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A Generalization of a Fixed Point Theorem of B. Fisher

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Let (X,d,E) and (Y,ρ,E) be two sequential complete metric spaces over a topological semifield E. It is proved that if T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities (1) and (2) below, that ST has a unique fixed point in X and TS has a unique fixed point in Y.

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1. Introduction

The notion of topological semifield has been introduced by the mathematicians M. Antonovski, V. Boltjanski and T. Sarymsakov in [1].

Let E be a topological semifield and K be the set of all its positive elements. Take any two elements x, y in E. If $y - x$ is in $\overline{K}$ (in K), this is denoted by $x << y$ ($x < y$). As proved in [1], every topological semifield E contains a subsemifield, so called the axis of E, isomorphic to the field R of real numbers.

The ordered triple (X,d,E) is called a metric space over the topological semifield, if there exists a mapping $d : X \times X \mapsto \overline{K}$ satisfying the usual axioms for a metric.

2. Main result

We shall prove the following theorem.

Theorem 1. *Let (X,d,E) and (Y,ρ,E) be sequential complete metric spaces over a topological semifield E. If T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities:*

$$\begin{aligned}
&(1) \qquad \rho(Tx, TSy) << a_1\rho(y, Tx) + b_1\rho(y, TSy) + c_1 d(x, Sy),\\
&(2) \qquad d(Sy, STx) << a_2 d(x, Sy) + b_2 d(x, STx) + c_2\rho(y, Tx)
\end{aligned}$$

for all x in X and y in Y, where a_i, b_i, c_i in $\overline{K}, a_i + b_i + c_i < 1, i = 1, 2$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Proof. Let x be an arbitrary point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively, by

$$(ST)^n x = x_n, \quad T(ST)^{n-1}x = y_n$$

for $n = 1, 2, \ldots$. Using inequality (2), we have

$$d(x_n, x_{n+1}) << a_2 d(x_n, x_n) + b_2 d(x_n, x_{n+1}) + c_2\rho(y_n, y_{n+1}),$$

which implies

$$d(x_n, x_{n+1}) << t_2\rho(y_n, y_{n+1}),$$

where $t_2 = c_2(1 - b_2)^{-1} < 1$. Using inequality (1), we have

$$\rho(y_n, y_{n+1}) << a_1\rho(y_n, y_n) + b_1\rho(y_n, y_{n+1}) + c_1 d(x_{n-1}, x_n),$$

which implies

$$\rho(y_n, y_{n+1}) << t_1 d(x_{n-1}, x_n),$$

where $t_1 = c_1(1 - b_1)^{-1} < 1$.

It follows that

$$d(x_n, x_{n+1}) << t_2\rho(y_n, y_{n+1}) << t_1 t_2 d(x_{n-1}, x_n) << \ldots << (t_1 t_2)^n d(x, x_1),$$

and since $0 << t_1 t_2 < 1$, $\{x_n\}$ is a Cauchy sequence in X and $\{y_n\}$ is a Cauchy sequence in Y. By using that (X, d, E) is a sequential complete metric space, we deduce that $\{x_n\}$ converges to a point z in X. Because (Y, ρ, E) is a sequential complete metric space, we deduce that $\{y_n\}$ converges to apoint w in Y.

Now, by using inequality (1), we have

$$\rho(Tz, y_n) << a_1\rho(y_{n-1}, Tz) + b_1\rho(y_{n-1}, y_n) + c_1 d(z, x_{n-1}).$$

Letting n tend to infinity, we have

$$(1 - a_1)\rho(Tz, w) << 0$$

and so, $Tz = w$, since $1 - a_1 > 0$. Similarly, we can prove that $Sw = z$ and

$$STz = Sw = z \quad \text{and} \quad TSw = Tz = w.$$

Thus, ST has a fixed point z and TS has a fixed point w.

Now, suppose that ST has a second fixed pint z'. Then by using the inequality (2), we have

$$d(STz', STz) << a_2 d(z, STz') + b_2 d(z, STz) + c_2 \rho(Tz', Tz),$$

or

$$(1 - a_2) d(z', z) << c_2 \rho(Tz', Tz),$$

which implies

$$d(z', z) << c_2 (1 - a_2)^{-1} \rho(Tz', Tz).$$

But by using inequality (1),

$$\rho(Tz, TSTz') << a_1 \rho(Tz', Tz) + b_1 \rho(Tz', TSTz') + c_1 d(z, STz'),$$

or

$$(1 - a_1) \rho(Tz, Tz') << c_1 d(z, z'),$$

which implies

$$\rho(Tz, Tz') << c_1 (1 - a_1)^{-1} d(z, z')$$

and so,

$$d(z', z) << c_1 c_2 (1 - a_1)^{-1} (1 - a_2)^{-1} d(z, z').$$

Since, $0 << c_1 c_2 (1 - a_1)^{-1} (1 - a_2)^{-1} < 1$, the uniqueness of z follows. Similarly, w is the unique fixed point of TS. This completes the proof of the theorem. ∎

Remark. In case $E = R$ in Theorem 1, we obtain Theorem 1 of Brian Fisher [2].

References

[1] M. Ya. Antonovski, V. G. Boltyanski, T. Sarymsakov. *Topological Semifields*, Tashkent, 1960 (In Russian).

[2] B. Fisher. Fixed point on two metric spaces, *Glasnik Mat.* **16 (36)**, 1981, 333-337.

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