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Two-Parameter Bifurcations of Multiple Steady States of a Reaction-Diffusion Equation

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Presented by Bl. Sendov

In this paper we study the number and properties of the solutions of the equation $\varepsilon u_{xx} = u(u-1)(u-\lambda), u(0) = 0, u(1) = \mu, \lambda \in (0,1)$, depending on the values of $\varepsilon > 0$ and $\mu \geq 0$. It is found that all solutions are either increasing or have only one critical value, a maximum. The positive quadrant of the (ε, μ)-plane is cut by bifurcation curves into regions with: a) one solution only (of the first type); b) one solution of the first type and even number of solutions of the second type (at least two); c) no solutions of the first type and one solution of the second type; d) no solutions of the first type and three or more (odd number) solutions of the second type. The geometry of the regions is investigated in details.

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0. Introduction

Smoller and Wasserman [5] investigated the existence and nonuniqueness of steady state solutions of the parabolic equation

$$(S) \qquad u_t = u_{xx} - f(u),$$

with homogeneous boundary conditions:

$$u(-L,t) = u(L,t) = 0,$$

where $f(u)$ is a cubic polynomial. This parabolic boundary value problem is known as the Nagumo equation and finds applications in the investigation of the FitzHugh - Nagumo model of nerve excitation. It is used in the modelling of myelinated nerve axons [1], [2] and in other applications.

With a proper change of variables this problem becomes equivalent to the following one:

$$u_t = \varepsilon u_{xx} - f(u),$$
$$u(0,t) = u(1,t) = 0, \text{ for } \varepsilon > 0.$$

The equation of the steady state solutions is then:

$$(SS) \qquad \begin{aligned} \varepsilon u_{xx} &= f(u), \\ u(0) &= u(1) = 0. \end{aligned}$$

Smoller and Wasserman [5] showed the existence of a bifurcation value of the parameter L. In terms of ε their result can be translated so: The trivial solution $u \equiv 0$ exists for all ε. There is an ε_0^* such that for $\varepsilon > \varepsilon_0^*$ there are no other steady state solutions; for $\varepsilon < \varepsilon_0^*$ and $\varepsilon = \varepsilon_0^*$ there exist respectively two and one nontrivial steady state solutions.

Since the work of Smoller and Wasserman [5], a number of papers have been devoted to the qualitative investigation of the solutions of the homogeneous problem (SS). To the best of our knowledge, however, the steady states of the nonhomogeneous problem have not been considered.

In this paper we investigate the steady state solutions of the parabolic equation (S) with nonhomogeneous boundary conditions, thus getting the problem:

$$(SSS) \qquad \begin{aligned} \varepsilon u_{xx} &= f(u), \\ u(0,t) &= 0, \\ u(1,t) &= \mu \geq 0, \end{aligned}$$

varying ε and μ. We break the problem into two distinct cases by looking for solutions with zero or one critical points since no other case is possible.

We use a modification (based on inverse functions) of the "time-map" method utilized by Smoller and Wasserman to find explicit solutions of the steady state equation. Although the method is not new (see [4] and [3]), our results are. We investigate the bifurcations of solutions of equations (1a), (1b) and (1c) (hereafter referred to as equation (1)) in a two - dimensional parameter space (ε, μ) while, in this context, in [5] their projection on the one-dimensional parameter space $\mu = 0$ is investigated. Thus, our results cannot be predicted from those in [5]. Really, the trivial and non-trivial solutions found by Smoller and Wasserman can be viewed as a nondecreasing solution and solutions with a maximum of (1) in the case $\mu = 0$. If the distribution of solutions of the first and the second type were to remain similar for all $\mu \geq 0$, there would be simply two regions so that in the first one (1) would have three solutions (one increasing and two with a maximum) while in the second one there would be only one solution, the nondecreasing one.

Our results show that this is not the case.

We find a variety of situations: a) a region with only one solution with a maximum; b) a region with three or more (odd number) solutions with a maximum; c) a region with one nondecreasing solution and two or more (even number) solutions with a maximum; d) a region with only one nondecreasing solution. The union of these regions constitutes the whole positive quadrant of the (ε, μ)-plane. As is seen, the cases a) and b) are newly emerged phenomena for large enough positive μ: regions in which nondecreasing solutions do not exist.

Therefore, our results are not a trivial generalization of the results in [5].

As we consider the steady states of the Nagumo equation, one could expect that the following presentation may have significant applications in nerve conduction theory or other fields as well. A further task is the stability analysis of these steady states and possible applications of it but this will be the the subject of a separate paper.

It is worthy of note that we not only establish the number of solutions in the different cases but also classify the solutions according to their monotonicity properties and point out explicitly the bifurcation curves.

The structure of this paper is as follows. In §1 we give the problem a tractable form. In §2 we find the regions of existence of nondecreasing solutions. Section 3 is similarly devoted to solutions with a maximum. All considerations in these two sections are done in the case $0 < \lambda < \frac{1}{2}$. In §4 it is shown that no solutions with more than one critical points exist. The rather short §5 is devoted to the case $1 > \lambda \geq \frac{1}{2}$. Sections 6 and 7 are conclusive.

1. Statement of the problem

We shall consider the following problem: Given values of $\lambda \in (0, 1), \varepsilon > 0$ and $\mu \geq 0$, how many solutions does the following boundary value problem

$$\varepsilon u_{xx} = u(u-1)(u-\lambda) = f(u), \tag{1a}$$

$$u(0) = 0, \tag{1b}$$

$$u(1) = \mu \geq 0, \tag{1c}$$

have? Does the number of solutions depend on the values of ε, μ and how?

