Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 13, 1999, Fasc. 3-4

Existence for Landesman-Lazer Boundary Value Problems via Variational Inequalities

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Presented by P. Kenderov

In the present paper we study the existence of solutions to some boundary value problems for noncoercitive semilinear elliptic equations. A variational inequalities approach allows us to obtain in the same way results for cases that usually are treated separately by various methods.

AMS Subj. Classification: 35J25, 35J35

Key Words: boundary value problems, noncoercitive semilinear elliptic equations, variational inequalities

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The present work treats the solvability of some boundary value problems for noncoercitive semilinear elliptic equations. Since the paper of Landesman and Lazer [1] treating the Dirichlet problem for equations of the form

$$Lu + g(x, u) = f(x),$$

where L is selfadjoined elliptic operator of second order with one dimensional kernel, numereous publications sharpening and extendig this result have appeared. Sufficient, and in some cases necessary conditions connect the behaviour of the nonlinear term $g(x, u)$ at infinity, the functions in the kernel of L and the right-hand side f. Such conditions are often called Landesman-Laser type conditions, or even a Landesman-Laser principle [2]. An extensive study of related topics is the article [3] (in particular sections III and IV). Most of the considerations are carried out by splitting argument and in the particular case when 0 is the first eigenvalue of L, it is usually supposed that L is selfadjoint or at least monotone. This however leaves out the case of the Dirichlet problem for general second order elliptic equations, this last case being treated separately, usually by means of sub- and supersolutions.

In the present paper we consider an abstract linear operator L from a subspace V of $L^2(\Omega)$ into its dual, subject to restrictions that hold for a number of elliptic problems (conditions L1 – L3 below). The main existence result (Theorem 2) is for bounded nonlinearities $g(x,u)$ under the assumptions that zero is the first eigenvalue of L and that the right-hand side (in the dual of V) satisfies appropriate condition of Landesman–Lazer type. This includes non-monotone operators with first eigenvalue zero and in particular non selfadjoined second order equations with non-smooth coefficients. The proof is based on the fact that variational inequalities for pseudomonotone operators arrising from the equations under consideration have solution in appropriate convex sets. Assuming that no such solution is a solution to the original equation, we arrive at a contradiction with the imposed Landesman–Lazer type conditions. This approach is in part inspired by [4, Th. 2.3] and is used in our previous work [5]. Theorem 1 represents a surjectivity results for linear operators ot the type considered with trivial kernel and a nonlinear term that has at most sublinear growth. It arrises naturally as a part of the proof of Theorem 2 but seems of interest by itself. Then Theorem 3 is a sharpening for monotone operators of the results thus obtained, related to the results in [3].

Let V be a Hilbert space with a norm $\|\cdot\|_V$ that is everywhere dense in $L^2(\Omega)$ for some bounded region $\Omega \subset R^n$ and with a compact inclusion. Identifying $L^2(\Omega)$ with its dual, we get

$$V \subset L^2(\Omega) \subset V^*, \tag{1}$$

where V^* is the dual of V. In the applications we take for V a closed subspace of $H^m(\Omega)$, such that $H_0^m(\Omega) \subset V$. We denote by $\langle\cdot,\cdot\rangle$ the pairing between the spaces V and V^*.

Let

$$L : V \to V^*$$

be a continuous linear operator satisfying the following properties:

L1. A Gårding type inequality

$$\langle Lu, u\rangle \geq \alpha\|u\|_V^2 - c_0\|u\|^2_{L^2(\Omega)},$$

where $\alpha, c_0 = \text{const}$; $\alpha > 0$. L1 implies that the operator L, as well as its adjoint $L^* : V \to V^*$ have kernels of equal finite dimension.

L2. The operator $L + \lambda$ is invertible for every $\lambda > 0$.

