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Integrability for a Nonlinear Electrodynamical Model of Three Conservation Laws [1]

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In this paper existence of simple states for a quasilinear hyperbolic system of PDEs describing the Josephson's effect in the quantum electronics is studied. A pure geometrical approach for finding simple states is discussed.

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1. Introduction

We deal with the following quasilinear system of PDEs

$$(1)\qquad \left|\begin{aligned} u_x &= -Cv_t - I_0(v)\sin w + [G_0(v) + G_1(v)\cos w]v \\ v_x &= -Lu_t \\ w_t &= (2\pi/\Phi_0)v, \end{aligned}\right.$$

where x is the space, t the time. In the physical sense the dependents u, v, w are defined as follow: u is the tunnel superconducting current, v is the electrical potential on the barrier, and w is the relative phase difference between the macroscopic quantum wave functions for two superconductors. The real functions $G_0(v)$ and $G_1(v)$ are sufficiently smooth in $\mathbf{R}$. Of physical point of view $G_0(v)$ and $G_1(v)$ mean the tunnel current and the interference current of one quasi-particle respectively. By $I_0 = I_0(v)$ we mean the critical Josephson's current which depends in the real case slightly on v. Therefore I_0 can be taken equal to constant (see Parmentier [6]). By C, L, Φ_0 denote the following real

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constants: C-capacitance, L-inductance, $\Phi_0 = h/2e$ is the magnetic flux quantum, where h is the Planck's constant, a nd e is the electron charge. This system consists of three Kirchhoff conservation laws concerning the Josephson's effect in the quantum electronics. Some similar physical effects arising in this model were discussed in [6] and [8]. In the existing literature it is customary to see the reduction of the considered problem into one dimensional sin-Gordon equation

$$(C\Phi_0/2\pi)w_{tt} - (\Phi_0/2\pi L)w_{xx} = -I_0 \sin w,$$

with the present parameter I_0 and $G_0 = G_1 = 0$. During the last decades this class of equations was studied by a number of authors applying the well known inverse scattering method. It is easy to see that the variables u and v take no place in the sin$-$Gordone equation that motivates us to pay attention to the system (1). Hyperbolic systems similar to the system under consideration were investigated by many mathematicians (cf. [1] and the references therein). An algebraic approach for solving conservation low systems was proposed by Grundland (cf. [2]) but for most general cases where the coefficients depend on t, x, u the Grundland's approach can no longer be applied. In this paper we apply a pure geometric method proposed by Tabov in [9] (see also the papers [4], [5], [10]).

In Section 2 we give the well known Grundland's lemma that is the necessary and sufficient condition for existence of simple states to the system under consideration.

In Section 3 we modify the approach represented in [9] where a two-dimensional case was considered.

2. Preliminaries

In this section we list the assumptions and recall the known existence result due to Grundland which is needed to develop our main result. Here we will modify slightly the method represented in [9] since it is specialized in solving the hyperbolic systems containing two equations. In the present case the system contains three equations and three dependents u, v, w. By analogy with [9] we look for two suitable vector fields η_1 and η_2 represented in the coordinate basis $\{\partial_t, \partial_x, \partial_y, \partial_u, \partial_v, \partial_w\}$, where $\partial_a \equiv \partial/\partial a$, and y stands for one of the partial derivative of u, v or w with respect to t and x. By these vector fields we determine a two dimensional distribution which is characterized with the special property involutivity. This means that the considered problem can be reduced to an involutive linear homogeneous system of PDEs,

$$\eta_1\Phi = 0\,, \quad \eta_2\Phi = 0, \tag{2}$$

where $\Phi = \Phi(z)$ $(z \in \mathbf{R}^6)$ is unknown scalar function. In the case when (2) is involutive taking into account the classical Frobenius theory we can find four functionally independent solutions Φ_k $(k = 1,2,3,4)$ of (2) so that from the system

$$\Phi^k(z) = c_k \ (k = 1,2,3,4) \tag{3}$$

by the classical implicit function theorem we derive the existence and uniqueness of the simple state, that is the solution constructed by Riemann invariants (see Jeffrey [3]).

Lemma 1. (Grundland [2]) *The system (1) possesses Riemann invariants if and only if the gradients* $\{\nabla u, \nabla v, \nabla w\}$ *are collinear.*

The Grundland's lemma can be interpreted in the following way.

