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Moment Preserving Approximations

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We discuss the problem of approximation of a given function f by elements from a linear subspace span $\{\psi_j\}$ of $L_2[a,b]$ of fixed dimension which preserves maximal number of moments of f with respect to another preassigned system of functions $\{\varphi_k\}$.

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1. Introduction

Let $[a,b]$ be a fixed interval and assume that $\mu(x)$ is a given weight function on $[a,b]$. Denote by $L_2[a,b]$ the space of all square integrable functions on $[a,b]$ with a weight μ and equipped with the norm

$$\|f\| := \left(\int_a^b \mu(x)f^2(x)\,dx\right)^{\frac{1}{2}}.$$

Let $\varphi_0(x), \varphi_1(x), \ldots$ be a given system of functions from $L_2[a,b]$. To any function $f \in L_2[a,b]$ let us put in correspondence the sequence of its *moments*

$$\mu_k(f) := \int_a^b \mu(x)f(x)\varphi_k(x)\,dx, \quad k = 0, 1, \ldots .$$

We consider here the problem of approximation of f on the basis of a finite data of moments $\{\mu_k\}_{k=0}^n$. The classical L_2-approximation of f by *generalized polynomials* $P(x) = a_0\varphi_0(x) + \cdots + a_n\varphi_n(x)$ is evidently equivalent to the construction of P that has the same moments as f. Indeed, the necessary and sufficient conditions for the polynomial $P(x) = a_0\varphi_0(x) + \cdots + a_n\varphi_n(x)$ to minimize

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$$\|f - \sum_{j=0}^{n} \alpha_j \varphi_j(x)\|$$

over all $\alpha_0, \alpha_1, \ldots, \alpha_n$ are

$$\int_a^b \mu(x)\Big[f(x) - \sum_{j=0}^{n} a_j \varphi_j(x)\Big]\varphi_k(x)\,dx = 0 \quad \text{for } k = 0, \ldots, n,$$

and clearly, this system of equations is equivalent to

$$\int_a^b \mu(x)\Big[a_0\varphi_0(x) + \cdots + a_n\varphi_n(x)\Big]\varphi_k(x)\,dx = \mu_k(f), \quad k = 0, \ldots, n,$$

i.e., to $\mu_k(P) = \mu_k(f)$, $k = 0, \ldots, n$. Thus the moment preserving approximation comes from L_2 minimization problems.

There are several important extensions of this classical question. One of them, which arose in the recent years in connection with the optimal recovery of functions, can be formulated roughly in the following way: Given a class $\mathcal{F} \subset L_2[a, b]$, construct a method of the form

$$f(x) \approx \sum_{k=0}^{n} \mu_k(f) c_k(x)$$

with some $\{c_k(x)\}$ which has a minimal error in $\mathcal{F}$. In case $\{\varphi_k\}$ is the trigonometric system $\{\cos kx, \sin kx\}$ such optimal methods have been constructed explicitly in [1] for $\mathcal{F} = W_p^r$ (where W_p^r is the Sobolev class of r times differentiable 2π-periodic functions with locally bounded rth derivative in L_p). In this case $\{\mu_k(f)\}$ are simply the Fourier coefficients of f and thus the problem reduces to reconstruction of functions from W_p^r on the basis of a finite data of their Fourier coefficients. A general problem of best recovery of functions in a Hilbert space on the basis of coefficients of expansions along a given orthogonal system was studied in [5].

We go here in another direction. Assume that $\psi_0(x), \ldots \psi_n(x)$ is another system of functions in $L_2[a, b]$. The problem is to construct an approximation

$$\sigma(f; x) := b_0\psi_0(x) + \cdots + b_n\psi_n(x)$$

which preserves the moments of f with respect to the preassigned system $\varphi_0(x)$, $\ldots$, $\varphi_n(x)$. Precisely, we want to find $b_0, \ldots, b_n$ such that

$$(1) \quad \int_a^b \mu(x)\Big[b_0\psi_0(x) + \cdots + b_n\psi_n(x)\Big]\varphi_k(x)\,dx = \mu_k(f) \quad \text{for } k = 0, \ldots, n.$$

Clearly, this system has a unique solution if and only if

$$\det \left\{(\psi_i, \varphi_k)\right\}_{i,k=0}^{n} \neq 0, \tag{2}$$

where we have used the common notation

$$(f,g) := \int_a^b \mu(x)f(x)g(x)\,dx$$

for the inner product.

