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Evaluation of Some Integrals Involving Hermite Polynomials

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In this paper we evaluate some integrals involving Hermite polynomials, using the method of the generating functions. Various particular cases are obtained.

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1. Introduction

Recently, a growing interest in "new" classes of polynomials generalizing the ordinary Hermite polynomials has been shown [1,2,8,9]. Their importance has been recognized in different contexts, such as the theory of multivariate Bessel functions [3], quantum and classical treatment of phase-space evolution [4], entangled oscillators [10,11], mixed states of light [7], etc.

There are many different forms of the generalized Hermite polynomials [5], among them we have

$$e^{xu+yv+zu^2+wv^2-kuv} = \sum_{m,n=0}^{+\infty} \frac{u^m v^n}{m!n!} H^{(2)}_{m,n}(x,z,y,w,k), \tag{1}$$

where the 5-variable polynomials $H^{(2)}_{m,n}(x,z,y,w,k)$ are explicitly given by

$$H^{(2)}_{m,n}(x,z,y,w,k) = \sum_{q=0}^{\min(m,n)} (-1)^q q! \binom{m}{q}\binom{n}{q} k^q\, H_{m-q}(x,z) H_{n-q}(y,w), \tag{2}$$

while the $H_n(x,y)$ belong to the Kampé de Fériet family and are specified by the generating function [5],

$$e^{xt+yt^2} = \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n(x,y). \tag{3}$$

The following identity holds:

$$(4) \qquad H^{(2)}_{m,n}\left(x, -\frac{1}{2}, x, -\frac{1}{2}, 0\right) = H_{m+n}(x),$$

where $H_m(x)$ are the ordinary Hermite polynomials derived from the generating function

$$(5) \qquad e^{xt-\frac{1}{2}t^2} = \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n(x).$$

In this paper we evaluate some integrals involving the Hermite polynomials using the method of the generating function. Various particular cases are obtained.

2. Integrals

In this section, we evaluate three different types of integrals involving Hermite polynomials.

(i) Integral of the form $\int_{-\infty}^{+\infty} \cos(ky) e^{-\frac{\alpha}{2}(y-a)^2} H_m(y) H_n(y)\, dy, \quad \alpha > 0$

Let,

$$(6) \qquad F_{m,n}(a,k) = \int_{-\infty}^{+\infty} \cos(ky)\, e^{-\frac{\alpha}{2}(y-a)^2}\, H_m(y)\, H_n(y) dy.$$

In order to evaluate (6) we multiply both sides by $\frac{t^m}{m!} \frac{r^n}{n!}$, then summing we obtain,

$$(7) \quad \sum_{m,n=0}^{+\infty} \frac{t^m}{m!} \frac{r^n}{n!} F_{m,n}(a,k) = e^{-\frac{1}{2}(t^2+r^2)} \int_{-\infty}^{+\infty} \cos(ky)\, e^{-\frac{\alpha}{2}(y-a)^2}\, e^{(t+r)y} dy$$

by using (5).

This result can be written as,

$$(8) \quad \sum_{m,n=0}^{+\infty} \frac{t^m}{m!} \frac{r^n}{n!} F_{m,n}(a,k) = e^{-\frac{1}{2}(t^2+r^2)} \mathrm{Re} \left\{ \int_{-\infty}^{+\infty} e^{iky}\, e^{-\frac{\alpha}{2}(y-a)^2}\, e^{(t+r)y} dy \right\},$$

and making a change of variable, we get

$$(9) \qquad \sum_{m,n=0}^{+\infty} \frac{t^m}{m!} \frac{r^n}{n!} F_{m,n}(a,k) = e^{-\frac{1}{2}(t^2+r^2)+(t+r)a}$$

$$\times\ \mathrm{Re} \left\{ e^{ika} \int_{-\infty}^{+\infty} e^{-\left[\frac{\alpha}{2}w^2 - (ik+t+r)w\right]}\, dw \right\}.$$

Evaluating the Gaussian w-integral of (9), we get

$$\sum_{m,n=0}^{+\infty} \frac{t^m}{m!}\frac{r^n}{n!} F_{m,n}(a,k) = \sqrt{\frac{2\pi}{\alpha}}\, e^{-k^2/2\alpha} \tag{10}$$

$$\times \operatorname{Re}\left\{ e^{ika} e^{(a+i\frac{k}{\alpha})t+(a+i\frac{k}{\alpha})r+\frac{t^2}{2}(\frac{1}{\alpha}-1)+\frac{r^2}{2}(\frac{1}{\alpha}-1)+\frac{1}{\alpha}rt} \right\}.$$