Equation (1) is equivalent to the following system:

$$\varepsilon v' = f(u),\quad u' = v, \tag{2a}$$

$$u(0) = 0,\quad u(1) = \mu. \tag{2b}$$

For a given value v_0, the system (2a) subject to the initial conditions

$$u(0) = 0,\quad v(0) = v_0, \tag{2c}$$

has a unique solution. Each solution of (1), if it exists, is a solution of (2a), (2c) for a properly chosen v_0. We pose the problem in the following way: *Given $\lambda, \varepsilon, \mu$, what is the number of different values v_0, for which the solution of (2a), (2c) is also a solution of (2a), (2b)?* Obviously, if the solution of (2a), (2c) does not exist, this number is zero.

We shall first find the form of the solutions of (2a), (2c) and later we will select out of these the ones satisfying (1).

If $u(x)$ is a solution of (2a), (2c), it cannot be decreasing on the whole interval [0,1], since $\mu > 0$. There must be a subinterval of [0,1] on which $u(x)$ is monotonously increasing. We can classify the solutions of (2a), (2c) according to the number of their intervals of monotonicity.

We first consider the case $0 < \lambda < \frac{1}{2}$ as it is more complicated.

2. Nondecreasing solutions: case $0 < \lambda < \frac{1}{2}$

2.1. Necessary and sufficient conditions for uniqueness

Assume that (2a), (2c) has a nondecreasing solution i.e. such that $u'(x) = v(x) \geq 0, x \in (0,1)$.

We shall call these *solutions of type 1.*

Then $v_0 > 0$ and since $v(x) = 0$ only at isolated values of x (uniqueness arguments), $u(x)$ is invertible in $[0,1]$. We can define $x(u)$ and then (2a),(2c) is equivalent to

$$\varepsilon\frac{dv}{du} = \frac{f(u)}{v},\quad \frac{dx}{du} = \frac{1}{v},\quad v(0) = v_0,\quad x(0) = 0,$$

which can be solved in the following way:

(3)
$$v = \sqrt{\frac{2}{\varepsilon}F(u) + v_0^2},$$
$$x = \int_0^u \frac{dw}{\sqrt{\frac{2}{\varepsilon}F(w) + v_0^2}} = \Upsilon(v_0^2, u),$$

where
$$F(u) = \int_0^u f(u)du = u^2\left[\frac{u^2}{4} - \frac{(1+\lambda)}{3}u + \frac{\lambda}{2}\right].$$

Taking the inverse of $\Upsilon(v_0^2, u)$, if v_0 is fixed, we obtain:

(4)
$$u(x) = \Upsilon^{-1}(v_0^2, x)$$
$$v(x) = \sqrt{\frac{2}{\varepsilon}F(\Upsilon^{-1}(v_0^2, x)) + v_0^2}.$$

As we see from (3), all the nondecreasing solutions of (1) are strictly increasing and formulae (4) give an expression for the solution of (2a), (2c) in the monotonous case considered. Obviously, (4) will also give the solution of (1) if and only if

(5)
$$\Upsilon(v_0^2, \mu) = 1.$$

Before attempting to clarify when (5) will hold, let us note that $F(u)$ has critical points at $u = 0$ (minimum), $u = \lambda$ (maximum), $u = 1$ (minimum) and that $F(u)$ has three roots. Because $0 < \lambda < \frac{1}{2}$, the roots of F are: $u_1 = 0$ (a double root), $u_2 \in (\lambda, 1), u_3 > 1$.

Theorem 1. *If $\mu \le u_2$ or $\mu \ge 1$, then (5) has a unique solution v_0 (and therefore (1) has a unique monotonously increasing solution). If $1 > \mu > u_2$, then (5) (and therefore, (1)) has a solution if and only if*
$$\Upsilon(-\frac{2}{\varepsilon}F(\mu), \mu) \ge 1.$$
If it exists, the solution is unique.

Proof. $\Upsilon(v_0^2, u)$ is defined on $D_2 = (0,\infty) \times [0, u_2] \bigcup (-\frac{2}{\varepsilon}F(1), \infty) \times [1, \infty) \bigcup W \subset R^2$, where $W = \{(v_0^2, u) : \frac{-2}{\varepsilon}F(u) < v_0^2 < \infty\}$. It is a decreasing function of v_0^2 (this justifies the uniqueness) and tends to zero when $v_0^2 \to \infty$. Obviously, $\Upsilon(v_0^2, \mu) = 1$ has a solution iff
$$\lim_{v_0^2 \to \eta} \Upsilon(v_0^2, \mu) \ge 1,$$

where

$$\eta = \begin{cases} 0, & \text{if } \mu \le u_2; \\ -\dfrac{2}{\varepsilon}F(1), & \mu \ge 1; \\ -\dfrac{2}{\varepsilon}F(\mu), & \mu \in (u_2, 1), \end{cases}$$

This makes the second statement of the theorem clear.

Also, since $\lim\limits_{v_0^2 \to 0, -\frac{2}{\varepsilon}F(1)} \Upsilon(v_0^2, \mu) = \infty$ (as 0 is a double root of F and 1 is a double root of $F - F(1)$), then in these cases there always exists a unique solution of (5). ■

Further we have to clarify on what conditions $\Upsilon(-\frac{2}{\varepsilon}F(\mu), \mu) \ge 1$ can hold.

Consider Υ as a function of ε too: $\Upsilon = \Upsilon(\varepsilon, v_0^2, u)$ with domain $D_3 = (0, \infty) \times D_2 \subset R^3$.