The construction of $\sigma(f;x)$ becomes extremely simple when the system $\psi_0,\ldots,\psi_n$ is *biorthogonal* to $\varphi_0,\ldots,\varphi_n$, that is, when

$$(\psi_j, \varphi_k) = \delta_{jk} \quad \text{for all} \quad j,k \in \{0,1,\ldots,n\}.$$

Then,

$$\det \left\{(\psi_i, \varphi_k)\right\}_{i,k=0}^{n} = 1$$

and the unique solution is obviously given by

$$\sigma(f;x) = \sum_{j=0}^{n} \mu_j(f)\psi_j(x). \tag{3}$$

Remark. In case $\varphi_0,\ldots,\varphi_n$ is the algebraic system $\{x^j\}$ and $\psi_0,\ldots,\psi_n$ is a system of splines of given degree r, the problem was studied in [4], [9], [8], [6], [7]. However, the simple construction based on the biorthogonality was not noticed there. Actually, all the quoted papers studied an optimal choice of the knots of the splines $\psi_0,\ldots,\psi_n$ which leads to an interesting extremal problem to which we shall return later.

Similarly, one could start from the L_p-minimization problem

$$\min_{\alpha_0,\ldots,\alpha_n} \int_a^b \mu(x)\left|f(x) - \sum_{k=0}^{n} \alpha_j\varphi_j(x)\right|^p dx$$

and consider approximations of the form $P(x) = b_0\psi_0(x) + \cdots + b_n\psi_n(x)$ that have the same L_p-generalized "moments", i.e., which satisfy the conditions

$$\int_a^b \mu(x)|f(x) - P(x)|^{p-2}(f(x) - P(x))\varphi_k(x)\,dx = 0, \quad k = 0,\ldots,n,$$

for a fixed p, $1 \leq p < \infty$.

2. Extremal problem

Let $\varphi_0, \varphi_1, \ldots$ be a preassigned sequence of functions in $L_2[a,b]$. We saw in the introduction that for every choice of $\{\psi_k\}_{k=0}^{n}$ satisfying (2), there exists a unique moment preserving approximation to f. Assume now that Ω is a collection of different systems $\{\psi_k\}_{k=0}^n$, all of them satisfying (2). Then, given f, what is the best choice of $\{\psi_k\}_{k=0}^n$ in Ω, in the sense that maximal number of moments $\{\mu_j\}$ are preserved? In order to answer this question, we first note the following simple characterization.

Lemma 1. *Let $n \geq 0$, $m > 0$ and $f \in L_2[a,b]$. Assume that the systems $\{\varphi_k\}_{k=0}^{n+m}$ and $\{\psi_k\}_{k=0}^{n+m}$ are biorthogonal in $[a,b]$. Then there exists an approximation $P \in$ span $\{\psi_k\}_{k=0}^n$ to f such that*

$$\mu_k(P) = \mu_k(f) \quad \text{for} \quad k = 0, \dots, n+m, \tag{4}$$

if and only if f is orthogonal to $\varphi_{n+1}(x), \dots, \varphi_{n+m}(x)$.

Proof. Indeed, as we mentioned already, the moment preserving approximation P is uniquely determined and it is given by

$$P(x) = \sum_{j=0}^{n+m} \mu_j(f)\psi_j(x).$$

If f is orthogonal to $\varphi_{n+1}(x), \dots, \varphi_{n+m}(x)$, then $\mu_j(f) = 0$ for $j = n+1, \dots, n+m$ and therefore P belongs to span $\{\psi_k\}_{k=0}^n$. The inverse statement, that the moment preserving implies the orthogonality of f to φ_j for $j = n+1, \dots, n+m$ is obvious. Indeed, let $P(x) = b_0\psi_0(x) + \cdots + b_n\psi_n(x)$ preserve the moments of f with respect to the system $\{\varphi_k\}_{k=0}^{n+m}$. Then

$$(f, \varphi_j) = \mu_j(f) = \mu_j(P) = (P, \varphi_j) = \sum_{k=0}^{n} b_k(\psi_k, \varphi_j) = 0$$

for $j = n+1, \dots, n+m$. The lemma is proved. ∎

Let us remark here that the approximation $P(x)$ is unique.

