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Orthogonal Systems of Rational Functions on the Segment and Quadratures of Gauss-type

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By means of rational interpolation we built the quadrature formulas generalizing classical Gauss formulas at Chebyshev nodes of the first and second type as well as at Jacobi nodes when $\alpha = -\beta = 1/2$.

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Introduction

The first complete studies of quadrature formulas by means of the properties of some special rational functions belong to B. Bojanov [2] and H.L. Loeb, Y. Werner [6]. Later, some authors, such as D.J. Newman [7], J.-E. Andersson [1], W. Gautshi [3], W. Van Assche, I. Vanherwegen [9], G. Lopez, J. Illan [8], studied quadrature formulas by means of rational approximation. In particular, they used rational interpolation and orthogonal systems of polynomials with a variable weight. But this research did not involve certain systems of orthogonal rational functions.

In the present paper we applied orthogonal properties on the segment $[-1,1]$ by weight $(1-x^2)^{\pm 1/2}$ of rational functions systems introduced into [5], we built a new orthogonal by weight $\sqrt{(1-x)/(1+x)}$ system of rational functions and studied the corresponding quadrature formulas of interpolation character.

1. Orthogonal rational function systems on the segment

Let $\{\alpha_k\}$ be an arbitrary sequence of complex numbers with $\alpha_0 = 0$, $\alpha_k < 1$, $k \in \mathbf{N}$, and define

$$(1) \qquad \zeta_0(z) \equiv 1,\ \zeta_n(z) = \frac{\sqrt{1-|\alpha_n|^2}}{1-\overline{\alpha}_n z} \prod_{k=0}^{n-1} \frac{z-\alpha_k}{1-\overline{\alpha}_k z}, \quad n \in \mathbf{N}.$$

1.1. M.M. Dzharbashyan and A.A. Kitbalyan [5] have introduced the following systems of functions:

$$(2) \qquad M_0^{(0)} \equiv 1, \quad M_n^{(0)}(x) = \frac{1}{2}\left(\zeta_n(e^{i\theta}) + \zeta_n(e^{-i\theta})\right), \quad n \in \mathbf{N};$$

$$(3) \qquad M_n^{(1)}(x) = \left(e^{i\theta}\zeta_n(e^{i\theta}) - e^{-i\theta}\zeta_n(e^{-i\theta})\right)/2i\sin\theta\,, \quad n \in \mathbf{N};\ x = \cos\theta.$$

The system of functions $\{M_n^{(0)}(x)\}_{n=0}^{+\infty}$ is orthogonal on the segment $[-1,1]$ with respect to the weight $(1-x^2)^{-1/2}$, the system of functions $\{M_n^{(1)}(x)\}_{n=1}^{+\infty}$ is orthogonal with respect to the weight $(1-x^2)^{1/2}$.

Lemma 1. *The functions $M_n^{(0)}(x)$ and $M_n^{(1)}(x)$ are rational of order n, $n \in \mathbf{N}$.*

P r o o f. It is not difficult to check that if $z = z \pm \sqrt{x^2-1}$, then

$$\frac{z-\alpha_k}{1-\overline{\alpha}_k z} = \frac{(1+|\alpha_k|^2)\,x - 2\mathrm{Re}\alpha_k \pm (1-|\alpha_k|^2)\sqrt{x^2-1}}{1+\overline{\alpha}_k^2 - 2\overline{\alpha}_k x}, \quad k = \overline{0, n-1};$$

$$\frac{z}{1-\overline{\alpha}_n z} = \frac{x - \overline{\alpha}_n\sqrt{x^2-1}}{1-2\overline{\alpha}_n x + \overline{\alpha}_n^2}.$$

Substitution of the obtained expressions in (2) and (3), leads to the conclusion of Lemma 1.