Theorem 2. *Given $\mu \in (u_2, 1)$, there exists $\varepsilon_1(\mu)$ such that equation (1) has a unique monotonously increasing solution for all $\varepsilon \ge \varepsilon_1(\mu)$ and no such solution otherwise. Besides, the bifurcation curve $(\varepsilon_1(\mu), \mu)$ is defined by*

$$\varepsilon_1(\mu) = 2/\left(\int_0^\mu \frac{dw}{\sqrt{F(w) - F(\mu)}}\right)^2. \tag{6}$$

Proof. It follows from Theorem 1 that

$$\Upsilon(\varepsilon, v_0^2, \mu) = 1 \tag{7}$$

has a unique solution v_0^2 iff

$$\sqrt{\frac{\varepsilon}{2}} \int_0^\mu \frac{dw}{\sqrt{F(w) - F(\mu)}} \ge 1. \tag{8}$$

Obviously, if $\varepsilon_1(\mu)$ satisfies (6), then (8) will be satisfied iff $\varepsilon \ge \varepsilon_1(\mu)$. For each such ε equation (7) will have a unique positive solution v_0. ■

Thus the existence of nondecreasing solutions has been completely investigated.

Remark 1. In comparison with the results of Smoller and Wasserman [5] for the one parameter problem, one observes a new phenomenon. For $\mu = 0$ the solution of type 1 exists for $\varepsilon > 0$ and this is still true for all $\mu \le u_2$. However, for $\mu \in (u_2, 1)$ no solutions of type 1 are present in the region

$$\mathcal{N} = \{(\varepsilon, \mu) : 0 < \varepsilon < \varepsilon_1(\mu), \mu \in (u_2, 1)\},$$

where other type solutions are present (as seen shortly). Further, for $\mu \geq 1$, solutions of type 1 exist again for all $\varepsilon > 0$.

2.2. The geometry of the region $\mathcal{N}$

Now we can describe in more details the region in the (ε, μ)-plane in which a unique solution of type 1 of (1) does not exist by using the following result of Smoller and Wasserman [1].

Theorem [SW]. *The function*

$$T(\mu) = \int_0^\mu \frac{dw}{\sqrt{F(w) - F(\mu)}}$$

defined for $\mu \in [u_2, 1]$ has a unique critical point, a minimum.

In [1] the statement of the theorem looks somewhat different from the above but the difference is only formal. (Really, v_0 is p in [1] and $\mu = \alpha(p)$ defined there takes values between u_2 and 1.)

Therefore, the bifurcation curve $(\varepsilon_1(\mu), \mu)$ has a unique critical point, a maximum, if $\mu \in (u_2, 1)$. When $\mu \to u_2$ or $\mu \to 1$, then $\varepsilon_1(\mu) \to 0$.

3. Solutions with a maximum (of type 2): the case $0 < \lambda < \frac{1}{2}$

3.1. Existence and nonuniqueness theorems

Consider now the case when $u'(x)$ has exactly one zero in $(0,1)$ at which it changes sign, i.e., $u(x)$ has one critical point $x_* \in (0,1), u_* = u(x_*), v_* = v(x_*) = 0$. Solutions of that type will be referred to as *type 2 solutions*.

Then $u(x)$ is invertible on $[0, x_*)$ and $[x_*, 1]$ separately and we can define the inverses $x_1(u)$ on $[0, u_*) = I_1$ and $x_2(u)$ on $[\mu, u_*] = I_2$. On these intervals (2a), (2c) is equivalent to:

$$\text{(9)}\qquad \begin{array}{ll} \varepsilon \dfrac{dv_1}{du} = \dfrac{f(u)}{v_1}, & \varepsilon \dfrac{dv_2}{du} = \dfrac{f(u)}{v_2}, \\ \dfrac{dx_1}{du} = \dfrac{1}{v_1}, \quad u \in I_1; & \dfrac{dx_2}{du} = \dfrac{1}{v_2}, \quad u \in I_2; \\ v_1(0) = v_0, & v_2(u_*) = v_1(u_*) = 0, \\ x_1(0) = 0, & x_2(u_*) = x_*. \end{array}$$

The equivalence is in the sense that every solution of (2a), (2c) u, v on $[0, x_*)$ and on $[x_*, 1]$ defines uniquely solutions $x_i, v_i, i = 1, 2$ of (9) on I_1 and I_2, and vice versa.

These two uncoupled systems (9) can be solved explicitly:

$$(10a)\qquad \begin{aligned} v_1(u) &= \operatorname{sign}(v_0)\sqrt{\frac{2}{\varepsilon}F(u)+v_0^2},\\ x_1(u) &= \operatorname{sign}(v_0)\int_0^u \frac{dw}{\sqrt{\frac{2}{\varepsilon}F(w)+v_0^2}} = \Upsilon(\varepsilon, v_0^2, u),\end{aligned}$$

if $u \in I_1$;

$$(10b)\qquad \begin{aligned} v_2(u) &= -\operatorname{sign}(v_0)\sqrt{\frac{2}{\varepsilon}(F(u)-F(u_*))},\\ x_2(u) &= x_* + \operatorname{sign}(v_0)\int_u^{u_*} \frac{dw}{\sqrt{\frac{2}{\varepsilon}(F(w)-F(u_*))}},\end{aligned}$$

if $u \in I_2$. The sign of v_0 is established in Lemma 1 below.

By taking the inverse functions $u(x)$, $x \in [0, x^*]$ and $u(x)$, $x \in [x^*, 1]$, we get the solution of (2a)-(2b), when it has a unique critical point x_* if such a solution exists.