Lemma 1 suggests a way of constructing such an approximation. First, let us note that if two systems $\{\hat{\varphi}_k\}_{k=0}^{n+m}$ and $\{\psi_k\}_{k=0}^{n+m}$ are biorthogonal in $[a,b]$, then each of them is linearly independent. Next, having any system $\{\varphi_k\}_{k=0}^{n+m}$ of linearly independent functions, we build another system $\hat{\varphi}_0, \dots \hat{\varphi}_{n+m}$ of functions

$$\hat{\varphi}_k(x) = \sum_{j=0}^{n+m} c_{kj}\varphi_j(x)$$

such that

$$(\hat{\varphi}_k, \psi_j) = \delta_{kj} \quad \text{for all } k, j \in \{0, \dots, n+m\}.$$

Clearly the constants c_{kj} are uniquely determined. We have

$$\hat{\mu}_k(f) := \int_a^b \mu(x) f(x) \hat{\varphi}_k(x)\, dx = \sum_{j=0}^{n+m} c_{kj}\mu_j(f).$$

Since the system $\{\hat{\varphi}_k\}_0^{n+m}$ is biorthogonal to $\{\psi_k\}_0^{n+m}\}$ then, by Lemma 1, the approximation

$$\hat{\sigma}(f;x) = \sum_{j=0}^{n} \hat{\mu}_j(f)\psi_j(x)$$

will preserve the moments $\{\hat{\mu}_j(f)\}_0^{n+m}$ if and only if $\hat{\mu}_j(f) = 0$ for $j = n+1,\ldots,n+m$. But if $\hat{\sigma}$ preserves $\{\hat{\mu}_j(f)\}_0^{n+m}$, it will preserve $\{\mu_j(f)\}_0^{n+m}$ as well. Indeed, since the functions $\{\hat{\varphi}_i(x)\}_0^{n+m}$ are biorthogonal to $\{\psi_j(f)\}_0^{n+m}$, they are linearly independent and thus they constitute a basis in span $\{\varphi_j(f)\}_0^{n+m}$. Therefore the conditions $\hat{\mu}_j(f) = \hat{\mu}_j(\hat{\sigma})$, which are equivalent to

$$\int_a^b (f(x) - \hat{\sigma}(x))\hat{\varphi}_j(x)\,dx = 0, \quad j = 0,\ldots,n+m,$$

show that $f - \hat{\sigma}$ is orthogonal to every function from span $\{\varphi_j(f)\}_0^{n+m}$ and particularly to $\varphi_0,\ldots,\varphi_{n+m}$, that is,

$$\int_a^b (f(x) - \hat{\sigma}(x))\varphi_j(x)\,dx = 0, \quad j = 0,\ldots,n+m,$$

which is actually $\mu_j(\hat{\sigma}) = \mu_j(f)$. Our claim is proved. ■

Thus we came to the following conclusion.

Theorem 1. *Let $\varphi_0,\ldots,\varphi_{n+m}$ be a given system of linearly independent functions in $L_2[a,b]$ and let $\psi_0,\ldots,\psi_n$ be functions from $L_2[a,b]$. Given $f \in L_2[a,b]$, there exists an approximation $b_0\psi_0(x)+\ldots+b_n\psi_n(x)$ to f which preserves all moments $\mu_0(f),\ldots,\mu_{n+m}(f)$ of f with respect to $\{\varphi_k\}_0^{n+m}$, if there exists an extension*

$$\psi_0,\ldots,\psi_n,\psi_{n+1},\ldots,\psi_{n+m}$$

of the system $\psi_0,\ldots,\psi_n$ such that f is orthogonal to the functions $\hat{\varphi}_{n+1},\ldots\hat{\varphi}_{n+m}$ from the corresponding system $\hat{\varphi}_0,\ldots\hat{\varphi}_{n+m}$ that is biorthogonal to $\psi_0,\ldots,\psi_{n+m}$. Moreover, the wanted approximation is given by

$$\hat{\mu}_0(f)\psi_0(x) + \cdots + \hat{\mu}_n(f)\psi_n(x).$$

Equipped with this tool, one may study the following extremal problem in various interesting situations.