Lemma 2. *If the numbers α_k, $k = \overline{1, n-1}$, are real or mutually complex-conjugate, with $\alpha_0 = 0$, $\alpha_n \in (-1,1)$, then the following expressions hold:*

$$(4) \qquad a)\ \ M_n^{(0)}(x) = \sqrt{\frac{\sqrt{1-a_n^2}}{1+a_n x}}\cos\mu_n(x),$$

where

$$\mu_n(x) = \arccos\frac{x-\alpha_n}{\sqrt{1-2\alpha_n x + \alpha_n^2}} + \sum_{k=1}^{n-1}\arccos\frac{x+a_k}{1+a_k x},$$

$$(5)\qquad a_k = -\frac{2\alpha_k}{1+\alpha_k^2}, \qquad k=\overline{1,n};$$

$$(6)\qquad b)\;\; M_n^{(1)}(x) = \sqrt{\frac{\sqrt{1-a_n^2}}{1+a_n x}}\sin\mu_{n+1}(x)\Big/\sqrt{1-x^2}\,,$$

where $\mu_{n+1}(x) = \arccos x + \mu_n(x)$.

Under these conditions, the functions $M_n^{(0)}(x)$ *and* $M_n^{(1)}(x)$ *each have* n *simple roots on the interval* $(-1,1)$.

Proof. If $z = x \pm \sqrt{x^2-1}$, then

$$(7)\qquad \frac{z-\alpha_k}{1-\alpha_k z} = \frac{x + a_k \pm \sqrt{1-a_k^2}\sqrt{x^2-1}}{1+a_k x} = e^{\pm i\gamma_k(x)},$$

where

$$\gamma_k(x) = \arccos\frac{x+a_k}{1+a_k x}, \quad k=\overline{1,n-1}.$$

In a similar way,

$$(8)\qquad \frac{\sqrt{1-|\alpha_n|^2 z}}{1-\overline{\alpha}_n z} = \frac{\sqrt{1-\alpha_n^2}\left(x-\alpha_n \pm \sqrt{x^2-1}\right)}{1-2x\alpha_n+\alpha_n^2} = \sqrt{\frac{\sqrt{1-a_n^2}}{1+a_n x}}e^{\pm i\gamma_n(x)},$$

where

$$\gamma_n(x) = \arccos\frac{x-\alpha_n}{\sqrt{1-2x\alpha_n+\alpha_n^2}}.$$

Assuming that $\alpha_k \in \mathbf{C}$ and $\alpha_{k+1} = \overline{\alpha}_k$, then we obtain as follows:

$$\frac{z-\alpha_k}{1-\overline{\alpha}_k z}\,\frac{z-\alpha_{k+1}}{1-\overline{\alpha}_{k+1} z} = \frac{z-\alpha_k}{1-\alpha_k z}\,\frac{z-\alpha_{k+1}}{1-\alpha_{k+1} z}.$$

Taking into account this equality, relations (7) and (8), we obtain the expression (4) from the relation (2).

In a similar way, we get formula (6).

The second part of Lemma 2 follows from the fact that $\mu_n(1) = 0$, $\mu_n(-1) = n\pi$ and since

$$\mu_n'(x) = \left(\frac{1-x\alpha_n}{1-2x\alpha_n+\alpha_n^2} + \sum_{k=1}^{n-1}\frac{1-a_k^2}{1+a_k x}\right)\frac{-1}{\sqrt{1-x^2}}, \quad \mu_n'(x) < 0,\; x\in[-1,1],$$

the function $\mu_n(x)$ decreases monotonically on the segment $[-1,1]$. ∎

It is worth mentioning that functions $M_n^{(0)}(x)$ and $M_n^{(1)}(x)$ have been determined by using the numbers a_k, $a_k \in \mathbf{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$, $k = \overline{1, n}$, and not by using the α_k, $|\alpha_k| < 1$, $k = \overline{0, n}$ numbers (see (5)).

