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# On Approximation by Modified Kantorovich Polynomials

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We establish some approximation properties by Kantorovich type polynomials for a function $f \in C^r[0,1], r = 0,1,2,\ldots$

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## 1. Introduction

By $C^r[0,1]$ ($C^0[0,1] = C[0,1]$), $r = 0,1,2,...$, we denote the set of all functions $f : [0,1] \to R$, with a continuous derivative of order $r$ on the interval $[0,1]$. For the function $f \in C^r[0,1]$, $r = 0,1,2,...$ the generalized Bernstein polynomial of $(n,r)$-th order was introduced by G.H. Kirov [2]:

$$B_{n,r}(f;x) = \sum_{k=0}^{n}\sum_{i=0}^{r} \frac{f^{(i)}(k/n)}{i!}(x-k/n)^i \binom{n}{k} x^k(1-x)^{n-k}. \tag{1}$$

The author proved, among others, the following theorem.

**Theorem A.** *Let $f \in C^r[0,1], r = 0,1,2,...$ and $B_{n,r}(f;x)$ be the generalized Bernstein polynomial of order $(n,r)$ for $f$. Then,*

$$\|f - B_{n,r}f\|_\infty = O(n^{-r/2}\omega(f^{(r)}; n^{-1/2})), \tag{2}$$

*where $\|g\|_\infty = \sup\{|g(x)| : x \in [0,1]\}$ for arbitrary $g \in C[0,1]$ and $\omega(g;s) = \sup\{|g(x) - g(y)| : x,y \in [0,1], |x-y| \le s\}$ is the modulus of continuity of the function $g$ in the segment $[0,1]$.*

The modulus of continuity in $L_p[0,1]$ of the function $f$ is the following function of $\delta \in [0,\infty)$ :

$$\omega(f;\delta)_p = \sup\{\|\Delta_h f\|_p : 0 < h \le \delta\}, \tag{3}$$

where $1 \le p < \infty$.

For a function $f \in L_p[0,1] (1 \le p < \infty)$ the Kantorovich polynomials are given by

$$B_n^*(f;x) = \sum_{k=0}^{n} p_{n,k}(x)(n+1)\int_{k/n+1}^{k+1/n+1} f(t)dt, \tag{4}$$

where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$.

Let us introduce the following Kantorovich type polynomials.

**Definition.** A *generalized Kantorovich polynomial* of $(n,r)$-th order for a function $f$, with $f^{(r)} \in L_p[0,1]$, $r = 0,1,2,\ldots$ is said to be the polynomial

$$B_{n,r}^*(f;x) = \sum_{k=0}^{n} p_{n,k}(x)(n+1)\int_{k/n+1}^{k+1/n+1} \sum_{i=0}^{r} \frac{f^{(i)}(t)}{i!}(x-t)^i dt. \tag{5}$$

For $r = 0$ from (4) and (5) it follows the equality $B_{n,0}^*(f;x) = B_n^*(f;x)$.

## 2. Main results

The object of this paper is to give similar results to Theorem A for the generalized Kantorovich polynomials.

**Theorem 1.** *Let $f \in C^r[0,1]$, $r = 1,2,3,\ldots$ and $B_{n,r}^*(f;x)$ be the generalized Kantorovich polynomial of order $(n,r)$ for $f$. Then,*

$$\|B_{n,r}^* f - f\|_\infty = O(n^{-r/2}\omega(f^{(r)}; n^{-1/2})). \tag{6}$$

**Theorem 2.** *Let $f \in C^r[0,1]$, $r = 1,2,3,\ldots$ and $B_{n,r}^*(f;x)$ be the generalized Kantorovich polynomial of order $(n,r)$ for the function $f$. Then,*

$$\|B_{n,r}^* f - f\|_p = O(n^{-r/2}\omega(f^{(r)}; n^{-1/2})_p), \tag{7}$$

*where $1 \le p < \infty$.*

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Using the modified Taylor's formula

$$f(x)=\sum_{i=0}^{n}\frac{f^{(i)}(t)}{i!}(x-t)^i+\frac{(x-t)^r}{(r-1)!}\int_0^1(1-u)^{r-1}\{f^{(r)}(t+u(x-t))-f^{(r)}(t)\}du$$