Lemma 1. *If a solution of type 2 exists then u_* is the unique solution of*

$$(11)\qquad \frac{2}{\varepsilon}F(u_*) = -v_0^2,$$

such that

$$(12)\qquad 1 > u_* > \max(\mu, u_2) > \lambda,$$

and

$$(13)\qquad 0 < v_0 < \sqrt{-\frac{2}{\varepsilon}F(1)}.$$

Proof. $0 = v_1^2(u_*) = \frac{2}{\varepsilon}F(u_*) + v_0^2$. This is possible only if $v_0^2 \le -\frac{2}{\varepsilon}F(1)$ (since 1 is an absolute minimum of F) and only for $u_* \in (u_2, u_3)$. The case $v_0 < 0$ is not possible since then $u_* < 0$ which contradicts the previous conclusion. Also, $v_0 \neq 0$, since $\Upsilon(\varepsilon, 0, u)$ is not defined and this implies $u_* \neq u_2$ or u_3. Also, u_* being positive, must be greater than μ and such that $f(u_*) < 0$. The latter implies that $u_* < 1$ and the validity of (13) follows. The uniqueness of u_* follows from the monotonicity of F on the interval $[u_2, 1]$. ■

Corollary 1. *If a solution of type 2 exists, it is one with a maximum.*

Corollary 2. *If $\mu \ge 1$, no solution of type 2 exists, no matter what $\varepsilon > 0$ is.*

Now, let us go back to the solution of (1).

Formulae (10a), (10b) give a solution of (1) iff $x_2(\mu) = 1$, i.e. (since, according to Lemma 1, sign $v_0 > 0$) iff for given ε and μ there exists v_0^2 satisfying (11) and (13) such that

$$(14)\quad \Psi(u_*,\mu) = \left[\int_0^{\mu} \frac{dw}{\sqrt{(F(w)-F(u_*))}} + 2\int_{\mu}^{u_*} \frac{dw}{\sqrt{(F(w)-F(u_*))}}\right] = \sqrt{\frac{2}{\varepsilon}}.$$

Remark 2. Since u_* has to obey (12), the domain of Ψ is $\{u \in (\max(\mu, u_2), 1), \mu \in [0,1)\}$.

Again, the problem is, for how many values of v_0^2 satisfying (13) equation (14) is fulfilled.

According to Corollary 2, we have to consider only the case $0 \le \mu < 1$.

Theorem 3. *Let $0 \le \mu < 1$. There exists a positive $\varepsilon_2(\mu) < \infty$ such that equation (1) has at least one solution of type 2 iff $\varepsilon \le \varepsilon_2(\mu)$. Besides, the bifurcation curve $(\varepsilon_2(\mu), \mu)$ is defined by*

$$(15)\quad \varepsilon_2(\mu) = \frac{2}{[\min_{u_* \in [\max(u_2,\mu),1]} \Psi(u_*,\mu)]^2}$$

and $\varepsilon_2(\mu) \to 0$ when $\mu \to 1$.

Proof. If $\mu \le u_2$, then the minimum in (15) has to be taken for $u_* \in (u_2, 1)$. Since $\Psi(u_*,\mu) \to \infty$ when $u_* \to u_2$ and when $u_* \to 1$, then the minimum in (15) does exist in $(u_2, 1)$ and is greater than $\eta_\mu = \dfrac{\mu^2}{(F(\lambda) - F(1))} > 0$. Also, if $u_2 < \mu < 1$, the minimum is taken for $u_* \in [\mu, 1]$, exists there and satisfies the same estimate. Therefore $0 < \varepsilon_2(\mu) < 1/\eta_\mu^2$. If $\varepsilon > \varepsilon_2(\mu)$, then $\Psi(u_*,\mu) > \sqrt{2/\varepsilon}$ for all possible values of v_0^2 satisfying (13) and (11), i.e., (14) cannot hold. If $\varepsilon \le \varepsilon_2(\mu)$, then $\min \Psi(u_*,\mu) \le \sqrt{2/\varepsilon}$. But $\Psi(u_*,\mu)$ grows infinitely when $u_* \to u_2$ or 1, i.e., when v_0^2 tends to 0 or $-\frac{\varepsilon}{2}F(1)$. Therefore, there is at least one v_0^2 such that (14) and (11) hold and therefore (1) has at least one solution of type 2 if $\mu < 1$ and $\varepsilon \le \varepsilon_2(\mu)$. Moreover, when $\mu \to 1$, the denominator tends to ∞ and therefore $\varepsilon_2(\mu) \to 0$. ■

Let us denote the region where solutions of type 2 exist by $\mathcal{K}$.

$$\mathcal{K} = \{(\varepsilon,\mu) : 0 < \varepsilon \le \varepsilon_2(\mu)\}.$$

Theorem 4. *$\mathcal{N} \subset \mathcal{K}$.*

Proof. One can write Ψ as

$$(16)\quad \Psi(u_*,\mu) = \left[\int_0^{u_*} \frac{dw}{\sqrt{(F(w)-F(u_*))}} + \int_{\mu}^{u_*} \frac{dw}{\sqrt{(F(w)-F(u_*))}}\right].$$

Obviously, we have to show that if $\mu \in [u_2, 1]$, then $\varepsilon_2(\mu) \ge \varepsilon_1(\mu)$.

