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# On the Number and Location of Limit Trajectories of a Class of Autonomous Systems

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The quasi-linear autonomous systems with parameters have some interesting representations. We are interested in the range of the values of the parameters of the system where limit trajectories exist or the range of values where these trajectories are missing. The existence of the limit cycle is proved.

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We consider the system

$$\begin{aligned} \dot{x}_1 &= ax_1 + bx_2 - x_1 f(x_1^2 + x_2^2) \\ \dot{x}_2 &= cx_1 + dx_2 - x_2 f(x_1^2 + x_2^2), \end{aligned} \tag{1}$$

where $f$ is defined and continuous on some interval $[0, R)$, and $f(0) = 0$. The class of differential equations (1) represents some models of engineering physical systems. In particular, the problem for oscillation stability of signals of electronic generators [4] results in a systems of differential equations of type (1). In addition, for the analysis of the moving pictures the problem of convergence with last point of object tracking leads to the system of the same class.

In polar coordinates system (1) gets the form

$$\begin{aligned} 2\dot{r} &= r[a + b + (a - d)\cos 2\varphi + (b + c)\sin 2\varphi - 2f(r^2)] \\ 2\dot{\varphi} &= c - b - (a - d)\sin 2\varphi + (b + c)\cos 2\varphi. \end{aligned} \tag{2}$$

Let now we consider the system (2), where

$$d = a, \quad bc < 0, \tag{3}$$

then the origin $(x_1, x_2) = (0, 0)$ is a singular point for (1) and it is of focal type. Our purpose is a more detailed study of the trajectories behaviour of (1) which are close to the zero solution. In particular, we are interested in the range of the values of parameters of the system (1), where there exist limit cycles or the range of values, where these trajectories are missing.

Integrating the second equation of (2), we get

$$\operatorname{tg}(\varphi_0 + \varphi) = \frac{1}{b}\sqrt{-bc}\operatorname{tg}\sqrt{-bc}t, \quad \varphi_0 = \varphi(0). \tag{4}$$

We substitute (4) into the first equation of (2) and obtain

$$\dot{r} = -\Phi(t)r - rf(r^2), \tag{5}$$

where

$$\Phi(t) = a + (b+c)\frac{\sqrt{-bc}\operatorname{tg}\sqrt{-bc}t}{b - c\operatorname{tg}^2\sqrt{-bc}t}, \tag{6}$$

the function $\Phi(t)$ is periodic with a period $\frac{\pi}{k}$, $k = \sqrt{-bc}$, $bc < 0$.

Let us denote

$$\sigma = -\int_0^{\pi/k} \Phi(t)dt. \tag{7}$$

Now we apply to (5) a well known transformation (see [1]) and consider the equation

$$\dot{u} = -\Phi(t)u - \frac{k}{\pi}\sigma u. \tag{8}$$

**Theorem 1.** *All the solutions to (8) are periodic with a period* $\frac{\pi}{k}$.

Indeed, if $u$ is a solution to (8), we have

$$\dot{u}\left(t+\frac{\pi}{k}\right) = -\Phi(t)u\left(t+\frac{\pi}{k}\right) - \frac{k}{\pi}\sigma u\left(t+\frac{\pi}{k}\right), \tag{9}$$

since $\Phi$ is periodic with a period $\frac{\pi}{k}$. Hence. $u\left(t+\frac{\pi}{k}\right)$ is also a solution to (8) and consequently, there exists a constant $C$ such that

$$u\left(t+\frac{\pi}{k}\right) = Cu(t). \tag{10}$$

On the other side, solving (8) we obtain

$$u(t) = u(0)\exp\left(-\int_0^t \Phi(t)dt - \frac{k}{\pi}\sigma t\right). \tag{11}$$

Therefore,

$$u\left(t+\frac{\pi}{k}\right) = u(0)\exp\left[-\int_0^{t+\frac{\pi}{k}} \Phi(t)dt - \frac{k}{\pi}\sigma\left(t+\frac{\pi}{k}\right)\right]. \tag{12}$$

It follows from (10),(11) and (12) that

$$C = \frac{u\left(t+\frac{\pi}{k}\right)}{u(t)} = 1, \tag{13}$$

i.e. $u(t)$ is a periodic function.

