omega))^* \longrightarrow L_{p'}(Q), \bar{q}_0 = \frac{2}{p'}$$

is a linear bounded operator, where, as usual, "$*$" denotes the topological dual. By interpolation we obtain that

$$\bar{G}\colon (B^{q_0}_{p'}(\Omega), (B^{\bar{q}_0}_{p}(\Omega))^*)_{\bar{\theta},p'} \longrightarrow B^{2(1-\bar{\theta}),1-\bar{\theta}}_{p'}(Q), \quad \bar{\theta} \in (0,1)$$

is linear bounded operator, mapping the initial data ψ ($\psi \in (B^{q_0}_{p'}(\Omega), (B^{\bar{q}_0}_{p}(\Omega))^*)_{\bar{\theta},p'}$ into the set u of solutions to (2.7) in the sense of Proposition 2.15 ($u \in B^{2(1-\bar{\theta}),1-\bar{\theta}}_{p'}$ (Q)).

Next we look for $\bar{\theta} \in (0,1)$ such that

$$(B^{\bar{p}}_{p}(\Omega))^* \subset (B^{q_0}_{p'}(\Omega), (B^{\bar{q}_0}_{p}(\Omega))^*)_{\bar{\theta},p'}.$$

The above inclusion will be satisfied if

$$(B^{\bar{p}}_{p}(\Omega))^* \subset (\overset{\circ}{B}{}^{q_0}_{p'}(\Omega), (B^{\bar{q}_0}_{p}(\Omega))^*)_{\bar{\theta},p'} = (B^{-q_0}_{p}(\Omega), B^{\bar{q}_0}_{p}(\Omega))^*_{\bar{\theta},p},$$

i.e. if $B^{\bar{p}}_{p}(\Omega) \supset B^{(1-\bar{\theta})(-q_0)+\bar{\theta}\bar{q}_0}_{p}(\Omega)$. The last inclusion will be true if $\bar{p} \le -q_0 + \bar{\theta}(q_0 + \bar{q}_0) = -\frac{2}{p} + 2\bar{\theta}$, i.e.if $\frac{\bar{p}}{2} + \frac{1}{p} \le \bar{\theta}(< 1)$. (Since $\bar{p} < \frac{1}{p'}$, we have $\frac{\bar{p}}{2} + \frac{1}{p} < \frac{1}{2p'} + \frac{1}{p} = \frac{1}{2}(1 + \frac{1}{p}) < 1$).

The trace on Σ of the obtained solution belongs to $B^{\mu,\nu}_{p'}(\Sigma)$, $\mu = 2\nu = 2 - 2\bar{\theta} - \frac{1}{p'}$. The condition (2.8) is equivalent to $\mu > 0$, i.e. to $\bar{\theta} < \frac{1}{2}(1 + \frac{1}{p})$. Hence, choosing $\bar{\theta} \in (\frac{\bar{p}}{2} + \frac{1}{p}, \frac{1}{2}(1 + \frac{1}{p}))$ we see that (2.8) is true.

Now we summarize the obtained results in the following

Proposition 2.17. *Under the assumptions for the domain Ω and for the operators A and B made in §1 and under the condition (2.6) imposed on p we have:*

where $u \in W_p^{2,1}(Q)$ is the solution to the following BVP

$$(3.2) \qquad \begin{aligned} Lu &= g \quad \text{in} \quad Q, g \in L_p(Q) \\ Bu &= 0 \quad \text{on} \quad \Sigma \\ u(x,0) &= 0 \quad \text{on} \quad \Omega. \end{aligned}$$

Integrating by parts we obtain

$$(3.3) \qquad \begin{aligned} \kappa(u(x,T)) &= \lim_{\delta\to 0}\int_{Q_\delta} w.Lu\,dx\,dt = \lim_{\delta\to 0}\int_{Q_\delta}(w.Lu - u.L^*w)dx\,dt = \\ &= \lim_{\delta\to 0}[\int_\Omega u(x,T-\delta).w(x,T-\delta)dx - \int_{\Sigma_\delta} w(x,t).Bu(x,t)dx\,dt] = \\ &= \lim_{\delta\to 0}\int_\Omega u(x,T-\delta).w(x,T-\delta)dx. \end{aligned}$$

Now let $\varrho \in (0,T)$ be fixed and let $u \in W_p^{2,1}(Q)$ satisfy $Bu = 0$ on $\Sigma \setminus \Sigma_\varrho$ and $u(x,0) = 0$ on Ω. Then

$$u_0(x,t) = \begin{cases} (t-T+\varrho).u(x,t) & \text{for} \quad T-\varrho \le t \le T \\ 0 & \text{for} \quad 0 \le t \le T-\varrho \end{cases}$$

belongs to $W_p^{2,1}(Q)$ and is the solution to (3.2) with appropriate $g \in L_p(Q)$. Hence, using (3.3) we obtain

$$\kappa(\varrho.u(x,T)) = \lim_{\delta\to 0}\int_\Omega u_0(x,T-\delta).w(x,T-\delta)dx,$$

i.e.

$$\kappa(u(x,T)) = \lim_{\delta\to 0}\frac{\varrho-\delta}{\varrho}.\int_\Omega u(x,T-\delta).w(x,T-\delta)dx,$$

which is the same as (3.3).