L3.

$$\text{meas}\{w(x) = 0\} = 0 \qquad \forall w \in \ker L\backslash\{0\}.$$

For the function $g(x, u)$ we suppose that it is defined on $\Omega \times R$ and satisfies the Karatheodory condition, i.e. it is measurable with respect to x for every u and is continuous with respect to u for almost all $x \in \Omega$. Furthermore in the sequal we use some or all of the following hypotheses:

H1.

$$ug(x, u) \geq -c(x)|u| - d(x),$$

where $c \in L^2(\Omega)$, $d \in L^1(\Omega)$, $c(x) \geq 0$, $d(x) \geq 0$;

H2.

$$|g(x, u)| \leq \gamma|u| + b(x)$$

for some function $b \in L^2(\Omega)$;

H3. $g(x, u)/u \to 0$ for $|u| \to \infty$ for almost all $x \in \Omega$; or even the next stronger hypothesis:

H*

$$|g(x, u)| \leq b(x)$$

for some function $b \in L^2(\Omega)$ which implies all of them.

Let

$$\overline{g}^+(x) = \limsup_{u \to +\infty} g(x, u), \quad \underline{g}^+(x) = \liminf_{u \to +\infty} g(x, u),$$

$$\overline{g}^-(x) = \limsup_{u \to -\infty} g(x, u), \quad \underline{g}^-(x) = \liminf_{u \to -\infty} g(x, u).$$

As usually, $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$ for arbitrary function u and by $\{u > 0\}$ and $\{u < 0\}$ we denote the sets $\{x \in \Omega : u(x) > 0\}$ and $\{x \in \Omega : u(x) < 0\}$ respectively.

Theorem 1. *Let* L1, L2 *and* H1 - H3 *hold. Furthermore, let* $\ker L = \{0\}$. *Then the equation*

(2) $$Lu + g(x, u) = f$$

has a solution for every $f \in V^*$.

Proof. Let

$$B_R = \{u \in V : ||u||_{L^2} \leq R\}.$$

It is obvious that B_R is convex and closed in V.

Under the assumptions L1 and H2 the operator

$$Au = Lu + g(x, u)$$

is pseudomonotone on V. Indeed,

$$Au = Lu + cu - cu + g(x, u)$$

and the operator $L + c$ is monotone for c sufficiently large positive. On the other hand H2 implies that $-cu + g(x, u)$ is a continuous operator from $L^2(\Omega)$ into $L^2(\Omega)$ and from the compact imbedding $V \subset L^2(\Omega)$, it follows that it is a compact operator from V into V^*, hence it is pseudomonotone. Now A is pseudomonotone as a sum of a monotone and a pseudomonotone operator [6, Ch 2, Remark 2.12].

The variational inequality

$$(3) \qquad \langle Au, v - u\rangle \geq \langle f, v - u\rangle \qquad \forall v \in B_R$$

has a solution for every $f \in V^*$ and every fixed $R > 0$. Indeed, H1 implies for every fixed $R > 0$

$$\begin{aligned} \langle Au, u\rangle &= \langle Lu, u\rangle + \int_\Omega g(x, u)u \\ &\geq \alpha\|u\|_V^2 - c_0\|u\|_{L^2}^2 - \int_\Omega c(x)|u| - \int_\Omega d(x) \\ &\geq \alpha\|u\|_V^2 - c_0\|u\|_{L^2}^2 - \|c\|_{L^2}\|u\|_{L^2} - \|d\|_{L^1} \end{aligned}$$

and obviously,

$$\frac{\langle Au, u\rangle}{\|u\|_V} \to +\infty$$

for $\|u\|_V \to \infty$, $u \in B_R$. Hence the operator A is coercitive on B_R for every fixed $R > 0$. Now we can apply the well known existence results for variational inequalities (cf. [6, Ch. 2, Th. 8.2]) whence it follows that for every R fixed there exists a solution $u_R \in B_R$ to the variational inequality (3), i.e.

$$\langle Au_R, v - u_R\rangle \geq \langle f, v - u_R\rangle \qquad \forall v \in B_R,$$

or equvalently

$$\langle Au_R - f, v - u_R\rangle \geq 0 \qquad \forall v \in B_R$$

holds. As it is well known the last inequality can be interpreted in term of subdifferentials of convex functions (cf. [6, Ch. 2, 8.5, 8.6]), i.e.