Take a fixed solution $U(t)$ of (8). It is seen from (11) that $U(t)$ can be chosen to be strictly positive and, being periodic, it has a positive lower bound $L_1$ and an upper bound $L_2$. Now, we perform a change of variables in (8) introducing a new function $S(t)$ instead of $r(t)$:

$$r(t) = U(t)S(t). \tag{14}$$

Taking into account (8) we obtain from (5),

$$\dot{S} = \frac{k}{\pi}\sigma S - Sf[((U(t)S)^2]. \tag{15}$$

Thus, by the transformation (14) we reduced the equation (5) to (15).

Denote

$$fj(s) = \sup_{t\in R} f[(U(t)S)^2]; \qquad fi(s) = \inf_{t\in R} f[(U(t)S^2]. \tag{16}$$

We are interested in the behaviour for $t \to \pm\infty$ of the solutions of (15) which are close to the zero solution at the initial moment. This behaviour depends on the parameters of (1) and some properties of $f$.

**Theorem 2.** *For the singular point $(x_1, x_2) = (0,0)$ of the system (1), where the conditions (3) are satisfied, the following assertions hold:*

1) *If $\sigma < 0$, then $(0,0)$ is a focus.*

2.1) *If $\sigma = 0$ and $f(w) > 0$ in some interval $0 < w < m$, then $(0,0)$ is a stable focus.*

2.2) *If $\sigma = 0$ and $f(w) < 0$ in some interval $0 < w < m$, then $(0,0)$ is a unstable focus.*

3.1) *If $\sigma > 0$ and $f(w) < 0$ for $w > 0$, then $(0,0)$ is unstable focus.*

3.2) *If $\sigma > 0$ and there exist $\mu > 0$ and $\delta > 0$ such that*

$$0 \leq fj(s) \leq \frac{k}{\pi}\sigma \quad \text{for } 0 < s < \mu,$$

$$fi(s) > \frac{k}{\pi}\sigma \quad \text{for } \mu < s < \mu + \delta,$$

*then there exists a periodic trajectory or a limit cycle.*

The proof of the theorem is analogous to that in the [6]. In the above theorem the problem about the number of the limit cycles and their locations is not solved. The answer of these questions depends on the function $f$.

<u>**Case 1.**</u> $f(x_1^2 + x_2^2) = x_1^2 + x_2^2$ and $d = a > 0$, $bc < 0$.

Then the system (1) takes the form:

$$\begin{aligned} \dot{x}_1 &= ax_1 + bx_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= cx_1 + ax_2 - x_2(x_1^2 + x_2^2), \end{aligned} \tag{17}$$

the class of differential equations (17) has some interesting representations. Thus, the problem of synchronization of an electronic generator [4] results in a system (17).

We will prove for the system (17) that there exists an unique limit cycle. Changing the variables $(x_1, x_2)$ by $(r, \varphi)$ in (17), and integrating the equation for $r(t)$, we get

$$r(t) = \frac{\exp\left(\int \Phi(t)dt\right)}{\sqrt{c + 2\int \exp\left(-2\int \Phi(t)dt\right) dt}}, \tag{18}$$

where $\Phi(t)$ is the function of (6).

For each $t > 0$ there is an unique integer $N(t)$ such that

$$0 \leq t - N(t)\frac{\pi}{k} < \frac{\pi}{k}. \tag{19}$$

Let

$$m = \max_{0 \leq t < \frac{\pi}{k}} \left| \int_0^t \phi(\tau)d\tau \right|. \tag{20}$$

Then we have

$$\int_0^t \Phi(\tau)d\tau = \int_0^{N(t)\frac{\pi}{k}} \Phi(\tau)d\tau + R(t), \tag{21}$$

where

$$R(t) = \int_{N(t)\frac{\pi}{k}}^{t} \Phi(\tau)d\tau = \int_{0}^{t-N(t)\frac{\pi}{k}} \Phi(\tau)d\tau, \tag{22}$$