Lemma 3. *If the conditions of Lemma 2 are fulfilled and $\alpha_n = 0$, then the function $M_n^{(0)}(x)$ is the rational cosine-function of Chebyshev-Markov and the function $M_n^{(1)}(x)$ is the sine-function of Chebyshev-Markov (see [11], p.47).*

Lemma 3 is direct consequence of Lemma 2. We emphasize that the sequence of the corresponding rational Chebyshev-Markov functions (cosine-function or sine-function) is, generally speaking, not an orthogonal system. However, each Chebyshev-Markov function can be regarded as the element of some orthogonal system of rational functions.

1.2. Let

$$(9)\qquad Q_n(x) = \frac{1}{2\pi i} \int\limits_{|t|=\rho_n} \frac{\zeta_n(t)}{t^2 - 2tx + 1} \frac{1+t}{t} dt, \ x \in [-1; 1],\ n = 0, 1, \ldots,$$

where the number ρ_n, $\rho_n > 1$, is chosen so that the points α_k^{-1}, $k = \overline{0, n}$, are outside the integration contour. Then, it is easy to observe that the integrand has two simple poles at the points of $x \pm \sqrt{x^2 - 1}$ in the circle $|t| < \rho_m$ and, consequently,

$$(10)\, Q_n(x) = \frac{1}{2i \sin\theta} \left(\left(1 + e^{-i\theta}\right) \zeta_n(e^{i\theta}) - \left(1 - e^{i\theta}\right) \zeta_n(e^{-i\theta}) \right), \quad x = \cos\theta.$$

Hence, with the help of equalities (7),(8) and (1) it is easy to see that the function $Q_n(z)$ is rational of n order.

Theorem 1. *The system of rational functions $Q_n(x)$, $n = 0, 1, \ldots$, is orthogonal on the segment $[-1, 1]$ with respect to the weight $\sqrt{(1-x)/(1+x)}$.*

P r o o f. Consider the integral

$$J_{mn} = \int\limits_{-1}^{1} \sqrt{\frac{1-x}{1+x}} Q_m(x) \overline{Q_n(x)} dx, \quad m, n = 0, 1, \ldots.$$

Let us use the representation (9) to get

$$J_{mn} = -\frac{1}{4\pi^2} \int\limits_{|t|=\rho_m} \zeta_m(t) \frac{1+t}{t} dt \int\limits_{|u|=\rho_n} \overline{\zeta}_n(u) \frac{1+u}{u} du$$

$$\times \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \frac{dx}{(t^2-2tx+1)(u^2-2ux+1)}. \tag{11}$$

It is not difficult to find that

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \frac{dx}{(t^2-2tx+1)(u^2-2ux+1)} = \frac{\pi}{(1+t)(1+u)(tu-1)}, |t|>1, |u|>1.$$

Substituting the achieved expression into (11) we find

$$J_{mn} = -\frac{1}{4\pi} \int_{|t|=\rho_m} \zeta_m(t)\frac{dt}{t} \int_{|u|=\rho_n} \overline{\zeta}_n(u)\frac{du}{u(tu-1)}.$$

The integrand of the inner integral has the singular point $u = 1/t$ in the circle $|u| < \rho_n$, $\rho_n > 1$. Consequently,

$$\int_{|u|=\rho_n} \overline{\zeta}_n(u)\frac{1}{tu-1}\frac{du}{u} = 2\pi i \overline{\zeta}_n\left(\frac{1}{t}\right), \quad n \geq 1,$$

and

$$J_{mn} = \frac{-i}{2\pi} \int_{|t|=\rho_m} \zeta_m(t)\overline{\zeta}_n\left(\frac{1}{t}\right)\frac{dt}{t}.$$

Taking a limit as $\rho_m \to 1$, we get

$$J_{mn} = \frac{-i}{2\pi} \int_{|t|=1} \zeta_m(t)\overline{\zeta}_n(t)\frac{dt}{t} = \frac{1}{2\pi} \int_{|t|=1} \zeta_m(t)\overline{\zeta}_n(t)|dt|.$$

Now, we can use the orthogonal system $\{\zeta_n(t)\}_0^{+\infty}$ on the unit circle (see [4]). Theorem 1 is proved. ■

Assume that $\alpha_k = 0$, $k = \overline{0,n}$, then it is evident that

$$Q_n(x) = \frac{\sin(n+1/2)\theta}{\sin\theta/2}, \quad x = \cos\theta,$$

i.e. $Q_n(x)$ is the well-known Jacobi polynomial.