For this purpose, observe that

$$\Psi(\mu,\mu) = [\frac{\varepsilon_1(\mu)}{2}]^{-\frac{1}{2}},$$

i.e.,

$$\sqrt{\frac{2}{\varepsilon_2(\mu)}} = \min_{u_* \in [\mu,1]} \Psi(u_*,\mu) \le \sqrt{\frac{2}{\varepsilon_1(\mu)}}. \qquad \blacksquare$$

Lemma 2. *If $1 > \mu > u_2$, then $\lim_{\substack{u_* \to \mu \\ u_* > \mu}} \frac{\partial \Psi}{\partial u_*}(u_*,\mu) = +\infty$, $\lim_{\substack{u_* \to 1 \\ u_* < 1}} \frac{\partial \Psi}{\partial u_*}(u_*,\mu) = +\infty$ and $\frac{\partial \Psi}{\partial u_*}$ is bounded in every subinterval of $(\mu, 1)$.*

P r o o f. We shall omit the $*$ sign for simplicity. Since F is a fourth order polynomial,

$$F(w)-F(u) = (u-w)[-f(u)+\frac{1}{2}(u-w)f'(u)-\frac{1}{6}(u-w)^2 f''(u)+\frac{1}{24}(u-w)^3 f'''(u)].$$

Using this representation and the substitution $v = \sqrt{u-w}$, we get:

$$\Psi(u,\mu) = 2\Big[\int_0^{\sqrt{u}} \frac{dv}{\sqrt{\eta(u,v)}} + \int_0^{\sqrt{u-\mu}} \frac{dv}{\sqrt{\eta(u,v)}}\Big],$$

where

$$\eta(u,v) = -f(u) + \frac{1}{2}v^2 f'(u) - \frac{1}{6}v^4 f''(u) + \frac{1}{24}v^6 f'''(u).$$

Then

$$\begin{aligned} \frac{\partial \Psi}{\partial u}(u,\mu) &= [-F(u)]^{-\frac{1}{2}} + [F(\mu) - F(u)]^{-\frac{1}{2}} \\ &+ \int_0^{\sqrt{u}} \frac{f'(u) - \frac{1}{2}v^2 f''(u) + \frac{1}{6}v^4 f'''(u)}{[\eta(u,v)]^{\frac{3}{2}}} dv \\ &+ \int_0^{\sqrt{u-\mu}} \frac{f'(u) - \frac{1}{2}v^2 f''(u) + \frac{1}{6}v^4 f'''(u)}{[\eta(u,v)]^{\frac{3}{2}}} dv. \end{aligned} \tag{17}$$

When $1 > \mu > u_2, F(\mu) \neq 0$. Therefore the first term on the right-hand side is bounded when $u \to \mu$. Moreover, $\eta(u,v) = \frac{F(u-v^2)-F(u)}{v^2} \neq 0, u \in (u_2, 1), v \in [0, \sqrt{u}]$, as is easy to see. It follows that the two integrals are bounded when $u \to \mu$ and that $\partial \Psi / \partial u$ is bounded in every subinterval of $(\mu, 1)$. It also follows that the behavior of $\partial \Psi / \partial u$ when $u \to \mu$ is determined by the second term which tends to $+\infty$.

When $u = 1$ the integrals are divergent at their lower limit and since $f'(1) > 0$, they diverge to $+\infty$. Hence

$$\lim_{\substack{u \to 1 \\ u < 1}} \frac{\partial \Psi}{\partial u} = +\infty.$$

■

Corollary 3. *For any $\mu, \Psi(u,\mu), u \in (\mu,1)$ has a finite number of extrema on the interval* $(\max(\mu,u_2),1)$ *(remember Remark 2).*

Indeed, if $\mu > u_2, \Psi$ is an increasing function in the intervals $(\mu,\mu+\delta)$ and $(1-\delta,1)$ for some δ. Besides, $\partial\Psi/\partial u$ is bounded in $[\mu+\delta,1-\delta]$ and therefore Ψ can have only a finite number of extrema. The same type of arguments holds for $\mu \in [0,u_2]$.

Lemma 3. *There exists $\tilde{\mu}$ such that if $1 > \mu > \tilde{\mu}$, then $\frac{\partial\Psi}{\partial u}(u,\mu) > 0$ for all $u \in [\mu,1)$.*

P r o o f. Since $-\eta'_u(1,0) = f'(1) > 0$, there exists μ_0 sufficiently close to 1 such that for all $u \in [\mu_0,1)$,

$$\int_0^{\sqrt{u-\mu_0}} -\frac{\eta'_u(u,v)}{(\sqrt{\eta(u,v)})^{\frac{3}{2}}}dv > 0.$$

It follows from (17) that in this case

$$\frac{\partial\Psi}{\partial u}(u,\mu) > \int_0^{\sqrt{u}} -\frac{\eta'_u(u,v)}{(\sqrt{\eta(u,v)})^{\frac{3}{2}}}dv.$$

As shown in Lemma 2, the integral on the right is positive if u is close enough to 1. Therefore there exists $\tilde{\mu}$ such that if $u \geq \tilde{\mu}$, then $\frac{\partial\Psi}{\partial u}(u,\mu) > 0$. ■

In the following statement $\bar{\mathcal{N}}$ is the closure of $\mathcal{N}$.

Theorem 5. *There is an even number of solutions of type 2 (at least two) in $\mathcal{K}\backslash\bar{\mathcal{N}}$ with the exception of some curves, defined below by (19), on which there may be an odd number of solutions (at least three) of type 2.*

P r o o f. We first consider the case $\mu \in [0,u_2]$. As was noted in the proof of Theorem 3, if $0 < \mu \leq u_2, \Psi(u_*,\mu) \to \infty$ when $u_* \to u_2$ or 1. Therefore $[\Psi(u_*,\mu)]^{-1} \to 0$ at both ends of the interval $[u_2,1]$ and since Ψ is continuous in $(u_2,1)$, $2[\Psi(u_*,\mu)]^{-2}$ takes all the values between 0 and $\varepsilon_2(\mu)$ at least twice, i.e. for each μ and $\varepsilon \in [0,\varepsilon_2(\mu)]$ there are at least two solutions of type 2.