Then (1) leads to,

$$\sum_{m,n=0}^{+\infty} \frac{t^m}{m!}\frac{r^n}{n!} F_{m,n}(a,k) = \sqrt{\frac{2\pi}{\alpha}}\, e^{-k^2/2\alpha} \tag{11}$$

$$\times \operatorname{Re}\left\{ e^{ika} \sum_{m,n=0}^{+\infty} \frac{t^m}{m!}\frac{r^n}{n!} H^{(2)}_{m,n}\left(a+i\frac{k}{\alpha}, \frac{1}{2}\left(\frac{1}{\alpha}-1\right), a+i\frac{k}{\alpha}, \frac{1}{2}\left(\frac{1}{\alpha}-1\right), -\frac{1}{\alpha}\right)\right\}.$$

Thus, finally we obtain the integral

$$\int_{-\infty}^{+\infty} \cos(ky)\, e^{-\frac{\alpha}{2}(y-a)^2}\, H_m(y)H_n(y)dy = \sqrt{\frac{2\pi}{\alpha}} e^{-k^2/2\alpha} \tag{12}$$

$$\times \operatorname{Re}\left\{ e^{ika}\, H^{(2)}_{m,n}\left(a+i\frac{k}{\alpha}, \frac{1}{2}\left(\frac{1}{\alpha}-1\right), a+i\frac{k}{\alpha}, \frac{1}{2}\left(\frac{1}{\alpha}-1\right), -\frac{1}{\alpha}\right)\right\}, \quad \alpha > 0.$$

(ii) Integral of the form

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{cos(kx+hy)\, e^{-\frac{\alpha}{2}x^2} H_m(x,y) e^{-\frac{\beta}{2}y^2}\, dx\, dy}, \quad \alpha > 0, \beta > 0$$

Let,

$$F_m(h,k) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cos(kx+hy)\, e^{-\frac{\alpha}{2}x^2} H_m(x,y) e^{-\frac{\beta}{2}y^2}\, dx\, dy. \tag{13}$$

If we multiply both side of (13) by $\frac{t^m}{m!}$, then summing and using (3), we get

$$\sum_{m=0}^{+\infty} \frac{t^m}{m!} F_m(h,k) = \operatorname{Re}\left\{ \int_{-\infty}^{+\infty} e^{-[\frac{\beta}{2}y^2-y(t^2+ih)]} \left[\int_{-\infty}^{+\infty} e^{-[\frac{\alpha}{2}x^2-x(t+ik)]}dx\right] dy \right\} \tag{14}$$

evaluating the inner integral, we have

$$(15)\quad \sum_{m=0}^{+\infty} \frac{t^m}{m!} F_m(h,k) = \sqrt{\frac{2\pi}{\alpha}} \mathrm{Re}\left\{ e^{\frac{1}{2\alpha}(t^2+2itk-k^2)} \int_{-\infty}^{+\infty} e^{-\left[\frac{\beta}{2}y^2 - y(t^2+ih)\right]} \right\} dy,$$

that is,

$$(16)\quad \sum_{m=0}^{+\infty} \frac{t^m}{m!} F_m(h,k) = \frac{2\pi}{\sqrt{\alpha\beta}} e^{-\frac{k^2}{2\alpha}-\frac{h^2}{2\beta}} \mathrm{Re}\left(e^{\frac{ik}{\alpha}t+\frac{1}{2\alpha}t^2} e^{\frac{ih}{\beta}t^2+\frac{1}{2\beta}t^4} \right)$$

using (3) again,

$$\sum_{m=0}^{+\infty} \frac{t^m}{m!} F_m(h,k) = \frac{2\pi}{\sqrt{\alpha\beta}} e^{-\frac{k^2}{2\alpha}-\frac{h^2}{2\beta}} \mathrm{Re}\left\{ \sum_{j,n=0}^{+\infty} \frac{t^{2j+n}}{j!n!} H_n\left(\frac{ik}{\alpha}, \frac{1}{2\alpha}\right) H_j\left(\frac{ih}{\beta}, \frac{1}{2\beta}\right) \right\}$$
(17)