The extremal problem: *Given a family Ω of systems $\psi_\lambda := \{\psi_1(\lambda;x),\ldots,\psi_{2n}(\lambda;x)\}$ depending of n parameters $\lambda := (\lambda_1,\ldots\lambda_n)$ and a function f, find a system ψ_{λ^*} such that the moment preserving approximation*

$$\sigma(x) = b_1\psi_1(\lambda^*;x) + \cdots + b_n\psi_n(\lambda^*;x)$$

to f with respect to a preassigned sequence $\varphi_1,\ldots,\varphi_{2n}$ has a "double precision", that is, satisfies the conditions

$$\mu_j(\sigma) = \mu_j(f) \quad \text{for } j = 1, \ldots, 2n.$$

The formulated problem resembles the famous problem of Gauss about the existence of a quadrature formula of double algebraic degree of precision. As we shall see later, there is a relation between these two problems.

3. Total positivity and moment preserving

We consider in this section our extremal problem in spaces defined by *totally positive kernels*. Let us first recall some definitions and notations.

We say that the function $K(x,t)$ is a *totally positive kernel* on $[a,b]\times[c,d]$, if

$$K\begin{bmatrix} x_1 & \cdots & x_n \\ t_1 & \cdots & t_n \end{bmatrix} := \det\ \{K(x_i,t_j)\}_{i=1,j=1}^{n\ \ n} \geq 0 \tag{5}$$

for each choice of the natural number n, the points $x_1 < \cdots < x_n$ from $[a,b]$ and $t_1 < \cdots < t_n$ from $[c,d]$. In case of strict inequality $K(x,t)$ is called *strictly totally positive kernel.*

One very important consequence from the strict total positivity of $K(x,t)$ is that the set of functions $K(x_1,t),\ldots,K(x_n,t)$ form a Tchebycheff system on $[c,d]$ for each $x_1 < \cdots < x_n$ from $[a,b]$. This means that any non-zero generalized polynomial

$$P(t) := a_1 K(x_1,t) + \cdots + a_n K(x_n,t)$$

has at most $n-1$ distinct zeros in $[c,d]$. Equivalently, the Lagrange interpolation problem

$$a_1 K(x_1,t_j) + \cdots + a_n K(x_n,t_j) = f_j, \quad j = 1,\ldots,n,$$

has a unique solution $a_1,\ldots,a_n$ for any fixed data $\{f_1,\ldots f_n\}$ and nodes $t_1 < \cdots < t_n$ in $[c,d]$.

Clearly the determinant (5) corresponds to the Lagrange interpolation problem. There are kernels $K(x,t)$ which have the property to preserve the sign of the determinant corresponding to the more general Hermite interpolation problem. Such kernels are called *extended totally positive* (abbreviated to ETP). In order to give the precise definition, we shall use the same notation

$$K\begin{bmatrix} x_1 & \cdots & x_n \\ t_1 & \cdots & t_n \end{bmatrix} \tag{6}$$

for sets of points $\{x_i\},\{t_j\}$ which are not necessarily distinct. for example, if $x_{i-1} < x_i = \cdots = x_{i+p} < x_{i+p+1}$ we interpret (6) as a determinant in which $K(x_{i+m},t)$ is replaced by

$$\frac{\partial^m}{\partial x^m} K(x,t)\Big|_{x=x_i}$$

for $m = 0, \ldots, p$. Similarly, if $t_{i-1} < t_i = \cdots = t_{i+p} < x_{i+p+1}$ then $K(x, t_{i+m})$ is replaced by

$$\frac{\partial^m}{\partial t^m} K(x,t)\Big|_{t=t_i}.$$

Definition 1. The function $K(x,t)$ is called *extended totally positive kernel on* $[a,b] \times [c,d]$, if

$$K\begin{bmatrix} x_1 & \cdots & x_n \\ t_1 & \cdots & t_n \end{bmatrix} > 0$$

for each choice of the points $x_1 \le \cdots \le x_n$ from $[a,b]$ and $t_1 \le \cdots \le t_n$ from $[c,d]$.

More about the totally positive kernels can be found in the book of Karlin [10].

Assume in what follows that $\varphi_1(x), \ldots, \varphi_M(x)$ are square integrable functions which form a Tchebycheff system on $[a,b]$ for any specified M and $K(x,t)$ is an extended totally positive kernel on $[a,b] \times [c,d]$.