and the equality $B_n^*(1;x)=1$, from the definition of the modulus of continuity and (5) of the generalized Kantorovich polynomial, for every $x\in[0,1]$ we obtain:

$$\begin{aligned}|f(x)-B_{n,r}^*(f;x)|&=\Big|\sum_{k=0}^{n}p_{n,k}(x)(n+1)\int_{k/n+1}^{k+1/n+1}f(x)dt\\&-\sum_{k=0}^{n}p_{n,k}(x)(n+1)\int_{k/n+1}^{k+1/n+1}\sum_{i=0}^{r}\frac{f^{(i)}(t)}{i!}(x-t)^i dt\Big|\\&=\Big|\sum_{k=0}^{n}p_{n,k}(x)(n+1)\cdot\int_{k/n+1}^{k+1/n+1}\frac{(x-t)^r}{(r-1)!}\\&\times\left(\int_0^1(1-u)^{r-1}\{f^{(r)}(t+u(x-t))-f^{(r)}(t)\}du\right)dt\Big| \qquad (8)\\&\le\sum_{k=0}^{n}p_{n,k}(x)(n+1)\cdot\int_{k/n+1}^{k+1/n+1}\frac{|x-t|^r}{(r-1)!}\\&\times\left(\int_0^1(1-u)^{r-1}\cdot\omega(f^{(r)};u|x-t|)du\right)dt.\end{aligned}$$

Since $\omega(g;\lambda s)\le(\lambda+1)\omega(g;s)$, $\lambda>0$, then

$$\omega(f^{(r)};u|x-t|\cdot n^{1/2}\cdot n^{-1/2})\le(u|x-t|\cdot n^{1/2}+1)\omega(f^{(r)};n^{-1/2}).$$

Thus we have the estimate

$$|f(x)-B_{n,r}^*(f;x)|\le\omega(f^{(r)};n^{-1/2})\cdot\sum_{k=0}^{n}p_{n,k}(x)(n+1)$$

$$\times\int_{k/n+1}^{k+1/n+1}\frac{|x-t|^r}{(r-1)!}\left(\int_0^1(1-u)^{r-1}\cdot\{u|x-t|n^{1/2}+1\}du\right)dt$$

$$=\omega(f^{(r)};n^{-1/2})\cdot\sum_{k=0}^{n}p_{n,k}(x)(n+1)\cdot\int_{k/n+1}^{k+1/n+1}\Big\{\frac{|x-t|^{r+1}}{(r-1)!}$$

$$\times\, n^{1/2}\int_0^1 u(1-u)^{r-1}du+\frac{|x-t|^r}{(r-1)!}\int_0^1(1-u)^{r-1}du\Bigg\}dt \tag{9}$$

$$=\omega(f^{(r)};n^{-1/2})\cdot\sum_{k=0}^{n}p_{n,k}(x)(n+1)\cdot\int_{k/n+1}^{k+1/n+1}\left\{\frac{|x-t|^{r+1}}{(r+1)!}n^{1/2}+\frac{|x-t|^r}{r!}\right\}dt$$

$$=\omega(f^{(r)};n^{-1/2})\cdot\Bigg\{\sum_{k=0}^{n}p_{n,k}(x)(n+1)\cdot\int_{k/n+1}^{k+1/n+1}\frac{|x-t|^{r+1}}{(r+1)!}\cdot n^{1/2}dt$$

$$+\sum_{k=0}^{n}p_{n,k}(x)(n+1)\cdot\int_{k/n+1}^{k+1/n+1}\frac{|x-t|^r}{r!}dt\Bigg\}.$$