making a change of index,

$$\sum_{m=0}^{+\infty} \frac{t^m}{m!} F_m(h,k) = \frac{2\pi}{\sqrt{\alpha\beta}} e^{-\frac{k^2}{2\alpha}-\frac{h^2}{2\beta}} \sum_{m,j=0}^{+\infty} \frac{t^m}{j!\,(m-2j)!}$$

$$(18)\quad \times \mathrm{Re}\left[H_j\left(\frac{ih}{\beta}, \frac{1}{2\beta}\right) H_{m-2j}\left(\frac{ik}{\alpha}, \frac{1}{2\alpha}\right) \right],$$

from (13) and (18), we establish that

$$(19)\quad \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cos(kx+hy) e^{-\frac{\alpha}{2}x^2} H_m(x,y) e^{-\frac{\beta}{2}y^2}\, dx\, dy$$

$$= \frac{2\pi}{\sqrt{\alpha\beta}} e^{-\frac{k^2}{2\alpha}-\frac{h^2}{2\beta}} \sum_{j=0}^{+\infty} \frac{m!}{j!\,(m-2j)!} \mathrm{Re}\left[H_j\left(\frac{ih}{\beta}, \frac{1}{2\beta}\right) H_{m-2j}\left(\frac{ik}{\alpha}, \frac{1}{2\alpha}\right) \right].$$

The result (16) may be written in the following forms:

$$\sum_{m=0}^{+\infty} \frac{t^m}{m!} F_m(h,k) = \frac{2\pi}{\sqrt{\alpha\beta}} e^{-\frac{k^2}{2\alpha}-\frac{h^2}{2\beta}} \sum_{m,j=0}^{+\infty} \frac{t^m}{j!\,(m-2j)!}$$

$$(20)\quad \times \mathrm{Re}\left[H_j\left(\frac{1}{2\alpha}, \frac{1}{2\beta}\right) H_{m-2j}\left(\frac{ik}{\alpha}, \frac{ih}{\beta}\right) \right]$$

and

$$\sum_{m=0}^{+\infty} \frac{t^m}{m!} F_m(h,k) = \frac{2\pi}{\sqrt{\alpha\beta}} e^{-\frac{k^2}{2\alpha}-\frac{h^2}{2\beta}} \sum_{m,j=0}^{+\infty} \frac{t^m}{j!(m-2j)!}$$

$$\times \operatorname{Re}\left[\left(\frac{ik}{\alpha}\right)^{m-2j} H_j\left(\frac{1}{2\alpha}+\frac{ih}{\beta},\frac{1}{2\beta}\right)\right] \tag{21}$$

therefore, from (13) and (20)-(21), we obtain respectively

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cos(kx+hy)e^{-\frac{\alpha}{2}x^2} H_m(x,y)e^{-\frac{\beta}{2}y^2}\,dx\,dy \tag{22}$$

$$= \frac{2\pi}{\sqrt{\alpha\beta}}\, e^{-\frac{k^2}{2\alpha}-\frac{h^2}{2\beta}} \sum_{j=0}^{+\infty} \frac{m!}{j!\,(m-2j)!} H_j\left(\frac{1}{2\alpha},\frac{1}{2\beta}\right) \operatorname{Re}\left[H_{m-2j}\left(\frac{ik}{\alpha},\frac{ih}{\beta}\right)\right],$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cos(kx+hy)e^{-\frac{\alpha}{2}x^2} H_m(x,y)e^{-\frac{\beta}{2}y^2}\,dx\,dy \tag{23}$$

$$= \frac{2\pi}{\sqrt{\alpha\beta}}\, e^{-\frac{k^2}{2\alpha}-\frac{h^2}{2\beta}} \sum_{j=0}^{+\infty} \frac{m!}{j!\,(m-2j)!} \operatorname{Re}\left[\left(\frac{ik}{\alpha}\right)^{m-2j} H_j\left(\frac{1}{2\alpha}+\frac{ih}{\beta},\frac{1}{2\beta}\right)\right],$$

$$\alpha > 0, \quad \beta > 0.$$

(iii) Integral of the form

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{e}^{-\frac{\alpha}{2}\mathbf{x}^2} \cos(\mathbf{hy})\mathbf{e}^{-\frac{\beta}{2}\mathbf{y}^2}\mathbf{H}_n(\mathbf{x},\mathbf{y})\mathbf{H}_m(\mathbf{x},\mathbf{y})\,\mathbf{dx}\,\mathbf{dy}, \quad \alpha > 0, \beta > 0$$