Lemma 4. *The function $Q_n(x)$ has n simple zeros on the interval $(-1,1)$ and the following expression representation holds*

$$Q_n(x) = \sqrt{2\frac{\sqrt{1-a_n^2}}{1+a_n x}} \sin\mu_{n+1/2}(x)/\sqrt{1-x},$$

where

$$\mu_{n+1/2}(x) = \frac{1}{2}\arccos x + \arccos\frac{x-\alpha_n}{\sqrt{1-2\alpha_n x+\alpha_n^2}} + \sum_{k=0}^{n-1}\arccos\frac{x+a_k}{1+a_k x}.$$

The proof of Lemma 4 is similar to that of Lemmas 2 and 3.

2. Quadratures of Gauss type

The quadrature formulas of this type, built by means of rational approximation, were discussed, as an example, in [3],[9],[10].

Let h be weight function on the segment $[-1,1]$, i.e. h be non-negative, integrable and $\int\limits_{-1}^{1} h(x)dx > 0$.

Let the numbers a_j, $j = \overline{1, n-1}$, satisfy the following condition: if at some j, $j = \overline{1, n-1}$, $\mathrm{Im}a_j \neq 0$, then also the complex conjugate $\overline{a_j}$ is among the numbers. Furthermore, if $a_j \in \mathbf{R}$, then $|a_j| < 1$.

Let us introduce the following symbols:

$$\mathbf{R}_{n-1}(a) = \left\{ P_{n-1}(x)/\prod_{j=1}^{n-1}(1+a_j x) \mid P_{n-1} \in \mathbf{P}_{n-1} \right\},$$

$$\mathbf{R}_{2n-1,2}(a) = \left\{ P_{2n-1}(x)/\prod_{j=1}^{n-1}(1+a_j x)^2 \mid P_{2n-1} \in \mathbf{P}_{2n-1} \right\},$$

where $\mathbf{P}_m$ is the set of algebraic polynomials of degree not greater than m. Thus, $\mathbf{R}_{n-1}(a)$ contains algebraic rational functions of order not higher than $n-1$ with poles at the points $-a_1^{-1}, -a_2^{-1}, \ldots, -a_{n-1}^{-1}$, $\mathbf{R}_{2n-1,2}(a)$ and is a set of rational functions of order not higher, than $2n-1$, with the same poles but of double multiplicity.

Then, let $q_n \in \mathbf{P}_n$ be polynomial orthogonal with respect to the weight $h(x)\prod\limits_{j=1}^{n-1}(1+a_j x)^{-2}$ on the segment $[-1,1]$. The polynomial q_n is known to have n simple roots on the interval $(-1,1)$:

$$-1 < x_1 < x_2 < \ldots < x_n < 1, \quad q_n(x_k) = 0, \ k = \overline{1, n}.$$

For any function f, defined on $(-1,1)$, let us built the interpolating rational function

$$L_{n-1}(x,f) = \sum_{k=1}^{n} f(x_k) l_k(x),$$

where

$$l_k(x) = t_n(x)/(x - x_k)t'_n(x_k),\ k = \overline{1,n};\quad t_n(x) = q_n(x)\prod_{j=1}^{n-1}(1 + a_j x)^{-1}.$$

It is easy to see that $L_{n-1}(x, f) \in \mathbf{R}_{n-1}(a)$ and for any function $r_{n-1} \in \mathbf{R}_{n-1}(a)$, $L_{n-1}(x, r_{n-1}) \equiv r_{n-1}(x)$.