$$-(Au_R - f) \in \partial I_{B_R}(u_R),$$

where

$$I_{B_R}(u) = \begin{cases} +\infty & \text{for} \quad \|u\|_{L^2} > R \\ 0 & \text{for} \quad \|u\|_{L^2} \leq R \end{cases}$$

is the indicator function of B_R and ∂ denotes the subdiferential with respect to V. Using the inclusions (1) (cf. also [7, Prop. 5.7]) it is readily seen that

$$\partial I_{B_R}(u) = \{\lambda u : \lambda \geq 0\}$$

whence it follows that

$$-(Lu_R + g(x, u_R) - f) = \lambda_R u_R$$

for some $\lambda_R \geq 0$, or

$$Lu_R + \lambda_R u_R + g(x, u_R) = f. \tag{4}$$

If $\lambda_R = 0$, then u_R is a solution to the equation (2). This is obviously the case whenever $\|u_R\|_{L^2} < R$. So, let us suppose that for no R the function u_R is a solution to the equation (2). This means that for every R there exist u_R with $\|u_R\|_{L^2} = R$ and $\lambda_R > 0$, such that (4) holds. Now it follows that

$$\langle Lu_R, u_R\rangle + \lambda_R R^2 \leq -\int_\Omega g(x, u_R)u_R dx + \langle f, u_R\rangle .$$

Now from L1 and H2 it follows that

$$\alpha\|u_R\|_V^2 - c_0 R^2 + \lambda_R R^2 \leq \|c\|_{L^2} R + C + \|f\|_{V^*}\|u_R\|_V,$$

and since $\|f\|_{V^*}\|u_R\|_V \leq \varepsilon\|f\|_{V^*}^2 + (4\varepsilon)^{-1}\|u_R\|_V^2$ we obtain

$$(\alpha - \varepsilon)\|u_R\|_V^2 + (\lambda_R - c_0)R^2 \leq C_\varepsilon(R + 1)$$

for every $R > 0$ for some constant C_ε. Hence for $\varepsilon < \alpha$ it follows

$$(\lambda_R - c_0)R^2 \leq C_\varepsilon(R + 1)$$

for every $R > 0$ and

$$\limsup_R (\lambda_R - c_0) \leq 0.$$

This means that the set of λ_R is bounded, i.e.

$$0 < \lambda_R \leq C \qquad \forall R > 0. \tag{5}$$

Now we get

$$(\alpha - \varepsilon)\|u_R\|_V^2 \leq C(R^2 + R + 1) \tag{6}$$

for a suitable constant C. Let now

$$w_R = \frac{u_R}{R}.$$

We have

$$\|w_R\|_{L^2} = 1 \tag{7}$$

and from (6) now it follows that

$$\|w_R\|_V \leq C \tag{8}$$

for some constant C. From (5), (8) and the compactness of the inclusion $V \subset L^2(\Omega)$ it follows that we can choose a sequence $R_\nu \to \infty$, such that for the corresponding sequences $\lambda_\nu = \lambda_{R_\nu}$ and $w_\nu = w_{R_\nu}$ we have

$$\lambda_\nu \to \overline{\lambda} \geq 0 \tag{9}$$

$$w_\nu \rightharpoonup w \qquad \text{weakly in } V \tag{10}$$

$$w_\nu \to w \qquad \text{strogly in } L^2(\Omega) \tag{11}$$

$$w_\nu(x) \to w(x) \qquad \text{almost everywhere in } \Omega. \tag{12}$$

Without any loss of generality, we can also suppose that there is a $k \in L^2(\Omega)$ such that

$$|w_\nu(x)| \leq k(x) \qquad \text{almost everywhere in } \Omega.$$

From the equation (4) it follows that for every $v \in V$ we have

$$\langle Lw_\nu, v\rangle + \lambda_\nu \int_\Omega w_\nu v dx + \int_\Omega \frac{g(x, u_\nu)}{R_\nu} v dx = \frac{1}{R_\nu} \langle f, v\rangle . \tag{13}$$