Lemma 1′. *The system (1) possesses Riemann invariants if and only if two of the following three equalities hold*

$$\begin{aligned} u_t v_x - u_x v_t &= 0 \\ u_t w_x - u_x w_t &= 0 \\ v_t w_x - v_x w_t &= 0. \end{aligned} \tag{4}$$

3. Involutivity in $\mathbf{R}^6$

Let us combine the system (1) with the conditions (4) (see Lemma 1'), that is

$$\left| \begin{aligned} u_x &= -Cv_t + F(v,w) \\ v_x &= -Lu_t \\ w_t &= Mv \\ u_t v_x - u_x v_t &= 0 \\ u_t w_x - u_x w_t &= 0, \end{aligned} \right. \tag{5}$$

where $F(v,w) \equiv -I_0(v)\sin w + [G_0(v) + G_1(v)\cos w]v$, $M \equiv 2\pi/\Phi_0$. *Using similar arguments as in [9] we state the following proposition.*

Proposition 1. *The vector function (u,v,w) is a solution of the system (1), constructed by means of Riemann invariants, if and only if (u,v,w) is a solution of the system (5).*

The above statement is a generalization of Theorem 1 in [9].

The system (5) is overdetermined since it consists of five equations, and the number of unknown functions equals three.

Let us write (5) in the form

$$(6) \qquad \left|\begin{array}{l} u_t = -L^{-1} v_x \\ v_t = -C^{-1}[F(v,w) - u_x] \\ w_t = Mv \\ u_t v_x - u_x v_t = 0 \\ u_t w_x - u_x w_t = 0. \end{array}\right.$$

Now using standard arguments as in [9, 10], we can express v_x by u_x, v and w. For this purpose we use the latter two equations of (6) yielding the following algebraic equation

$$(7) \qquad v_x^2 = (L/C)(y^2 - yF),$$

where we have set $y \equiv u_x$. Solving (7) we get

$$(8) \qquad v_x = \epsilon\sqrt{(L/C)[y^2 - yF]}$$

($\epsilon = +1$ or -1), i.e. $v_x = K(y,v,w)$, where $K(y,v,w)$ stands for the right-hand side of (8). Hence we obtain

$$(9) \qquad \left|\begin{array}{l} u_t = -L^{-1} K \\ v_t = -C^{-1}[F(v,w) - u_x] \\ w_t = Mv. \end{array}\right.$$

For (9) we fix the initial condition

$$u(0,x) = u^0(x) \in J_u, \quad v(0,x) = v^0(x) \in J_v,$$
$$w(0,x) = w^0(x) \in J_w, \quad y(0,x) = y^0(x) \in J_y, \ \text{for} \ x \in \mathbf{R}^1,$$

where J_u, J_v, J_w, J_y are open subintervals of $\mathbf{R}^1$. Next we solve (9) under the initial condition defined above. For this purpose we introduce the following assumption.

H.

(i) $F \in C^1(J_v \times J_w)$;

(ii) $(L/C)[r^2 - rF(p,q)] > 0$ where $LC > 0$, (r,p,q) belongs to the bounded domain $\Omega \subset \mathbf{R}^1 \times \mathbf{R}^1 \times \mathbf{R}^1$ defined as follows:

$$\Omega \equiv \{(r,p,q) : p \in J_v, \ q \in J_w, \ r \in J_y\};$$

(iii) $r[rK_r(r,p,q) - K(r,p,q)] \neq 0$ for $(r,p,q) \in \Omega$.

In what follows it is assumed that H holds. Next we give a statement which we need in the main result to prove the existence and uniqueness of the simple state for (1).

Proposition 2. *There exists three dimensional distribution $\theta(z)$ in $\mathbf{R}^6$.*

Proof. A most general approach for solving of (5) is based on the representing of the considered system by Pfaff differential forms defined on $\mathbf{R}^6$. Thus the system (9) reduces to the following Pfaff differential system

$$(10)\qquad \left|\begin{aligned} &\omega^1 \equiv du + (K/L)dt - ydx = 0\\ &\omega^2 \equiv dv - (F-y)C^{-1}dt - Kdx = 0\\ &\omega^3 \equiv dw - Mvdt - CMvK/(F-y)dx = 0. \end{aligned}\right.$$