Now, for a function f, integrable with weight h on the segment $[-1, 1]$, we examine the quadrature formula:

$$\int_{-1}^{1} h(x)f(x)dx \approx \sum_{k=1}^{n} A_k f(x_x), \tag{12}$$

where

$$A_k = \int_{-1}^{1} h(x)l_k(x)dx = \frac{1}{t'_n(x_k)}\int_{-1}^{1} h(x)\frac{t_n(x)}{x - x_k}dx, \quad k = \overline{1,n}.$$

The following properties are analogous to the known theorems of Gauss quadrature formulas (see, for example, [10]).

The quadrature formula (12) has the following properties:

1) It is exact for any rational function $r_{n-1} \in \mathbf{R}_{n-1}$ and $r_{2n-1} \in \mathbf{R}_{2n-1,2}$;

2) The coefficients A_k, $k = \overline{1,n}$ are positive and

3) $A_k = \frac{1}{t_n^2(x_k)} \int_{-1}^{1} h(x)\frac{t_n^2(x)}{(x-x_k)^2}dx, \qquad k = \overline{1,n};$

4) The following equality holds

$$\sum_{k=1}^{n} A_k = \int_{-1}^{1} h(x)dx.$$

If $f \in C[-1, 1]$, then the following inequality holds for the quadrature formula (12):

$$\left|\int_{-1}^{1} h(x)f(x)dx - \sum_{k=1}^{n} A_k f(x_k)\right| \leq 2R_{2n-1}(f, a)\int_{-1}^{1} h(x)dx,$$

where $R_{2n-1}(f,a) = \inf\limits_{r_{2n-1}\in \mathbf{R}_{2n-1,2}} \|f(x) - r_{2n-1}(x)\|_{C[-1,1]}$ is the best approximation of the function f by means of rational functions from $\mathbf{R}_{2n-1,2}$ on the segment $[-1,1]$.

3. Special cases of Gauss-type quadratures

Let the numbers a_k, $k = \overline{1,n}$, be real and $a_k \in (-1,1)$, or mutually complex-conjugate, with $a_0 = a_n = 0$.

3.1. Let us denote by m_n the rational function of Chebyshev-Markov:

$$m_n(x) = \cos \mu_n(x),$$

where

$$\mu_n'(x) = -\lambda_n(x)/\sqrt{1-x^2},\ \lambda_n(x) = \sum_{k=1}^{n} \frac{\sqrt{1-a_k^2}}{1+a_k x}.$$

The m_n function has n simple zeros on the interval $(-1,1)$ (see [11], p. 48): $-1 < x_n < x_{n-1} < \ldots < x_1 < 1$, $m_n(x_k) = 0$, $k = \overline{1,n}$. For any function $f \in C[-1,1]$ we shall construct the quadrature formula:

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx \approx \sum_{k=1}^{n} A_k f(x_k), \tag{13}$$

where

$$A_k = \frac{1}{m_n'(x_k)} \int_{-1}^{1} \frac{m_n(x)}{x-x_k} \frac{dx}{\sqrt{1-x^2}} = (-1)^k \frac{\sqrt{1-x_k^2}}{\lambda_n(x_k)} \int_{-1}^{1} \frac{\cos \mu_n(x)}{x-x_k} \frac{dx}{\sqrt{1-x^2}},\ k = \overline{1,n}. \tag{14}$$

Theorem 2. *The quadrature formula (13) has the following form*

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx \approx \sum_{k=1}^{n} \frac{\pi}{\lambda_n(x_k)} f(x_k) \tag{15}$$

and for its remainder the following estimate is valid

$$\left| \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx - \sum_{k=1}^{n} \frac{\pi}{\lambda_n(x_k)} f(x_k) \right| \le 2\pi R_{2n-1}(f,a). \tag{16}$$