Using the elementary inequality $(a+b)^r\le 2^{r-1}\cdot(a^r+b^r)$ for $a,b\ge 0$ and $r=1,2,3,\ldots$, we have:

$$\int_{k/n+1}^{k+1/n+1}|x-t|^r dt\le 2^{r-1}\int_{k/n+1}^{k+1/n+1}|x-k/n|^r dt$$

$$+\,2^{r-1}\int_{k/n+1}^{k+1/n+1}|k/n-t|^r dt=2^{r-1}\cdot n^{-r}\cdot\int_{k/n+1}^{k+1/n+1}|k-nx|^r dt$$

$$+\,2^{r-1}\cdot\left\{\int_{k/n+1}^{k/n}(k/n-t)^r dt+\int_{k/n}^{k+1/n+1}(t-k/n)^r dt\right\} \tag{10}$$

$$=2^{r-1}\cdot n^{-r}\cdot\frac{1}{n+1}|k-nx|^r$$

$$+\,2^{r-1}\cdot\left\{\left(\frac{k}{n(n+1)}\right)^{r+1}\cdot\frac{1}{r+1}+\left(\frac{n-k}{n(n+1)}\right)^{r+1}\cdot\frac{1}{r+1}\right\}$$

$$\le 2^{r-1}\cdot n^{-r}\cdot\frac{1}{n+1}|k-nx|^r+\frac{2^r}{r+1}\cdot(n+1)^{-(r+1)}.$$

So,

$$\sum_{k=0}^{n}p_{n,k}(x)(n+1)\int_{k/n+1}^{k+1/n+1}\frac{|x-t|^r}{r!}dt$$

$$\le\sum_{k=0}^{n}p_{n,k}(x)\cdot\frac{2^{r-1}}{r!}\cdot n^{-r}|k-nx|^r+\sum_{k=0}^{n}p_{n,k}(x)\cdot\frac{2^r}{(r+1)!}\cdot(n+1)^{-r} \tag{11}$$

$$=\frac{2^{r-1}}{r!}\cdot n^{-r}\sum_{k=0}^{n}p_{n,k}(x)|k-nx|^r+\frac{2^r}{(r+1)!}\cdot(n+1)^{-r}.$$

Using the Cauchy inequality and the obvious identity

$$\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}=1,$$

we get

$$\sum_{k=0}^{n}|k-nx|^r p_{n,k}(x)\le (S_{2r}(x))^{1/2},\quad x\in[0,1], \tag{12}$$

where $S_m(x)=\sum_{k=0}^{n}(k-nx)^m p_{n,k}(x)$, $m=0,1,2,...$

On the other hand, it is known ([3], p.248), that for every $x\in[0,1]$

$$|S_m(x)|\le K(m)\cdot n^{[m/2]}, \tag{13}$$

where $K(m)$ is a constant, depending on $m$, and $[m/2]$ is the integer part of $m/2$.

From (11)-(13) the estimate

$$\sum_{k=0}^{n}p_{n,k}(x)(n+1)\int_{k/n+1}^{k+1/n+1}\frac{|x-t|^r}{r!}dt$$
$$\le\frac{2^{r-1}}{r!}\cdot n^{-r}\cdot\sqrt{K(2r)}\cdot n^{r/2}+\frac{2^r}{(r+1)!}\cdot n^{-r} \tag{14}$$
$$=n^{-r/2}\cdot\left\{\frac{2^{r-1}}{r!}\cdot\sqrt{K(2r)}+\frac{2^r}{(r+1)!}\cdot n^{-r/2}\right\}=O(n^{-r/2})$$

follows.

In a similar way, we obtain the following estimate

$$\sum_{k=0}^{n}p_{n,k}(x)(n+1)n^{1/2}\int_{k/n+1}^{k+1/n+1}\frac{|x-t|^{r+1}}{(r+1)!}dt \tag{15}$$
$$\le n^{-(r+1)/2}\cdot\left\{\frac{2^r}{(r+1)!}\cdot\sqrt{K(2r+2)}+\frac{2^{r+1}}{(r+2)!}\cdot n^{-(r+1)/2}\right\}\cdot n^{1/2}=O(n^{-r/2}).$$

The estimates (14)-(15) together with (9) implies (6) and the proof of the theorem is completed. ■

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For $p=1$ we have

$$\int_0^1(1-u)^{r-1}|f^{(r)}(t+u(x-t))-f^{(r)}(t)|du$$

$$\leq \int_0^1 |f^{(r)}(t+u(x-t)) - f^{(r)}(t)|du \leq \omega(f^{(r)}; |x-t|)_1 \tag{16}$$
$$\leq (|x-t|n^{1/2}+1)\cdot\omega(f^{(r)}; n^{-1/2})_1.$$