When $\mu > u_2, \Psi(\mu,\mu)$ is well defined and if $\varepsilon_1(\mu) < \varepsilon_2(\mu)$ (i.e., if for a given μ there is an ε such that $(\varepsilon,\mu) \in \mathcal{K} \setminus \bar{\mathcal{N}}$), then $\min_{u_* \in [\mu,1]} \Psi(u_*,\mu) < \Psi(\mu,\mu)$. Therefore in such a case there exists at least one interval $J = [u^1,u^2]$ such that $\Psi(u_1,\mu) = \Psi(u_2,\mu) = \Psi(\mu,\mu)$ and $\Psi(u,\mu) < \Psi(\mu,\mu), u \in J$, i.e.,

$$\varepsilon_2(\mu) \geq 2[\Psi(u,\mu)]^{-2} > \varepsilon_1(\mu), u \in J,$$

and

$$2[\Psi(u^i,\mu)]^{-2} = \varepsilon_1(\mu),\ i = 1,2.$$

Therefore, for each $\varepsilon \in (\varepsilon_1(\mu), \varepsilon_2(\mu))$ there exist at least two values of $u \in J$ such that $2[\Psi(u,\mu)]^{-2} = \varepsilon$, i.e., there are at least 2 solutions of type 2.

That the number of possible solutions is finite, is guaranteed by the fact that $\Psi(u_*, \cdot)$ can only have a finite number of extrema (Corollary 3).

For a given μ and ε let $u_*^k, k = 1, ..., m$, be all the values of u_* satisfying (14). Suppose that m is odd. Obviously, it follows that ε is such that for some u_*^s,

$$\begin{aligned} &\varepsilon = 2[\Psi(u_*^s, \mu)]^{-2}, \\ &\frac{\partial \Psi}{\partial u_*}(u_*^s, \mu) = 0, \\ &\frac{\partial^2 \Psi}{\partial u_*^2}(u_*^s, \mu) \neq 0. \end{aligned} \tag{18}$$

Using the Implicit Function Theorem, we can solve the second equation of (18) for u_*^s and define the bifurcation curves

$$\varepsilon^s(\mu) = 2[\Psi(u_*^s(\mu), \mu]^{-2}, \quad \frac{\partial \Psi}{\partial u_*}(u_*^s(\mu), \mu) = 0. \tag{19}$$

On these curves (1) can possibly have an odd number of solutions. ■

Theorem 6. *There is an odd number of solutions of type 2 in $\mathcal{N}$ with the exception of some curves defined below by (20) on which there may be an even number (at least 2) of solutions of type 2.*

Proof. Lemma 2 shows that $\Psi(u_*, \cdot)$ is an increasing function in a neighborhood of μ and of 1 if $\mu \in (u_2, 1)$. There are the following possibilities:

1) $\Psi(u, \mu)$ is increasing for all $u_* \in (u_2, \mu)$. Then (14) can have only one solution and therefore (1) has one solution of type 2.

2) $\Psi(u_*, \mu)$ has at least one local minimum. Then (14) may have three or more (odd number) solutions for ε in certain ranges. **Figure 1** illustrates some possible cases (see next pages).

There may be an even number of solutions for some ε only if $\sqrt{2/\varepsilon}$ is the value at a maximum or minimum of $\Psi(u_*, \mu)$. Therefore, given μ, for all but a finite number of $\varepsilon \in [0, \varepsilon_1(\mu)]$, equation (1) has an odd number of solutions. An even number of solutions (at least 2) is possible for $\varepsilon(\mu)$ satisfying the equations

$$\varepsilon(\mu) = 2[\Psi(u_*, \mu)]^{-2},$$

where

$$\frac{\partial \Psi}{\partial u_*}(u_*, \mu) = 0,$$
$$\frac{\partial^2 \Psi}{\partial u_*^2}(u_*, \mu) \neq 0.$$

Using the Implicit Function Theorem we can solve for $u_*(\mu)$ and define the bifurcation curves

$$\varepsilon(\mu) = 2[\Psi(u_*(\mu), \mu)]^{-2}, \ \frac{\partial \Psi}{\partial u_*}(u_*(\mu), \mu) = 0 \tag{20}$$

on which (1) may have an even number of solutions. ∎

The following result establishes the existence of another bifurcation curve situated in $\mathcal{N}$.

Theorem 7. *For each $\mu \in (u_2, 1)$ there exists $\varepsilon_0(\mu) \leq \varepsilon_1(\mu)$ such that Equation (1) has a unique solution of type 2 iff $0 < \varepsilon \leq \varepsilon_0(\mu)$.*

Proof. Let $\mu \in (u_2, 1)$. As was already noted, $\Psi(u_*, \mu)$ has a finite number of extrema. Let $\tilde{u}_1, ..., \tilde{u}_p$ be the maximums and let $\Psi(\tilde{u}_m) \geq \Psi(\tilde{u}_i), i = 1, ..., p$. Since $\Psi(u, \mu)$ grows infinitely when $u \to 1$, there exists $\bar{u}(\mu)$ such that $\Psi(\bar{u}(\mu)) = \Psi(\tilde{u}_m)$ and Ψ is increasing in $[\bar{u}(\mu), 1]$. It follows that for all $\varepsilon \leq 2[\Psi(\bar{u}(\mu), \mu)]^{-2} = \varepsilon_0(\mu)$, equation (1) has a unique solution of type 2. ∎

Let $\mathcal{L} = \{(\varepsilon, \mu) : 0 < \varepsilon \leq \varepsilon_0(\mu), u_2 < \mu < 1\}$.

Now the obvious question is: can (1) actually have more than one solution of type 2 in $\mathcal{N}$?

The answer is "yes" and is supplied by the following theorem/

Theorem 8. $\mathcal{N} \setminus \mathcal{L} \neq \emptyset$.