Finally, let $\varphi_n(x,t)$ be C^∞ on Σ for each natural n, $\varphi_n \equiv 0$ on $\Sigma\setminus\Sigma_{1/n}$, $\lim_{n\to\infty}\varphi_n = \varphi$ in the L_p–norm and v_n be the solution to (2.1), corresponding to φ_n. By continuity argument we have $\lim_{n\to\infty} v_n(x,T) = v(x,T)$ in $B_p^{\bar{p}}(\Omega)$. Hence, using (3.3) we obtain

$$\begin{aligned} \kappa(v(x,T)) &= \lim_{n\to\infty}\kappa(v_n(x,T)) = \lim_{n\to\infty}\lim_{\delta\to 0}\int_\Omega v_n(x,T-\delta).w(x,T-\delta)dx \\ &= \lim_{n\to\infty}\lim_{\delta\to 0}\int_{\Sigma_\delta} w(x,t).\varphi_n(x,t)\,d\Sigma = \lim_{n\to\infty}\int_\Sigma w(x,t).\varphi_n(x,t)\,d\Sigma \\ &= \int_\Sigma w(x,t).\varphi(x,t)\,d\Sigma. \end{aligned}$$

Proposition 3.2. *If $w \in L_{p'}(Q)$ is the solution of (3.1) for $\kappa \in (B_p^{\bar{p}}(\Omega))^*$ and $w = 0$ in Q, then $\kappa = 0$.*

Proof. Let $z_1 \in W_p^{2,1}(Q)$ and $z_2 \in B_p^{2(1-\theta),1-\theta}(Q)$ (where $\frac{1}{2}(1-\frac{1}{p}) < \theta < 1-\frac{n+2}{2p}$, cf. §2) be the solutions to

$$\begin{aligned} Lz_1 &= g \quad \text{in} \quad Q, g \in L_p(Q) \\ Bz_1 &= 0 \quad \text{on} \quad \Sigma \\ z_1(x,0) &= 0 \quad \text{on} \quad \Omega \end{aligned} \qquad \text{and} \qquad \begin{aligned} Lz_2 &= 0 \quad \text{in} \quad Q \\ Bz_2 &= \varphi \quad \text{on} \quad \Sigma, \varphi \in L_p(\Sigma) \\ z_2(x,0) &= 0 \quad \text{on} \quad \Omega \end{aligned}$$

respectively. Let $z = z_1 + z_2$. Then $z \in B_p^{2(1-\theta),1-\theta}(Q)$ is the solution to

$$\begin{aligned} Lz &= g \quad \text{in} \quad Q, g \in L_p(Q) \\ Bz &= \varphi \quad \text{on} \quad \Sigma, \varphi \in L_p(\Sigma) \\ z(x,0) &= 0 \quad \text{on} \quad \Omega \end{aligned}$$

and depends continuously on $g \in L_p(Q)$ and $\varphi \in L_p(\Sigma)$ because of the analogous properties of z_1 and z_2.

Now let $\xi(x) \in C^2(\Omega)$ be arbitrary. Let $\Phi(x,t) \in C^2(\mathbf{R}^{n+1})$ satisfy $\{t = 0\} \cup \operatorname{supp}\Phi = \emptyset$ and $\Phi(x,T) = \xi(x)$ on Ω. Let $\chi_n(t) \in C^\infty[0,T]$ satisfy $\chi_n \equiv 1$ on $[0, T-\frac{2}{n}]$ and $\chi_n \equiv 0$ on $[T-\frac{1}{n}, T]$ for $n \in \mathbf{N}$. Let v_n be the solution to

$$\begin{aligned} Lv_n &= \chi_n.L\Phi \quad \text{in} \quad Q \\ Bv_n &= \chi_n.B\Phi \quad \text{on} \quad \Sigma \\ v_n(x,0) &= 0 \quad \text{on} \quad \Omega \end{aligned}$$

for every $n \in \mathbf{N}$. Since $\lim_{n\to\infty} \chi_n.L\Phi = L\Phi$ in $L_p(Q)$ and $\lim_{n\to\infty} \chi_n.B\Phi = B\Phi$ in $L_p(\Sigma)$, by continuity argument we obtain $\lim_{n\to\infty} v_n(x,T) = \Phi(x,T) = \xi(x)$ in $B_p^{\bar{p}}(\Omega)$. Using the same argument as in the proof of Proposition 3.1 we obtain

$$\kappa(\xi(x)) = \lim_{n\to\infty}(v_n(x,T)) = \lim_{n\to\infty} \int_\Sigma w.Bv_n = 0$$