Then, by (8) and (16), we obtain

$$|f(x) - B^*_{n,r}(f;x)| \leq \omega(f^{(r)}; n^{-1/2})_1$$

$$\times \sum_{k=0}^{n} p_{n,k}(x)(n+1)\int_{k/n+1}^{k+1/n+1} \frac{|x-t|^r}{(r-1)!}\left(|x-t|\cdot n^{1/2}+1\right)dt \tag{17}$$

$$= \omega(f^{(r)}; n^{-1/2})_1 \cdot \sum_{k=0}^{n} p_{n,k}(x)(n+1)\int_{k/n+1}^{k+1/n+1}\left\{\frac{|x-t|^{r+1}}{(r-1)!}n^{1/2} + \frac{|x-t|^r}{(r-1)!}\right\}dt.$$

The next estimates are carried out analogously to the estimates (10)-(13) for the preceding theorem. Then, by (17), we get:

$$|f(x) - B^*_{n,r}(f;x)| \leq \omega(f^{(r)}; n^{-1/2})_1 \cdot \left\{ n^{-r/2}\cdot\frac{r(r+1)\cdot 2^r}{(r+1)!}\sqrt{K(2r+2)}\right.$$

$$\left. + n^{-r-(1/2)}\cdot\frac{r(r+1)\cdot 2^{r+1}}{(r+2)!} + n^{-r/2}\cdot\frac{2^{r-1}r}{r!}\sqrt{K(2r)} + n^{-r}\cdot\frac{2^r r}{(r+1)!}\right\}$$
$$= O(n^{-r/2}\cdot\omega(f^{(r)}; n^{-1/2})_1). \tag{18}$$

Hence, $\|B^*_{n,r}f - f\|_1 = O(n^{-r/2}\cdot\omega(f^{(r)}; n^{-1/2})_1)$ and the proof is complete for $p=1$.

If $1<p<\infty$ then, by Hölder's inequality, we have

$$\int_0^1 (1-u)^{r-1}|f^{(r)}(t+u(x-t)) - f^{(r)}(t)|du$$

$$\leq \left\{\int_0^1 |f^{(r)}(t+u(x-t)) - f^{(r)}(t)|^p du\right\}^{1/p} \cdot \left\{\int_0^1 (1-u)^{p(r-1)/(p-1)}du\right\}^{p/p-1}$$
$$\leq \omega(f^{(r)}; |x-t|)_p \cdot \left(\frac{p-1}{pr-1}\right)^{p/p-1} \tag{19}$$
$$\leq (|x-t|n^{1/2}+1)\omega(f^{(r)}; n^{-1/2})_p \cdot \left(\frac{p-1}{pr-1}\right)^{p/p-1}.$$

Then we have the estimate

$$|f(x) - B^*_{n,r}(f;x)| \le \omega(f^{(r)}; n^{-1/2})_p \cdot \left(\frac{p-1}{pr-1}\right)^{p/p-1}$$

$$\times \sum_{k=0}^{n} p_{n,k}(x)(n+1) \cdot \int_{k/n+1}^{k+1/n+1} \left\{ \frac{|x-t|^{r+1}}{(r-1)!} \cdot n^{1/2} + \frac{|x-t|^r}{(r-1)!} \right\} dt$$

$$= O(n^{-r/2} \cdot \omega(f^{(r)}; n^{-1/2})_p), \tag{20}$$

using similar calculation to (18).

Therefore, (20) implies $\|B^*_{n,r}f - f\|_p = O(n^{-r/2} \cdot \omega(f^{(r)}; n^{-1/2})_p)$, which was to be proved. ■

**Remark.** Our theorems are established for $r = 1, 2, 3, \ldots$, because the problems of characterization of $\|B^*f - f\|_p$ were treated by many authors. We mention the following result [1, p.117].

**Theorem B.** *Let $f \in L_p[0,1]$ and $\varphi^2(x) = x(1-x)$. Then,*

$$\|B^*_n f - f\|_p \le M[\omega^2_\varphi(f; n^{-1/2})_p + n^{-1}\|f\|_p], \tag{21}$$

*where $\omega^2_\varphi(f;\delta) = \sup\{\|\Delta^2_{h\varphi} f\|_p : 0 < h \le \delta\}$ is the Ditzian-Totik modulus of smoothness.*

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