$\Phi(t)$ being a periodic fuction. Using (19) and (20) we get

$$|R(t)| \leq m. \tag{23}$$

Taking into account (7) and the periodicity of $\Phi(t)$, we obtain from (23)

$$\int_{0}^{t} \Phi(\tau)d\tau = N(t)\sigma + R(t). \tag{24}$$

Let us now consider the positive expression

$$r(t) = \frac{\exp\left(-\int_{0}^{t}\Phi(\tau)d\tau\right)}{\sqrt{r_0^{-2} + 2\int_{0}^{t}\exp\left(-2\int_{0}^{\theta}\Phi(\tau)d\tau\right)d\theta}}. \tag{25}$$

In view of (24), the last formula gets the form

$$r(t) = \frac{\exp(-R(t) - N(t)\sigma)}{\sqrt{r_0^{-2} + 2\int_{0}^{t}\exp[-2(R(\theta) + N(\theta)\sigma]d\theta}}. \tag{26}$$

Then consider the function

$$g(t) = -N(t)\sigma - R(t). \tag{27}$$

**Lemma 1.**

$$\left|g(t) + \frac{k}{\pi}\sigma\right| \leq m - \sigma \tag{28}$$

*for erery t.*

Proof. Applying (7), (21) and (27) this inequality follows directly:

$$\left|g(t) + \frac{k}{\pi}\sigma\right| \leq \left|\left(-N(t) + \frac{k}{\pi}t\right)\sigma - R(t)\right|$$

$$\leq -\frac{k}{\pi}\left[t - N(t)\frac{\pi}{k}\right]\sigma + m \leq m - \sigma.$$

**Lemma 2.** *The following estimations*

$$r_1(t) \leq r(t) \leq r_2(t) \tag{29}$$

*hold, where $r_1(t)$ and $r_2(t)$ are functions such that*

$$\lim_{t\to\infty} r_1(t) = \sqrt{-\frac{k}{\pi}\sigma}\, e^{-2(m-\sigma)} \tag{30}$$

$$\lim_{t\to\infty} r_2(t) = \sqrt{-\frac{k}{\pi}\sigma}\, e^{2(m-\sigma)} \tag{31}$$

*as $t \to \infty$.*

Proof. Using (27) and Lemma 1, for $r(t)$ we obtain the following inequality

$$r(t) = \frac{e^{g(t)}}{\sqrt{r_0^{-2} + 2\int\limits_0^t e^{2g(\theta)}d\theta}} \leq \frac{e^{2(m-\sigma)}}{\sqrt{-\frac{\pi}{k\sigma} + A_1 e^{\frac{2nk}{\pi}\sigma t}}} = r_2(t), \tag{32}$$

where

$$A_1 = \frac{\pi}{k\sigma} + r_0^{-2}e^{2(m-\sigma)}. \tag{33}$$

It is easy to see that

$$r_2(t) \to \sqrt{-\frac{k\sigma}{\pi}}\, e^{2(m-\sigma)}$$

with $t \to \infty$.

Similarly, we find the following lower bound for $r(t)$:

$$r(t) \geq \frac{e^{-2(m-\sigma)}}{\sqrt{-\frac{\pi}{k\sigma} + A_2 e^{\frac{2k}{\pi}\sigma t}}},$$

where

$$A_2 = \frac{\pi}{k\sigma} + r_0^{-2}e^{-2(m-\sigma)}. \tag{34}$$

Let us denote

$$\frac{e^{-2(m-\sigma)}}{\sqrt{-\frac{\pi}{k\sigma} + A_2 e^{\frac{2nk}{\pi}\sigma t}}} = r_1(t).$$

Now,

$$r_1(t) \to \sqrt{-\frac{k\sigma}{\pi}e^{-2(m-\sigma)}}$$

as $t \to \infty$.

Hence we obtain

$$r_1(t) \le r(t) \le r_2(t),$$

where $\lim_{t\to\infty} r_1(t)$ and $\lim_{t\to\infty} r_2(t)$ exists.