Let,

$$F_{m,n}(h) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{\alpha}{2}x^2} \cos(hy)e^{-\frac{\beta}{2}y^2} H_n(x,y)H_m(x,y)\,dx\,dy \tag{24}$$

by a similiar procedure to the previous sections we have

$$\sum_{m,n=0}^{+\infty} \frac{t^m}{m!}\frac{u^n}{n!}F_{m,n}(h) = \frac{2\pi}{\sqrt{\alpha\beta}}\, e^{-\frac{h^2}{2\beta}} \operatorname{Re}\left[e^{\left(\frac{1}{2\alpha}+\frac{ih}{\beta}\right)t^2+\left(\frac{1}{2\alpha}+\frac{ih}{\beta}\right)u^2+\frac{tu}{\alpha}}\; e^{\frac{t^4}{2\beta}+\frac{u^4}{2\beta}+\frac{t^2u^2}{\beta}}\right],$$
(25)

which using (1) equals to

$$\sum_{m,n=0}^{+\infty} \frac{t^m}{m!}\frac{u^n}{n!}F_{m,n}(h) = \frac{2\pi}{\sqrt{\alpha\beta}}\, e^{-\frac{h^2}{2\beta}} \sum_{m,n,k,j=0}^{+\infty} \frac{t^m}{(m-2k)!k!}\frac{u^n}{(n-2j)!j!} \tag{26}$$

$$\times \operatorname{Re}\left[H^{(2)}_{m-2k,n-2j}\left(0,\frac{1}{2\alpha}+\frac{ih}{\beta},0,\frac{1}{2\alpha}+\frac{ih}{\beta},-\frac{1}{\alpha}\right)H^{(2)}_{k,j}\left(0,\frac{1}{2\beta},0,\frac{1}{2\beta},-\frac{1}{\beta}\right)\right],$$

hence

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{\alpha}{2}x^2}\cos(hy)e^{-\frac{\beta}{2}y^2}H_n(x,y)H_m(x,y)\,dx\,dy = \frac{2\pi}{\sqrt{\alpha\beta}}\,e^{-\frac{h^2}{2\beta}}$$

$$\times \sum_{k,j=0}^{+\infty}\frac{m!}{(m-2k)!k!}\frac{n!}{(n-2j)!j!}\operatorname{Re}\left[H^{(2)}_{m-2k,n-2j}\left(0,\frac{1}{2\alpha}+\frac{ih}{\beta},0,\frac{1}{2\alpha}+\frac{ih}{\beta},-\frac{1}{\alpha}\right)\right.$$

$$\left.\times H^{(2)}_{k,j}\left(0,\frac{1}{2\beta},0,\frac{1}{2\beta},-\frac{1}{\beta}\right)\right],\quad \alpha>0,\beta>0. \tag{27}$$

3. Particular cases

For $a=0$, (12) reduces to

$$\int_{-\infty}^{+\infty}\cos(ky)e^{-\frac{\alpha}{2}y^2}H_m(y)\,H_n(y)\,dy = \sqrt{\frac{2\pi}{\alpha}}\,e^{-\frac{k^2}{2\alpha}}$$

$$\times \operatorname{Re}\left\{H^{(2)}_{m,n}\left(\frac{ik}{\alpha},\frac{1}{2}\left(\frac{1}{\alpha}-1\right),\frac{ik}{\alpha},\frac{1}{2}\left(\frac{1}{\alpha}-1\right),-\frac{1}{\alpha}\right)\right\},\quad \alpha>0. \tag{28}$$

Further, let $\alpha=1$, we have

$$\int_{-\infty}^{+\infty}\cos(ky)e^{-y^2/2}\,H_m(y)\,H_n(y)\,dy = \sqrt{2\pi}\,e^{-\frac{k^2}{2}}$$

$$\times \operatorname{Re}\left\{H^{(2)}_{m,n}(ik,0,ik,0,-1)\right\}, \tag{29}$$

and in view of the result

$$e^{x(u+v)-uv} = \sum_{m,n=0}^{+\infty}\frac{u^m}{m!}\frac{v^n}{n!}\,H_{m,n}(x) \tag{30}$$

and the equation (1), we get

$$\int_{-\infty}^{+\infty}\cos(ky)e^{-y^2/2}\,H_m(y)\,H_n(y)\,dy$$

$$= \begin{cases}\sqrt{2\pi}\,e^{-k^2/2}(-1)^{\frac{m+n}{2}}H_{m,n}(k) & \text{if } m+n=\text{even},\\ 0 & \text{if } m+n=\text{odd}.\end{cases} \tag{31}$$

This result has been given recently by Dattoli and Torre [6].