From (9) – (11) it follows

$$\langle Lw_\nu, v\rangle \to \langle Lw, v\rangle \tag{14}$$

$$\lambda_\nu \int_\Omega w_\nu v dx \to \overline{\lambda} \int_\Omega w v dx \tag{15}$$

$$\frac{1}{R_\nu} \langle f, v\rangle \to 0. \tag{16}$$

From H2 it follows

$$\left\| \frac{g(x, u_\nu)}{R_\nu} \right\|_{L^2} \leq \left\| \frac{\gamma |u_\nu| + b}{R_\nu} \right\|_{L^2} \leq \frac{\gamma}{R_\nu} \|u_\nu\|_{L^2} + \frac{\|b\|_{L^2}}{R_\nu} \leq C.$$

On the other hand H3 implies that

$$\frac{g(x, R_\nu w_\nu(x))}{R_\nu} \to 0$$

for almost all $x \in \Omega$. Indeed, for x such that $w(x) \neq 0$ we have

$$\frac{g(x, R_\nu w_\nu(x))}{R_\nu} = \frac{g(x, R_\nu w_\nu(x))}{R_\nu w_\nu(x)} w_\nu(x) \to 0$$

and on the set $w(x) = 0$ we have

$$\left|\frac{g(x, R_\nu w_\nu(x))}{R_\nu}\right| \leq \gamma |w_\nu(x)| + \frac{b(x)}{R_\nu} \to 0.$$

This implies that

$$\frac{g(x, u_\nu(x))}{R_\nu} \to 0$$

weakly in $L^2(\Omega)$ (cf. [6, Ch. 1, Lemma 1.3], i.e.

$$\int_\Omega \frac{g(x, u_\nu(x))}{R_\nu} v dx \to 0 \tag{17}$$

holds for every v. Now (13) - (16) and (17) imply

$$\langle Lw + \overline{\lambda} w, v \rangle = 0 \qquad \forall v \in V,$$

i.e.

$$Lw + \overline{\lambda} w = 0.$$

Since (7) and (11) imply $\|w\|_{L^2} = 1$ and $\overline{\lambda} \geq 0$, from L2 it follows that $\overline{\lambda} = 0$, i.e.

$$w \in \ker L \backslash \{0\} \tag{18}$$

and this contradicts the assumption that $\ker L = \{0\}$.

Theorem 2. *Let* L1 - L3 *and* H* *hold. Let $f \in V^*$ be such that for every $w \in \ker L \backslash \{0\}$ there exists a $W \in \ker L^*$ satisfying*

$$\int_\Omega wW dx > 0 \tag{19}$$

and

$$\langle f, W \rangle < \int_{\{w>0\}} (\underline{g}^+ W^+ - \overline{g}^+ W^-) dx + \int_{\{w<0\}} (\underline{g}^- W^+ - \overline{g}^- W^-) dx. \tag{20}$$

Then the equation

$$Lu + g(x, u) = f$$

has a solution.

P r o o f. We repeat verbatim all the arguments in the proof of Theorem 1 until obtaining (18). Now for every $W \in \ker L^*$ we have

$$\langle f, W\rangle = \lambda_\nu R_\nu \langle w_\nu, W\rangle + \int_\Omega g(x, u_\nu) W dx.$$

Let us now take for W this corresponding to w element in $\ker L^*$ satisfying (19) and (20). From (19) and (11) it follows that $\langle w_\nu, W\rangle > 0$ for all sufficiently large ν, whence

$$\liminf_{\nu\to\infty} \lambda_\nu R_\nu \langle w_\nu, W\rangle \geq 0.$$

Furthermore, H* implies

$$g(x, u_\nu)W \geq -b(x)|W(x)| \in L^1(\Omega)$$

whence by Fatou's lemma

$$\liminf_{\nu\to\infty} \int_\Omega g(x, u_\nu) W dx \geq \int_\Omega \liminf_{\nu\to\infty} g(x, u_\nu) W dx.$$

These last inequalities imply

$$\begin{aligned}
\liminf_{\nu\to\infty} \langle f, W\rangle &= \liminf_{\nu\to\infty} \left\{ \lambda_\nu R_\nu \langle w_\nu, W\rangle + \int_\Omega g(x, u_\nu) W dx \right\} \\
&\geq \liminf_{\nu\to\infty} \lambda_\nu R_\nu \langle w_\nu, W\rangle + \liminf_{\nu\to\infty} \int_\Omega g(x, u_\nu) W dx \\
&\geq \liminf_{\nu\to\infty} \int_\Omega g(x, u_\nu) W dx.
\end{aligned}$$