Proof. Obviously, we have to prove that there exists $\mu \in (u_2, 1)$ such that $\Psi(u_*, \mu)$ has at least one minimum in $(\mu, 1)$. As we know,

$$\Psi(u_*, \mu) \to \infty \ \text{ when } \ \mu \to u_2 \ \text{ and } \ u_* \to \mu.$$

Therefore if μ and u_* are close enough to u_2 and $u_2 > \hat{u} = \dfrac{1 - u_2}{2}$, then

$$\Psi(u_*, \mu) > 2 \int_0^{\hat{u}} \frac{dw}{\sqrt{F(w) - F(\hat{u})}} > \Psi(\hat{u}, \mu),$$

which gives the proof since $\Psi(u_*, \mu) \to \infty, u_* \to 1$. ∎

3.2. On the geometry of the regions $\mathcal{K}, \mathcal{N}$ and $\mathcal{L}$

Theorem 9. *If $\mu \in [0, u_2]$, $\varepsilon_2(\mu)$ is an increasing function of μ.*

P r o o f. $\varepsilon_2(\mu) = 2\left[\min_{u_*\in[u_2,1]} \Psi(u_*,\mu)\right]^{-2}$.

From the representation (16) of Ψ it is clear that when μ increases, Ψ decreases. Therefore $\varepsilon_2(\mu)$ increases. ∎

Theorem 10. *There exist nonempty neighborhoods of* u_2 : $I_1 = (u_2, \mu_k)$ *and of 1:* $I_2 = (\mu_{\mathcal{N}}, 1)$ *such that if* $\mu \in I_1$, *then* $\varepsilon_2(\mu) > \varepsilon_1(\mu)$ *(i.e., the case* $\mu \in (u_2, 1)$ *in the proof of Theorem 5 is not a fiction !) and if* $\mu \in I_2$, *then* $\varepsilon_2(\mu) = \varepsilon_1(\mu)$ *($\mathcal{K}$ and $\mathcal{N}$ merge from some point on).*

P r o o f. The existence of I_1 was, in fact, already established in Theorem 8 since we proved there that for $\mu > u_2$ close enough to u_2, $\Psi(\mu,\mu) > \min\Psi$. The existence of I_2 follows from Lemma 3 which means that for all $\mu \in (\tilde{\mu}, 1)$, $\Psi(u_*,\mu)$ is an increasing function, i.e., $\Psi(\mu,\mu) = \min\Psi(u_*,\mu)$, i.e. $\varepsilon_1(\mu) = \varepsilon_2(\mu)$. ∎

Figure 2 (see next pages) provides a schematic picture of the regions $\mathcal{K}, \mathcal{N}$ and $\mathcal{L}$ predicted by these theoretical results.

4. Solutions with more than one critical point

Let us consider now the possibility of existence of a solution of (1) having more than one critical point at which v changes sign, i.e., $u(x)$ has at least one minimum and one maximum, denoted by $u(x_{\min}) = u_{\min}$, $u(x_{\max}) = u_{\max}$, $v(x_{\min}) = v(x_{\max}) = 0$.

But since

$$v^2(u) - v_0^2 = \frac{2}{\varepsilon}F(u),$$

it follows

$$F(u_{\min}) = F(u_{\max}) = -\frac{\varepsilon}{2}v_0^2,$$

and therefore $u_2 < u_{\min} < 1$ and $u_{\max} > 1$.

But then:

$$u''(x_{\min}) = f(u_{\min}) < 0$$

and

$$u''(x_{\max}) = f(u_{\max}) > 0,$$

which is impossible.

Therefore (1) can have no solutions with more than one critical point.

5. The case $1 > \lambda \geq \frac{1}{2}$

In the case $\lambda > \frac{1}{2}$, $F(u)$ has no real roots other than $u = 0$ and when $\lambda = \frac{1}{2}$, $u_2 = u_3 = 1$. Going through the same type of arguments as in Section 1.2.1, we see that (1.1) has a unique solution of type 1 for all values of ε and μ.

Also, since $F(u) \geq 0$ for all u, (11) can never be true. Therefore, there are no solutions of type 2 if $1 > \lambda > \frac{1}{2}$.

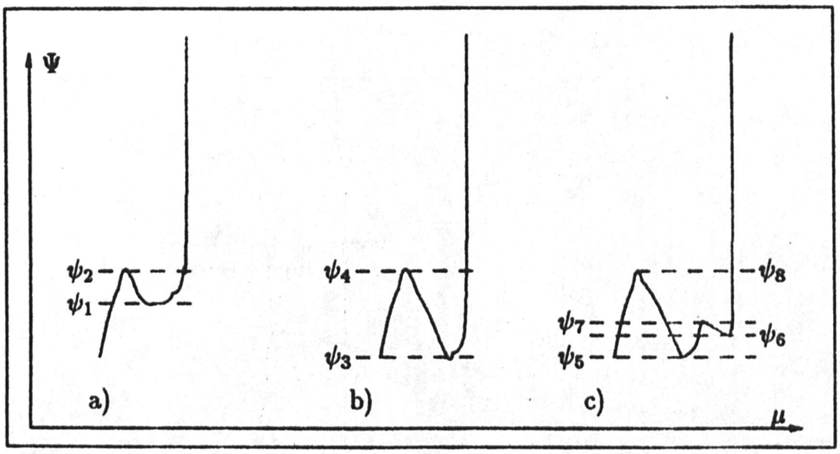

Figure 1:

Suppose that for a given μ, Ψ has the following features: **a)** 2 extrema. Then for $\varepsilon \in (\frac{2}{\psi_2^2}, \frac{2}{\psi_1^2})$ there are 3 solutions of type 2; for $\varepsilon = \frac{2}{\psi_2^2}, \frac{2}{\psi_1^2}$ there are 2 solutions of type 2; for $\varepsilon \in (0, \frac{2}{\psi_2^2}) \cup (\frac{2}{\psi_1^2}, \varepsilon_1(\mu))$ 1 solution; **b)** 2 extrema. The situation is similar but there is only one interval for ε with 1 solution of type 2; **c)** 4 extrema. There are 5 solutions of type 2 for $\varepsilon \in (\frac{2}{\psi_7^2}, \frac{2}{\psi_6^2})$; 3 solutions for $\varepsilon \in (\frac{2}{\psi_6^2}, \frac{2}{\psi_5^2}) \cup (\frac{2}{\psi_8^2}, \frac{2}{\psi_7^2})$; 1 solution for $\varepsilon \in (0, \frac{2}{\psi_8^2})$.