Since $C^2(\Omega)$ is dense in $B_p^{\bar{p}}(\Omega)$, we have $\kappa = 0$.

Before proving Theorems 1.2 and 1.4 we give the existence result.

Proposition 3.3. *The problem (P) is solvable, i.e. there is $\hat{\varphi} \in U$ such that*

$$\|u(\cdot,T;\hat{\varphi}) - z(\cdot)\|_{B_p^{\bar{p}}(\Omega)} = \min_{\varphi\in U} \|u(\cdot,T;\varphi) - z(\cdot)\|_{B_p^{\bar{p}}(\Omega)}.$$

The proof is based on standard weak compactness and week lower semi-continuity argument (using Theorem 7 on p.671 in [GW], the fact that $L_\infty(\Sigma) \subset L_p(\Sigma)$ for $p \geq 1$ and Propositions 2.17 and 3.1).

We shall need the following two results.

Lemma 3.4. ([M]). *Let $w \in C^\infty(\bar{Q} \times [0,T))$ be a solution of $L^*w = 0$. If $w|_\gamma = \left.\frac{\partial w}{\partial \nu}\right|_\gamma = 0$ for some nonvoid open subset $\gamma \subset \Sigma$, then $w = 0$ in the points $(x,t) \in Q$ which can be connected with a point of γ by a horizontal line (a line on which $t =$const).*

Lemma 3.5. ([Tan]). *Let $w \in C^\infty(\bar{Q} \times [0,T))$ satisfy $L^*w = 0$ and $Bw = 0$. If all data and $\partial\Omega$ are analytic, then $w|_\Sigma$ is analytic too.*

P r o o f o f T h e o r e m 1.2. Let $\gamma \subset \Sigma$ have positive measure. We shall prove that

$$\{u(\cdot,T;\varphi) | \varphi \in L_\infty(\Sigma), \mathrm{supp}\varphi \subset \gamma\} \tag{3.4}$$

is dense in $B_p^{\bar{p}}(\Omega)$. Assuming the contrary, we can find $\kappa \in (B_p^{\bar{p}}(\Omega))^*$, $\kappa \neq 0$, such that $\kappa(u(x,T;\varphi)) = 0$ for each $\varphi \in L_\infty(\Sigma)$ which satisfies $\mathrm{supp}\varphi \subset \gamma$. According to Proposition 3.1

$$0 = \kappa(u(x,T;\varphi)) = \int_\Sigma w(x,t).\varphi(x,t)d\Sigma = \int_\gamma w(x,t).\varphi(x,t)d\Sigma,$$

where $w \in L_{p'}(Q)$ is the solution of (3.1). Hence $w|_\gamma = 0$. Since $w|_\Sigma$ is analytic (Lemma 3.5) we have $w|_\Sigma = 0$. Since $Bw = 0$ on Σ, $\left.\frac{\partial w}{\partial \nu}\right|_\Sigma = 0$. Hence $w = 0$ in Q (Lemma 3.4). By Proposition 3.2 we get $\kappa = 0$ which contradicts the choice of κ.

Now let us assume that a solution $\hat{\varphi}$ of (P) (which exists by Proposition 3.3) is not "bang–bang", i.e. there are some $\delta > 0$ and $\gamma \subset \Sigma$ of positive measure such that $|\hat{\varphi}| \leq 1 - \delta$ on γ. Let $\varphi_1 \in L_\infty(\Sigma)$ be such that $\mathrm{supp}\varphi_1 \subset \gamma$ and

$$\|u(\cdot,T;\hat{\varphi}+\varphi_1) - z(\cdot)\|_{B_p^{\bar{p}}(\Omega)} \leq \frac{1}{2}\|u(\cdot,T;\hat{\varphi}) - z(\cdot)\|_{B_p^{\bar{p}}(\Omega)}$$