We obtain on the set $\{W > 0\}$

$$\liminf_{\nu\to\infty} g(x, u_\nu) W = W \liminf_{\nu\to\infty} g(x, u_\nu) = W\{\underline{g}^+ \chi_{\{w>0\}} + \underline{g}^- \chi_{\{w<0\}}\}$$

and on the set $\{W > 0\}$

$$\begin{aligned}
\liminf_{\nu\to\infty} g(x, u_\nu) W &= -W \liminf_{\nu\to\infty} (-g(x, u_\nu)) = -W(-\limsup_{\nu\to\infty} g(x, u_\nu)) \\
&= W \limsup_{\nu\to\infty} g(x, u_\nu) = W\{\overline{g}^+ \chi_{\{w>0\}} + \overline{g}^- \chi_{\{w<0\}}\},
\end{aligned}$$

where χ denotes the charactristic function of the corresponding set. By virtue of L3 the last two inequalities give

$$\liminf_{\nu\to\infty} g(x, u_\nu) W = \chi_{\{w>0\}}[W^+ \underline{g}^+ - W^- \overline{g}^+] + \chi_{\{w<0\}}[W^+ \underline{g}^- - W^- \overline{g}^-]$$

almost everywhere in Ω and finally

$$\langle f, W\rangle \geq \int_{\{w>0\}} [W^+ \underline{g}^+ - W^- \overline{g}^+] + \int_{\{w<0\}} [W^+ \underline{g}^- - W^- \overline{g}^-]$$

which contradicts the hypothesis (20) of the theorem.

Remark 1. Let $\dim \ker L = 1$. Then $\dim \ker L^* = 1$ also. Let w and W be two elements in $\ker L$ and $\ker L^*$ respectively, such that

$$\int_\Omega wW dx > 0.$$

Then the condition (20) takes the form

$$\int_{\{w>0\}} (\overline{g}^- W^+ - \underline{g}^- W^-) dx + \int_{\{w<0\}} (\overline{g}^+ W^+ - \underline{g}^+ W^-) dx < \langle f, W\rangle$$

$$< \int_{\{w>0\}} (\underline{g}^+ W^+ - \overline{g}^+ W^-) dx + \int_{\{w<0\}} (\underline{g}^- W^+ - \overline{g}^- W^-) dx.$$

In the particular case

$$g^{\pm} = \lim_{u \to \pm\infty} g(u)$$

the condition becomes

$$g^- \int_{\{w>0\}} W dx + g^+ \int_{\{w<0\}} W dx < \langle f, W\rangle < g^+ \int_{\{w>0\}} W dx + g^- \int_{\{w<0\}} W dx.$$

If there is a function $W \in \ker L^*$, such that

$$\int_{\{w>0\}} W dx > \int_{\{w<0\}} W dx,$$

then it is enough to have $g^+ > g^-$ in order to ensure the existence for some $f \in V^*$.

Remark 2. Let L be a second order elliptic operator. As it is well known, under appropriate regularity assumptions, the eigenfunctions corresponding to the first eigenvalue of the Dirichlet problem for L and L^* have constant sign in Ω. If we denote them by w and W respectively and take both positive, then (19) is automaticaly satisfied and (20) becomes

$$\int_\Omega \overline{g}^- W dx < \langle f, W\rangle < \int_\Omega \underline{g}^+ W dx.$$

Similar arguments apply to some Neuman problems, when the kernels consist of constant functions only.

Remark 3. Let L be a monotone operator, i.e. $\langle Lu, u\rangle \geq 0\ \forall u \in V$. Then

$$\ker L = \ker L^*$$

and for a function W corresponding to $w \in Ker L\backslash\{0\}$ we can choose w itself. In this case (19) is satisfied and (20) becomes

$$\langle f, w\rangle < \int_{\{w>0\}} \underline{g}^+ w dx + \int_{\{w<0\}} \overline{g}^- w dx \qquad \forall w \in \ker L\backslash\{0\}. \tag{21}$$

Remark 4. The condition H* was used only to assure the hypothesis of the Fatou's lemma. In particular cases it can be replaced by less stringent conditions. For instance in the case of monotone operator L we have the following result.