From the above considerations it follows, that the limit set $L(C^+)$ is included in the closed set

$$\{r/r_1(\infty) \le r \le r_2(\infty)\}. \tag{35}$$

Thus, according to the Poincare-Bendixson theorem [1] there exists a periodic trajectory or a limit cycle. Now, applying to system (17) the criterion of Dulac for an annular region [2], we obtain that the existing limit cycle is only one.

Summarizing, we have the following theorem.

**Theorem 3.** *Let in the class of quasi-linear autonomous system (17) the conditions*

$$d = a > 0, \quad bc < 0 \tag{35}$$

*are satisfied. Then for the system (17) there exists an unique limit cycle, which is located between two circumferences*

$$r_1 = \sqrt{a}e^{-2(m-\sigma)} \tag{36}$$

*and*

$$r_2 = \sqrt{a}e^{2(m-\sigma)}. \tag{37}$$

*Thus the limit trajectory satisfies the inequalities*

$$ae^{-4(m+a\frac{\pi}{k})} \le x^2 + y^2 \le a^{4(m+a\frac{\pi}{k})}, \quad k = \sqrt{-bc}, \quad bc < 0.$$

Now let us consider the system (1), where

$$a = -d, \quad a^2 + bc < 0, \tag{38}$$

and

$$a = -d = 0, \quad bc < 0, \tag{39}$$

then the origin $(x_1, x_2) = (0,0)$ is a singular point for (1) and it is of center type for the corresponding linear system. It is known (see [2]) that for the system (1), the origin is either a focus or a center. Our purpose is a more detailed study.

Thus, we have the following theorem.

**Theorem 4.** *For the singular point $(x_1, x_2) = (0,0)$ of the system (1) the following assertions hold:*

*4.1) If $d = -a$, $a^2 + bc < 0$ and $f(w) > 0$ in some interval $0 < w < m$, then $(0,0)$ is a stable focus [7].*

*4.2) If $d = -a$, $a^2 + bc < 0$ and $f(w) < 0$ in some interval $0 < w < m$, then $(0,0)$ is unstable focus.*

*4.3) If $d = -a = 0$, $bc < 0$, then the limit set $L(C^+)$ of the system (1) is either an unique point which is the sinqular point for (1), or is a periodical trajectory, [3].*

Case 2. $f(x_1^2 + x_2^2) = \ln(x_1^2 + x_2^2)$, $x_1^2 + x_2^2 > 0$ and $d = -a = 0$, $bc < 0$.

In this case, when the singular point $(0,0)$ is of center type for the linear system corresponding to (1), the canonical form of the system (1) is:

$$\begin{aligned} \dot{x}_1 &= -\beta x_2 - x_1 \ln\left(\frac{x_1^2}{\beta^2} + \frac{x_2^2}{\beta^2}\right) \\ \dot{x}_2 &= -\beta x_1 - x_1 \ln\left(\frac{x_1^2}{\beta^2} + \frac{x_2^2}{\beta^2}\right), \quad \beta \neq 0. \end{aligned} \tag{40}$$

Integrating (40) after changing $(x_1, x_2)$ with $(r, \varphi)$, we get the following form:

$$\begin{aligned} \varphi &= t.\beta \\ r &= \beta e^{\ln \frac{r_0}{\beta} e^{-2t}}, \quad \beta > 0. \end{aligned} \tag{41}$$

From the second formula of (41) we obtain the following results:

a) If $0 < r_0 < \beta$, then $\lim\limits_{t\to-\infty} r(t) = 0$, $\lim\limits_{t\to+\infty} r(t) = \beta$.

b) If $r_0 > \beta$, then $\lim\limits_{t\to-\infty} r(t) = +\infty$, $\lim\limits_{t\to+\infty} r(t) = \beta$.

c) The circumference $r = \beta$ is a phase trajectory too. The trajectories start either from a neighbourhood of the origin $O(0,0)$ or from infinity when $t \to -\infty$. Thus the circumference $r = \beta$ is the limit cycle.

Case 3. $f(x_1^2 + x_2^2) = x_1^2 + x_2^2$, and $a = -d = 0, b = -c = 1$ is considered in [5].

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