(the set (3.4) being dense in $B_p^{\bar{p}}(\Omega)$). Let $\eta \in (0, \frac{\delta}{\|\varphi_1\|_{L_\infty(\Omega)}})$. We have

$$\|u(\cdot,T;\hat{\varphi}+\eta.\varphi_1) - z(\cdot)\|_{B_p^{\bar{p}}(\Omega)} \leq (1-\eta)\|u(\cdot,T;\hat{\varphi}) - z(\cdot)\|_{B_p^{\bar{p}}(\Omega)} +$$

$$+\eta\|u(\cdot,T;\hat{\varphi}+\varphi_1) - z(\cdot)\|_{B_p^{\bar{p}}(\Omega)} < \|u(\cdot,T;\hat{\varphi}) - z(\cdot)\|_{B_p^{\bar{p}}(\Omega)}.$$

Since $\hat{\varphi} + \eta\varphi_1 \in U$, this contradicts the optimality of $\hat{\varphi}$.

Corollary 3.6. *Let $p > \max\{3, n+1\}$ and $\bar{p} \in (\frac{n}{p}, 1 - \frac{1}{p})$ hold. Let $z(\cdot) \notin \{u(\cdot, t; \varphi) | \varphi \in U\}$. Then if the data and $\Gamma = \partial\Omega$ are analytic, problem (P) has unique solution.*

P r o o f. Only the uniqueness has to be proved. It can be derived directly from the "bang–bang" principle by assuming the contrary.

P r o o f o f C o r o l l a r y 1.4. Given $\varepsilon \in (0,1)$ we choose $p > (n+1)/(1-\varepsilon), \bar{p} = \varepsilon + n/p$ and we have $B_p^{\bar{p}}(\Omega) \subset C^{0,\varepsilon}(\bar{\Omega})$ by the Sobolev embedding theorem. Since $(C^{0,\varepsilon}(\bar{\Omega}))^* \subset (B_p^{\bar{p}}(\Omega))^*$, the same argument as in Proposition 3.3 yields existence of solution to the problem (P'). By Proposition 3.1 the set (3.4) is dense in $B_p^{\bar{p}}(\Omega)$ and because $B_p^{\bar{p}}(\Omega)$ is dense in $C^{0,\varepsilon}(\bar{\Omega})$, we conclude that (3.4) is dense in $C^{0,\varepsilon}(\bar{\Omega})$. From here on, the proof is the same as the proof of Theorem 1.2.

P r o o f o f T h e o r e m 1.5. (see the proof of Theorem 8 on p.672 in [GW]). Since $z(\cdot) \notin \{u(\cdot, T; \varphi) | \varphi \in U\}$, there is a nonzero $\kappa \in (B_p^{\bar{p}}(\Omega))^*$ separating $\{u(\cdot, T; \varphi) | \varphi \in U\}$ and the open ball with centre $z(\cdot)$ and radius $\rho = \min_{\varphi \in U} \|u(\cdot, T; \varphi) - z(\cdot)\|_{B_p^{\bar{p}}(\Omega)} > 0$, i.e. there is a real l such that

$$\kappa(u(\cdot, T; \varphi)) \leq l \quad \text{for each} \quad \varphi \in U$$

and $\kappa(\tilde{z}(\cdot)) > l$ for each $\tilde{z}(\cdot) \in B_p^{\bar{p}}(\Omega)$ satisfying $\|\tilde{z}(\cdot) - z(\cdot)\|_{B_p^{\bar{p}}(\Omega)} < \rho$. Let $\hat{\varphi}$ be a solution of (P). Since $\|u(\cdot, T; \hat{\varphi}) - z(\cdot)\|_{B_p^{\bar{p}}(\Omega)} = \rho$, we have $\kappa(u(\cdot, T; \hat{\varphi})) \geq l$. Hence

$$\kappa(u(\cdot, T; \hat{\varphi})) = \sup_{\varphi \in U} \kappa(u(\cdot, T; \varphi)).$$

By Proposition 3.1 we obtain

$$\int_\Sigma w(x,t).\hat{\varphi}(x,t) d\Sigma = \sup_{\varphi \in U} \int_\Sigma w(x,t).\varphi(x,t) d\Sigma,$$

where w is the solution of (3.1). Let $\gamma \subset \Sigma$ be the set of zeros of $w(x,t)|_\Sigma$. It is clear that $|\hat{\varphi}(x,t)| = 1$ a.e. on $\Sigma \setminus \gamma$. If γ has internal points, as in the proof of Theorem 1.2, but using only Lemma 3.4 and the reverse uniqueness theorem (cf. [F], Ch.VI, §7), we obtain $w = 0$. By Proposition 3.2 we have $\kappa = 0$ which is a contradiction.