Theorem 3. *Let the monotone operator L be such that* L1 *holds. Let $g(x, u)$ satisfy* H1 *and* H2. *Then for every $f \in V^*$, such that (21) holds the equation*

$$Lu + g(x, u) = f$$

has at least one solution.

P r o o f. The proof is identical with that of Theorem 1 until obtaining (14) - (16). Now H1 and H2 imply (cf. [3, (3.24)])

$$\begin{aligned} ug(x,u) &= ug(x,u) + c|u| + d - c|u| - d \\ &= |ug(x,u) + c|u| + d| - c|u| - d \\ &\geq |u|\,|g(x,u)| - 2c|u| - d \\ &\geq \frac{1}{\gamma}|g(x,u)|(|g(x,u)| - b(x)) - 2c|u| - d \\ &\geq \frac{1}{\gamma'}|g(x,u)|^2 - 2c|u| - \tilde{d}, \end{aligned}$$

where $\gamma' < \gamma$ and $\tilde{d} \in L^1(\Omega)$. Now from (4) it follows

$$\langle Lu_\nu, u_\nu\rangle + \lambda_\nu R_\nu^2 + \frac{1}{\gamma'}\int_\Omega |g(x,u_\nu)|^2 dx - 2\int_\Omega c|u_\nu| dx - \int_\Omega \tilde{d}(x) dx$$

$$\leq \langle Lu_\nu, u_\nu\rangle + \lambda_\nu R_\nu^2 + \int_\Omega u_\nu g(x,u_\nu) dx = \langle f, u_\nu\rangle$$

and since $\langle Lu_\nu, u_\nu \rangle \geq 0$ and $\lambda_\nu > 0$ we get

$$\frac{1}{\gamma'}\int_\Omega |g(x,u_\nu)|^2 dx \leq \|f\|_{V^*}\|u_\nu\|_V + 2\|c\|_{L^2(\Omega)}\|u_\nu\|_{L^2(\Omega)} + \|\tilde{d}\|_{L^1(\Omega)}.$$

Dividing with R_ν^2 gives

$$\frac{1}{\gamma'}\int_\Omega \frac{|g(x,u_\nu)|^2}{R_\nu^2} dx \leq \frac{1}{R_\nu}(C\|f\|_{V^*} + 2\|c\|_{L^2(\Omega)} + \frac{\|\tilde{d}\|_{L^1(\Omega)}}{R_\nu}) \to 0$$

for $\nu \to \infty$ so instead of (17) we now have

$$\left\| \frac{g(x,u_\nu)}{R_\nu} \right\|_{L^2} \to 0.$$

This again gives

$$Lw + \overline{\lambda} w = 0.$$

Now $\|w\|_{L^2} = 1$ and the fact that L is monotone imply $\overline{\lambda} = 0$ and $w \in \ker L \backslash \{0\}$. The proof continues as in [3]. In more details, Proposition II.4 implies that under the assumptions H1 and H2 we have

$$\liminf_{\nu \to \infty} \int_\Omega w_\nu g(x,u_\nu) dx \geq \int_{\{w>0\}} \underline{g}^+ w dx + \int_{\{w<0\}} \overline{g}^- w dx$$

the right-hand side being a number in $(-\infty, +\infty]$. From (4) we obtain

$$\langle Lu_\nu, w_\nu \rangle + \lambda_\nu R_\nu \int_\Omega w_\nu^2 dx + \int_\Omega w_\nu g(x,u_\nu) dx = \langle f, w_\nu \rangle$$

whence dropping the nonnegative terms we obtain

$$\langle f, w \rangle = \liminf_{\nu \to \infty} \langle f, w_\nu \rangle \geq \liminf_{\nu \to \infty} \int_\Omega w_\nu g(x,u_\nu) dx$$

$$\geq \int_{\{w>0\}} \underline{g}^+ w dx + \int_{\{w<0\}} \overline{g}^- w dx,$$

in contrary to the asummptions (21).

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Received: 02.03.1998