If $k=0$ in (12), (19), (22) and (23) we obtain respectively:

$$\int_{-\infty}^{+\infty} e^{-\frac{\alpha}{2}(y-a)^2} H_m(y)H_n(y)dy = \sqrt{\frac{2\pi}{\alpha}} \tag{32}$$

$$\times H_{m,n}^{(2)}\left(a,\frac{1}{2}\left(\frac{1}{\alpha}-1\right),a,\frac{1}{2}\left(\frac{1}{\alpha}-1\right),-\frac{1}{\alpha}\right), \quad \alpha>0.$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{\alpha}{2}x^2} H_m(x,y)\cos(hy)\, e^{-\frac{\beta}{2}y^2}dx\, dy = \frac{2\pi}{\sqrt{\alpha\beta}} e^{-\frac{h^2}{2\beta}} \tag{33}$$

$$\times \sum_{j=0}^{+\infty} \frac{m!}{(m-2j)!j!}\mathrm{Re}\left[H_{m-2j}\left(0,\frac{1}{2\alpha}\right)H_j\left(\frac{ih}{\beta},\frac{1}{2\beta}\right)\right], \quad \alpha>0,\beta>0.$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{\alpha}{2}x^2} H_m(x,y)\cos(hy)\, e^{-\frac{\beta}{2}y^2}dx\, dy = \frac{2\pi}{\sqrt{\alpha\beta}} e^{-\frac{h^2}{2\beta}} \tag{34}$$

$$\times \sum_{j=0}^{+\infty} \frac{m!}{j!(m-2j)!}H_j\left(\frac{1}{2\alpha},\frac{1}{2\beta}\right)\mathrm{Re}\left[H_{m-2j}\left(0,\frac{ih}{\beta}\right)\right], \quad \alpha>0,\beta>0.$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{\alpha}{2}x^2} H_m(x,y)\cos(hy)\, e^{-\frac{\beta}{2}y^2}dx\, dy \tag{35}$$

$$=\begin{cases} \frac{2\pi}{\sqrt{\alpha\beta}} e^{-\frac{h^2}{2\beta}} \frac{m!}{\Gamma(\frac{m}{2}+1)}\mathrm{Re}\left[H_{\frac{m}{2}}\left(\frac{1}{2\alpha}+\frac{ih}{\beta},\frac{1}{2\beta}\right)\right] & \text{if } m=\text{even}, \\ 0 & \text{if } m=\text{odd}, \end{cases} \quad \alpha>0,\beta>0.$$

For $h=0$, (35) and (27) reduce to

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{\alpha}{2}x^2-\frac{\beta}{2}y^2} H_m(x,y)\, dx\, dy \tag{36}$$

$$=\begin{cases} \frac{2\pi}{\sqrt{\alpha\beta}} \frac{m!}{\Gamma(\frac{m}{2}+1)} H_{\frac{m}{2}}\left(\frac{1}{2\alpha},\frac{1}{2\beta}\right) & \text{if } m=\text{even}, \\ 0 & \text{if } m=\text{odd}, \end{cases} \quad \alpha>0,\beta>0.$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{\alpha}{2}x^2-\frac{\beta}{2}y^2} H_n(x,y)H_m(x,y)\, dx\, dy$$

$$=\frac{2\pi}{\sqrt{\alpha\beta}}\sum_{k,j=0}^{+\infty}\frac{m!}{(m-2k)!k!}\frac{n!}{(n-2j)!j!}\left[H_{m-2k,n-2j}^{(2)}\left(0,\frac{1}{2\alpha},0,\frac{1}{2\alpha},-\frac{1}{\alpha}\right)\right.$$

$$\times \; H^{(2)}_{k,j}\left(0,\frac{1}{2\beta},0,\frac{1}{2\beta},-\frac{1}{\beta}\right)\Bigg], \quad \alpha > 0, \beta > 0. \tag{37}$$

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