Theorem 2. *Let $\nu_1, \ldots, \nu_n$ be any given ordered set of odd natural numbers and $N := \nu_1 + \cdots + \nu_n$. For every integrable function $g(t) \ge 0$ on $[c,d]$ there exists a unique set of points $c \le t_1 < \cdots < t_n \le d$ such that the linear combination of N functions*

$$\sigma(x) = \sum_{i=1}^{n} \sum_{j=0}^{\nu_i - 1} b_{ij} \frac{\partial^j}{\partial t^j} K(x, t_i)$$

preserves $N + n$ moments of f,

$$f(x) := \int_c^d K(x,t) g(t)\, dt,$$

with respect to $\varphi_1(x), \ldots, \varphi_{N+n}(x)$, i.e., $\mu_j(\sigma) = \mu_j(f)$ for $j = 1, \ldots, N+n$. Moreover, the points $\{t_i\}$ coincide with the nodes and the numbers $\{b_{ij}\}$ coincide with the coefficients of the Gaussian quadrature formula with weight $g(t)$

$$\int_c^d g(t) F(t)\, dt \approx \sum_{i=1}^{n} \sum_{j=0}^{\nu_i - 1} b_{ij} F^{(j)}(t_i)$$

of type $(\nu_1, \ldots, \nu_n)$ for the extended Tchebycheff system

$$\Big\{ \int_a^b K(x,t) \varphi_j(x)\, dx, \quad j = 1, \ldots, N+n \Big\}.$$

There is no element from the linear span of $\mathcal{K}_N:=\{\frac{\partial^j}{\partial t^j}K(x,t_i)\}_{i=1,j=0}^{n\ \ \nu_i-1}$ *which preserves more than* $N+n$ *moments of* f.

Proof. We shall use Theorem 1 to construct the extremal moment preserving approximation. In order to do this, for each choice of the set $t_1,\dots<t_n$ in $[c,d]$ we extend the system

$$\left\{\frac{\partial^j}{\partial t^j}K(x,t_i),\ i=1,\dots,n,\ j=0,\dots,\nu_i-1\right\}$$

to

$$\{\psi_j(x)\}_1^{N+n}=\left\{\frac{\partial^j}{\partial t^j}K(x,t_i),\ i=1,\dots,n,\ j=0,\dots,\nu_i\right\}.$$

Since $K(x,t)$ is ETP-kernel, $\{\psi_j(x)\}_1^{N+n}$ form an extended Tchebycheff system on $[a,b]$ (see [11]). Following the instructions from Theorem 1, let us now construct the system of functions $\{\hat{\varphi}_j\}_1^{N+n}$ from $\Phi:=\operatorname{span}\{\varphi_j\}_1^{N+n}$ which are biorthogonal to $\{\psi_j\}_1^{N+n}$. Note first that the mapping

$$\varphi\to\int_a^b K(x,t)\varphi(x)\,dx$$

defines one-to-one correspondence between the elements of Φ and the space of functions

$$K_\Phi:=\left\{\int_a^b K(x,t)\varphi(x)\,dx:\varphi\in\Phi\right\}.$$

Given i,j and the points $t_1<\dots<t_n$ in $[a,b]$, there exists a unique generalized polynomial $P_{ij}(t)\in K_\Phi$ satisfying the Hermite interpolation conditions

$$P_{ij}^{(\lambda)}(t_k)=\delta_{ik}\delta_{j\lambda}\quad\text{for all}\ \ k\in\{1,\dots,n\}\ \ \text{and}\ \ \lambda\in 0,\dots,v_k.$$

We claim that the functions $\{\hat{\varphi}_k\}_1^{N+n}$ (denoted for convenience by $\hat{\varphi}_{ij}$, $i=1,\dots,n$, $j=0,\dots,\nu_i$) are defined by the equalities

$$P_{ij}(t)=\int_a^b K(x,t)\hat{\varphi}_{ij}(x)\,dx\quad\text{for}\ \ i=1,\dots,n,\ j=0,\dots,\nu_i.$$

Set for simplicity

$$P_k(t):=P_{k,\nu_k}(t),\quad \hat{\varphi}_k(x):=\hat{\varphi}_{k,\nu_k}(x)\quad\text{for}\ \ k=1,\dots,n.$$

Because of the existence and uniqueness of $\{\hat{\varphi}_j\}_1^{N+n}$, we need only check the orthogonality conditions

$$\int_a^b\frac{\partial^\lambda}{\partial t^\lambda}K(x,t_k)\hat{\varphi}_{ij}(x)\,dx=\delta_{ik}\delta_{j\lambda}$$

for the corresponding i, k, j, λ, and they evidently follow from the definition of $\hat{\varphi}_{ij}$.