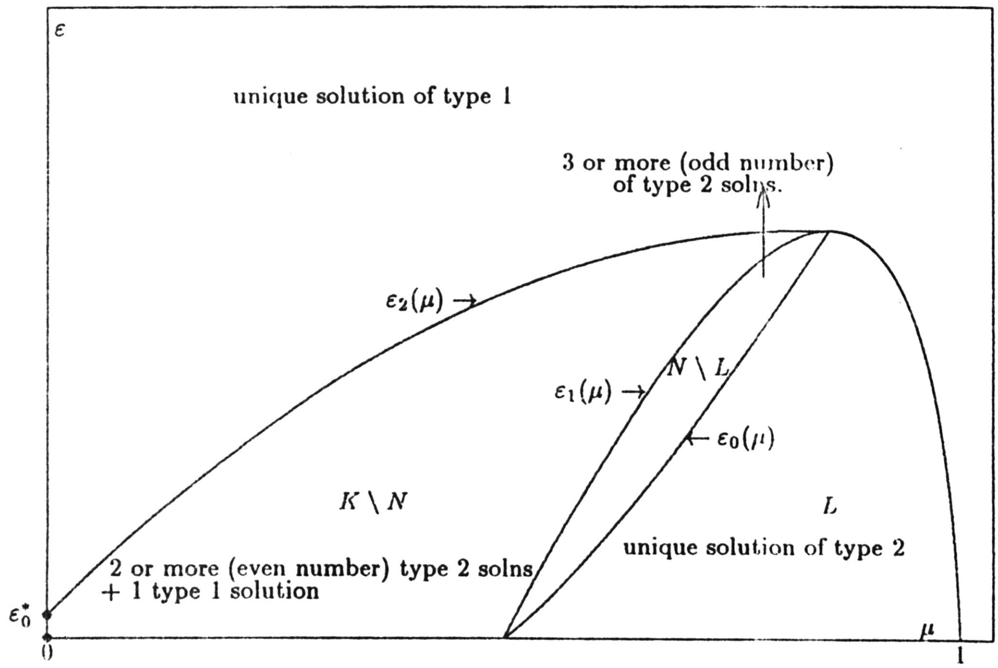

Figure 2:

Positioning of the regions $\mathcal{K} = \{\varepsilon \leq \varepsilon_2(\mu)\}$, $\mathcal{N} = \{\varepsilon \leq \varepsilon_1(\mu)\}$, $\mathcal{L} = \{\varepsilon \leq \varepsilon_0(\mu)\}$.

6. Summary of the results

To summarise, we have the following results in the (ε, μ) - plane (look at **Figure 2**):

a) Outside of region $\mathcal{K}$ there is a unique solution of (1) which is of type 1.

b) In region $\mathcal{K}$ there is an odd number of solutions of (1) as follows:

b1) In $\mathcal{K} \setminus \mathcal{N}$ there are three or more (odd number of) solutions: one solution of type 1 and two or more (even number) solutions of type 2.

b2) In $\mathcal{N} \setminus \mathcal{L}$ there are no solutions of type 1 and three or more (odd number) solutions of type 2.

b3) In $\mathcal{L}$ there is a unique solution of type 2.

b4) On the curves defined by (19) in $\mathcal{K} \setminus \mathcal{N}$ the number of solutions of type 2 is even and on the curves defined by (20) in $\mathcal{N}$ the number of solutions of type 2 is odd because on them two solutions merge into one.

If we take $\mu = 0$, we get the case of Smoller and Wasserman [5] in which it is only possible to have either one solution of type 1 (for $\varepsilon > \varepsilon_0^*$) or three solutions (one of type 1 and two of type 2, for $\varepsilon < \varepsilon_0^*$).

But, for example, if we take μ close enough to u_2 and $\mu > u_2$, then for $\varepsilon < \varepsilon_0(\mu)$ there is one solution (of type 2), for $\varepsilon_0(\mu) < \varepsilon < \varepsilon_1(\mu)$ there are three or more (odd number) solutions of type 2, for $\varepsilon_1(\mu) < \varepsilon < \varepsilon_2(\mu)$ there is one solution of type 1 and two or more (even number) solutions of type 2, and for $\varepsilon_2(\mu) < \varepsilon$ there is one solution of type 1.

Or, if we take μ close enough to 1, $\mu < 1$, then for $\varepsilon < \varepsilon_2(\mu)$ there is a unique solution of type 2 and for $\varepsilon_2(\mu) < \varepsilon$ there is a unique solution of type 1.

In conclusion, if we vary μ, we get a much richer bifurcation structure.

7. Some remarks

We have designed a numerical procedure to calculate numerical approximations of the multiple steady states of equation (S) with nonhomogeneous boundary conditions which will be described elsewhere. A very interesting and hard problem to be solved in relation to our results is that of the stability of each of the steady states. It is very interesting to know whether it is possible to have more than one stable steady state or which one, the type 1 or type 2 solution is stable in case that both are present. We think that solving these problems will have a significant effect on the theory of nerve excitation in particular or, at least, will supply a very interesting example of a stability problem with multiple steady states in general.

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