Since the differential forms in (10) are linearly independent, we can define a codistribution $\Lambda(z)$ by the linear hull of the set $\{\omega^i\}_{i=1}^3$; $z \equiv (t,x,y,u,v,w) \in \mathbf{R}^6$. Further one can choose the differential forms

$$\begin{aligned} \tilde{\omega}^1 &= \omega^1\\ \tilde{\omega}^2 &= CMv\omega^2 - (F-y)\omega^3\\ \tilde{\omega}^3 &= K\omega^1 - y\omega^2, \end{aligned}$$

belonging to $\Lambda(z)$, thus the system

$$(11)\qquad \left|\begin{aligned} &\tilde{\omega}^1 \equiv du + (K/L)dt - ydx = 0\\ &\tilde{\omega}^2 \equiv CMvdv - (F-y)dw = 0\\ &\tilde{\omega}^3 \equiv Kdu - ydv = 0 \end{aligned}\right.$$

is equivalent to the previous. It is easy to see that the vector fields

$$\begin{aligned} \xi_1 &= (0,0,1,0,0,0)\\ \xi_2 &= (0,K,0,yK,K^2,-LMyv)\\ \xi_3 &= (yL,K,0,0,0,0) \end{aligned}$$

solve (11) in the sense that

$$\tilde{\omega}^i(\xi_j) = 0 \ \ (i = 1,2,3;\ \ j = 1,2,3).$$

Hence there exists a three-dimensional distribution $\theta(z)$ defined as the linear hull of ξ_i $(i = 1,2,3)$ since these vector fields are linearly independent. ■

The following statement is a generalization of those shown in [9].

Proposition 3. *For $k = 1,2,3$ there exists exactly one (up to a scalar multiplier) vector field $\eta = \tilde{\eta}_k$, satisfying the system*

$$\omega^k(\eta) = 0, \quad \partial\omega^k(\xi_j,\eta) = 0 \quad (k = 1,2,3;\ j = 1,2,3). \tag{12}$$

Moreover, the rank of the system of vector fields $\{\tilde{\eta}_k\}$ $(k = 1,2,3)$ is equal two.

Proof. We look for the unknown vector fields η solving the systems (12). In order to specify the above vector fields $\tilde{\eta}_k$ $(k = 1,2,3,4)$, we can use the known classical formula

$$\partial\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]), \tag{13}$$

where X, Y are some vector fields, $[X,Y]$ stands for their comutator, and ω is a differential form. Next we define the following differential two-forms

$$\partial\omega^1 = dy \wedge du^1 - dK \wedge du^3, \quad \partial\omega^2 = d(u^2 f_1/u^1) \wedge dt - d(f_1/u^1) \wedge dx,$$

$$\partial\omega^3 = d(yu^2) \wedge dt - dy \wedge dx, \quad \partial\omega^4 = d(u^2 f_3) \wedge dt - df_3 \wedge dx,$$

and the commutators

$$\begin{aligned}
[\xi_1,\xi_2] &= K_y\partial_x + (yK_y + K)\partial_u + 2KK_y\partial_v - LMv\partial_w \\
&= (0, K_y, 0, yK_y + K, 2KK_y, -LMv) = \partial_y\xi_2, \\
[\xi_1,\xi_3] &= L\partial_t + K_y\partial_x = (L, K_y, 0,0,0,0) = \partial_y\xi_3, \\
[\xi_2,\xi_3] &= (K^2K_v - LMyvK_w)\partial_x \\
&= (0, K^2K_v - LMyvK_w, 0,0,0,0) = (K^2\partial_v - LMyv\partial_w)\xi_3.
\end{aligned}$$

Thus, if we express η_k $(k = 1,2,3)$ by $\eta_k = a_k^1\xi_1 + a_k^2\xi_2 + a_k^3\xi_3$ (for all $k = 1,2,3$), therefore the coefficients a_k^j $(k = 1,2,3;\ j = 1,2,3)$ staying before the basis vector fields ξ_i $(i = 1,2,3)$ can be calculated by the commutators defined above. Hence,

$$\begin{aligned}
\tilde{\eta}_1 &= -y(K^2K_v - LMyvK_w)\xi_1 + (yK_y - K)\xi_2 + K\xi_3 \\
&= (LyK,\ yKK_y,\ -y(K^2K_v - LMyvK_w),\ yK_y - K, \\
&\qquad K^2(yK_y - K),\ LMyv(yK_y - K)), \\
\tilde{\eta}_2 &= \xi_3 = (yL, K, 0,0,0,0), \\
\tilde{\eta}_3 &= \xi_3 = (yL, K, 0,0,0,0).
\end{aligned}$$