Proof. Let us calculate the integral

$$J_{nk} = \int_{-1}^{1} \frac{m_n(x)}{x - x_k} \frac{dx}{\sqrt{1 - x^2}}.$$

Let us substitute $x = (1 - y^2)/(1 + y^2)$. Denote by $M_n(y) = m_n((1 - y^2)/(1 + y^2))$. As it is known, see [11], p.47, the function M_n is the Bernstein rational function on the real axis and has zeros at the points $\pm y_k$, $y_k = \sqrt{(1 - x_k)/(1 + x_k)}$, $k = \overline{1, n}$. We get

$$J_{nk} = -\frac{1 + y_k^2}{2} \int_{-\infty}^{\infty} \frac{M_n(y)}{y^2 - y_k^2} dy.$$

Let us evaluate the integral

$$J_n(z) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{M_n(y)}{y^2 - z^2} dy, \qquad z \in \mathbf{C},\ \mathrm{Im} z > 0.$$

From [11] we derive

$$M_n(y) = \frac{1}{2} \left(\prod_{m=1}^{n} \frac{y - z_m}{y - \overline{z}_m} + \prod_{m=1}^{n} \frac{y - \overline{z}_m}{y - z_m} \right),$$

where z_k are the roots of equation $y^2 + \frac{1+a_k}{1-a_k} = 0$, and $\mathrm{Im} z > 0$, $k = \overline{1, n}$. Let us also emphasize that the numbers z_k, $k = \overline{1, n}$, will be arranged symmetrically with respect to the imaginary axis.

Evidently,

$$J_n(z) = \frac{\pi i}{2} \left(\operatorname*{res}_{y=z} \frac{1}{y^2 - z^2} \prod_{m=1}^{n} \frac{y - z_m}{y - \overline{z}_m} - \operatorname*{res}_{y=-z} \frac{1}{y^2 - z^2} \prod_{m=1}^{n} \frac{y - \overline{z}_m}{y - z_m} \right)$$

$$= \frac{\pi i}{2z} \prod_{m=1}^{n} \frac{z - z_m}{z - \overline{z}_m}.$$

Then,

$$J_{nk} = -\frac{1 + y_k^2}{2} \lim_{\substack{z \to y_k, \\ \mathrm{Im} z > 0}} \cdot J_n(z) = -\pi i \frac{1 + y_k^2}{2 y_k} \prod_{m=1}^{n} \frac{y_k - z_m}{y_k - \overline{z}_m}.$$

Denote

$$\prod_{m=1}^{n} \frac{y_k - z_m}{y_k - \overline{z}_m} = \omega_{n,k}.$$

From the fact that $\mu_n(x_k) = \frac{\pi}{2} + \pi k$, $k = \overline{1,n}$, it follows that $\omega_{n,k} + \overline{\omega}_{n,k} = 0$, $\frac{1}{2i}(\omega_{n,k} - \overline{\omega}_{n,k}) = (-1)^k$.

In this case we find that $\omega_{n,k} = i(-1)^k$.

Thus,

$$J_{nk} = (-1)^k \pi \frac{1 + y_k^2}{2y_k} = \frac{(-1)^k \pi}{\sqrt{1 - x_k^2}}.$$

Then formula (15) is a consequence of relations (13) and (14).

Estimate (16) is a direct consequence of Theorem 2 considering that the function $m_n(x) = M_n^{(0)}(x)$ is a member of orthogonal system of rational functions $M_0^{(0)}(x)$, $M_1^{(0)}(x)$,..., $M_n^{(0)}(x)$, on the segment $[-1, 1]$ according to the weight $(1 - x^2)^{-1/2}$ and the numbers a_k, $k = \overline{1,n}$.