Remark 3.7. Since $C^{0,\varepsilon}(\bar{\Omega})$, $\varepsilon > 0$, is dense in $C(\bar{\Omega})$ and has stronger norm, Theorem 3 on p. 667 in [GW] is a corollary of Theorem 1.2.

References

[B] A.G. B u t k o v s k i y, *Distributed Control Systems*, Elsevier, N. York (1969).

[BL] J. B e r g h, J. L ö f s t r ö m, *Interpolation Spaces. An Introduction*, Springer, Berlin (1976).

[C] E. C a s a s, Pontryagin's principle for state-constrained boundary control problems of semilinear parabolic equations, *SIAM J. Control and Optim.*, **35**, No 4 (1997), 1297-1327.

[F] A. F r i e d m a n, *Partial Differential Equations of Parabolic Type*, Prentice–Hall, Inc. (1964).

[G1] P. G r i s v a r d, Commutativité de deux foncteurs d'interpolation et applications, *J. Math. Pures et Appl.*, **45** (1966), 143-290.

[G2] P. G r i s v a r d, Equations differentielles abstraites, *Ann. Scient. Ec. Norm. Sup., 4e série*, **2** (1969), 311-395.

[Gl] K. G l a s h o f f, Optimal control of one-dimensional linear parabolic differential equations, *Lect. Notes Math.*, **477** (1975), 102-120.

[GS] K. G l a s h o f f, E. S a c h s, On theoretical and numerical aspects of the bang–bang principle, *Numer. Math.*, **29** (1977), 93-113.

[GW] K. G l a s h o f f, N. W e c k, Boundary control of parabolic differential equations in arbitrary dimensions: supremum–norm problems, *SIAM J. Control and Optim.*, **14**, No 4 (1976), 667-681.

[HW] W. H a c k b u s c h, Th. W i l l, A numerical method for a parabolic bang–bang problem, *Control and Cybernetics*, **12**, No 3-4 (1983), 99-116.

[IT] I. I o r d a n o v, Ts. T s a c h e v, A note on the bang–bang principle for a parabolic optimal control problem, In: *Proc. Sixteenth Internat. School "Applications of Mathematics in Technics", Varna, Aug. 28 - Sept. 6, 1990.*

[KY] K. K o z o n o, M. Y a m a z a k i, Semilinear heat equations and the Navier–Stokes equation with distributions in new function spaces as initial data, *Commun. in Partial Diff. Eq-s*, **19**, No 5-6 (1994), 959-1014.

[L] J.-L. L i o n s, Contrôle optimale de systèmes gouvernés par des équations aux dérivées partielles, Dunod, Paris (1968).

[LM1] J.-L. L i o n s, E. M a g e n e s, *Problèmes aux limites non homogènes et applications*, **1**, Dunod, Paris (1968).

[LM2] J.-L. L i o n s, E. M a g e n e s, *Problèmes aux limites non homogènes et applications*, **2**, Dunod, Paris (1968).

[LM3] J.-L. L i o n s, E. M a g e n e s, Problemi ai limiti non omogenei, III, *Ann. S.N.S. Pisa, serie III*, **XV** (1961), 41-103.

[LSU] O.A. L a d i j e n s k a y a, V.A. S o l o n n i k o v, N.N. U r a l ' t s e v a, *Linear and Quasilinear Equations of Parabolic Type* (in Russian), Moscow (1967).

[M] S. M i z o h a t a, Unicité du prolongment des solutions pour quelques opérateurs différentiels paraboliques, *Mem. Coll. Sci. Kyoto Univ.*, **A 31** (1958), 219-239.

[R] F. R i b a u d, Semilinear parabolic equations with distributions as initial data, *Discrete and Continuous Dynamical Systems*, **3**, No 3 (1997), 305-316.

[SW] E.J.P.G. S c h m i d t, N. W e c k, On the boundary behaviour of solutions to elliptic and parabolic equations – with applications to boundary control for parabolic equations, *SIAM J. Control and Optim.*, **16** (1978), 593-598.

[Tan] H. T a n a b e, On differentiability and analyticity of solutions of weighted elliptic boundary value problems, *Osaka J. Math.*, **2** (1965), 163-190.

[Tay] M. T a y l o r, *Pseudodifferential Operators*, Princeton Univ. Press, Princeton (1981).

* *Faculty of Mathematics and Informatics*
Sofia University
5ª, James Boucher Str.
1126 Sofia, BULGARIA

Received: 07.07.1998

** *Institute of Mathematics and Informatics – Bulg. Acad. Sci.*
Acad. G. Bonchev Str., Block 8
1113 Sofia, BULGARIA
e-mail: tsachev@math.bas.bg