It is easily seen that the above stated set of three vector fields $\tilde{\eta}_k$ $(k = 1,2,3)$ has rank two. This complete the proof. ∎

Proposition 4. *There exists a two dimensional involutive subdistribution $\theta_1(z)$ of $\theta(z)$ resolving the differential system (11).*

Proof. Using Proposition 3, it easily follows that there exists the basis $\{\bar{\eta}_1, \bar{\eta}_2\}$ determined by the vector fields $\bar{\eta}_1 = \tilde{\eta}_1$ and $\bar{\eta}_2 = \tilde{\eta}_2$ whose linear hull forms a two dimensional subdistribution $\theta_1(z)$ of $\theta(z)$. On the other hand we get

$$[\bar{\eta}_1, \bar{\eta}_2] = -(K^2K_v - LMyvK_w)(yL, K, 0, 0, 0, 0) = A(y, v, w)\bar{\eta}_2,$$

where $A(y, v, w) \equiv -(K^2K_v - LMyvK_w)$. Consequently, θ_1 is involutive which completes the proof. ∎

4. The main result

Our main result is based on the following statement.

Theorem 1. *The system (9) possesses a unique simple state if and only if there exists two linearly independent involutive vector fields belonging to $\theta(z)$.*

This is a particular case of the general Pfaff problem (cf. Schouten, van der Kulk [7]). By this statement the considered initial problem reduces to the problem of finding suitable vector fields belonging to $\theta(z)$ such that their linear hull defines an involutive subdistribution $\theta_1(z)$ of $\theta(z)$, i.e. $\theta_1(z) \subset \theta(z)$, and such that by the system (2) we can get a solution of (9) hence (1) will be solved as well.

Proof. Proposition 2 suggests us that in order to solve the system (10), it suffices to find all two dimensional involutive subdistributions of $\theta(z)$. For that purpose we fix some $x_0 \neq 0$. Thus we have the initial data

$$y(0, x^0) = y^{00} \neq 0,\ u(0, x^0) = u^{00},\ \ v(0, x^0) = v^{00},\ \ w(0, x^0) = w^{00}.$$

Having in mind Proposition 4, it follows that $\theta_1(z)$ is involutive, and according to Frobenius' theorem, it is completely integrable, i.e. the system of linear first order partial differential equations

$$(14)\quad \left|\begin{array}{l} \bar{\eta}_1\ \Phi \equiv LyK)\Phi_t + yKK_y\Phi_x - y(K^2K_v - LMyvK_w)\Phi_y \\ +yK(yK_y - K)\Phi_u + K^2(yK_y - K)\Phi_v - LMyv(yK_y - K)\Phi_w = 0 \\ \bar{\eta}_2\ \Phi \equiv yL\Phi_t + K\Phi_x = 0, \end{array}\right.$$

has four functionally independent solutions. They can be obtained by solving the system

$$(15)\qquad \begin{aligned} dx/dt &= K/Ly \\ dy/dt &= (-K^2K_v + LMyvK_w)/L(yK_y - K) \\ du/dt &= K/L \\ dv/dt &= [y - F(v,w)]C^{-1} \\ dw/dt &= Mv \end{aligned}$$

under the same initial condition given above. Hence we obtain four integrals in the implicit form

$$(16)\qquad \Phi_j(t,x,y,u,v,w) = \Phi_j(0,x^0,y^{00},u^{00},v^{00},w^{00}) \ (j = 1,2,3,4).$$

The system (16) yields four implicit functions $y = y(t,x)$, $u = u(t,x)$, $v = v(t,x)$, $w = w(t,x)$ defined in some neighbourhood of the point $(0,x^0)$. Thus we conclude that $(y(t,x),u(t,x),v(t,x),w(t,x))$ is the only solution for the system (9) satisfying the initial condition given above. Consequently, the only simple state to (1) exists in the implicit form

$$u = u(t,x), \quad v = v(t,x), \quad w = w(t,x),$$

and $u(0,x) = u^0(x)$, $v(0,x) = v^0(x)$, $w(0,x) = w^0(x)$. This completes the proof. ■

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