3.2. Let ν_n be the rational sine-function of Chebyshev-Markov (see [11], p.49):

$$\nu_n(x) = \sin \mu_{n+1}(x)/\sqrt{1 - x^2},$$

where

$$\mu'_{n+1}(x) = -\lambda_{n+1}(x)/\sqrt{1 - x^2}, \quad \lambda_{n+1}(x) = 1 + \sum_{k=1}^{n} \frac{\sqrt{1 - a_k^2}}{1 + a_k x}.$$

Then the function ν_n is rational of n order and has n simple zeros on the interval $(-1, 1)$, $-1 < x_n < x_{n-1} < \ldots < x_1 < 1$.

For any function $f \in C[-1, 1]$ let us construct the quadrature formula:

$$\int_{-1}^{1} \sqrt{1 - x^2} f(x) dx \approx \sum_{k=1}^{n} A_k f(x_k), \tag{17}$$

where

$$A_k = \frac{\sqrt{1 - x^2}}{\nu'_n(x_k)} \int_{-1}^{1} \frac{\nu_n(x)}{x - x_k} dx = (-1)^{k+1} \frac{\sqrt{1 - x_k^2}}{\lambda_{n+1}(x_k)} \int_{-1}^{1} \frac{\sin \mu_{n+1}(x)}{x - x_k} dx, \quad k = \overline{1,n}.$$

Theorem 3. *The quadrature formula (17) is given by*

$$\int_{-1}^{1} \sqrt{1 - x^2} f(x) dx \approx \pi \sum_{k=1}^{n} \frac{1 - x_k^2}{\lambda_{n+1}(x_k)} f(x_k) \tag{18}$$

and for its remainder the following estimate holds

$$\left| \int_{-1}^{1} \sqrt{1 - x^2} f(x) dx - \pi \sum_{k=1}^{n} \frac{1 - x_k^2}{\lambda_{n+1}(x_k)} f(x_k) \right| \leq \pi R_{2n-1}(f, a).$$

B. Samokysh in his work [12] built the quadrature formula with Chebyshev weight of the second type optimal in H_2. It turns out, that the quadrature formula, deduced by B. Samokysh is a special case of formula (18). It is the case when $x_k = a_k$, $k = \overline{1,n}$, n is an odd number, and such numbers as $a_1, a_2, \ldots, a_n$ do exist and are the only ones.

3.3. Let $Q_n(x) = \sqrt{2}\sin\mu_{n+1/2}(x)/\sqrt{1-x}$ be the rational function of Jacobi type orthogonal with respect to the weight $\sqrt{(1-x)/(1+x)}$ on the segment $[-1,1]$, where

$$\mu'_{n+1/2}(x) = -\lambda_{n+1/2}(x)/\sqrt{1-x^2},\ \lambda_{n+1}(x) = \frac{1}{2} + \sum_{k=1}^{n}\frac{\sqrt{1-a_k^2}}{1+a_k x}$$

(see Theorem 1). According to Lemma 4, the function Q_n has n simple zeros on the interval $(-1,1)$, $-1 < x_n < x_{n-1} < \ldots < x_1 < 1$.

For $f \in C[-1,1]$ let us construct the quadrature formula:

$$\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}f(x)dx \approx \sum_{k=1}^{n} A_k f(x_k), \tag{19}$$

where

$$A_k = (-1)^{k+1}\frac{(1-x_k)\sqrt{1+x_k}}{\lambda_{n+1/2}(x_k)}\int_{-1}^{1}\frac{\sin\mu_{n+1/2}(x)}{\sqrt{1+x}(x-x_k)}dx.$$

Theorem 4. *The quadrature formula (19) is as follows*

$$\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}f(x)dx \approx \pi\sum_{k=1}^{n}\frac{1-x_k}{\lambda_{n+1/2}(x_k)}f(x_k)$$

and for its remainder the following estimate holds

$$\left|\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}f(x)dx - \pi\sum_{k=1}^{n}\frac{1-x_k}{\lambda_{n+1/2}(x_k)}f(x_k)\right| \le 2\pi R_{2n-1}(f,a).$$

Theorems 3 and 4 are proved like Theorem 2.

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