Having defined $\hat{\varphi}_{ij}$, we look now, in view of Theorem 1, for points $t_1, \ldots, t_n$ annihilating the moments $\hat{\mu}_{N+1}(f), \ldots, \hat{\mu}_{N+n}(f)$ of f with respect to $\hat{\varphi}_1, \ldots \hat{\varphi}_n$. This leads to the conditions

$$\begin{aligned} 0 = \hat{\mu}_{N+k}(f) &= \int_a^b f(x)\hat{\varphi}_k(x)\, dx \\ &= \int_a^b \int_c^d K(x,t)g(t)\, dt\hat{\varphi}_k(x)\, dx \\ &= \int_c^d \int_a^b K(x,t)\hat{\varphi}_k(x)\, dx\, g(t)\, dt \\ &= \int_c^d P_k(t)g(t)\, dt \end{aligned}$$

for $k = 1, \ldots, n$. But the coefficients B_{ij} of the quadrature formula

$$\int_c^d g(t)F(t)\, dt \approx \sum_{i=1}^{n} \sum_{j=0}^{\nu_i} B_{ij}F^{(j)}(t_i),$$

which is exact for each $f \in K_\Phi$, are given by

$$B_{ij} = \int_c^d P_{ij}(t)g(t)\, dt.$$

In addition, it is known from an extension [3] of the Krein theorem (see [11]) that there exists a unique set of points $t_1, \cdots < t_n$ in $[c,d]$ that annihilates the coefficients B_{k,ν_k} for $k = 1, \ldots, n$, and these points are the nodes of the Gaussian quadrature of type $(\nu_1, \ldots \nu_n)$ in K_Φ with a weight g. The existence part of the theorem is proved. The uniqueness follows from the uniqueness of the Gaussian nodes. The equality $b_{i,j} = B_{i,j}$ is clear, since $b_{ij} = \hat{\mu}_{ij}(f)$ and the integral expression of $\hat{\mu}_{ij}(f)$ equals B_{ij}, by the Gaussian quadrature formula.

It remains to show the extremality of the element σ constructed above. In order to do this, assume that there is a certain element s from $\mathcal{K}_N$ that preserves $N+n+1$ moments of f, i.e., that $\mu_j(f) = \mu_j(s) = 0$ for $j = 1, \ldots, N+n+1$. But taking into account the form of $f(x)$ this assumption leads to existence of constants $\{c_{ij}\}$ which satisfy the system of equations

$$\int_a^b \Big\{ \int_c^d K(x,t)g(t)\, dt - \sum_{i=1}^{n} \sum_{j=0}^{\nu_i - 1} c_{ij} \frac{\partial^j}{\partial t^j} K(x,t_i) \Big\} \varphi_k\, dx = 0$$

for $k = 1, \ldots, N+n+1$. The expression $M_N(x)$ in the curl brackets is called *extended monospline.* It is known (see for example [2]) that $M_N(x)$ has no more than $N+n$ sign changes in (a,b). Then we can construct a generalized polynomial $q(x)=d_1\varphi_1(x)+\cdots+d_{N+n}\varphi_{N+n}(x)+\varphi_{N+n+1}(x)$ which changes sign at the same points as $M_N(x)$ does, and only there (see [10]). Then $M_N(x)q(x)$ would have a constant sign on (a,b) and thus $\int_a^b M_N(x)q(x)\,dx\neq 0$ while, from the system above, the integral should be zero, a contradiction. The theorem is proved. ∎

Remark. Following the algorithm described in Theorem 1, we constructed an element $\sigma(x)$ from N-dimensional linear space that preserves $N+n$ moments of f. Now one may easily come to the suggestion to prove Theorem 2 directly, just verifying that σ satisfies the conditions $\mu_j(\sigma)=\mu_j(f)$, for $j=1,\ldots,N+n$. This really leads to a short proof but it does not give any idea how the form of σ was discovered. However, knowing the form of σ one could extend the approach to the following more general situation.

Let X and T be given linear functionals defined on the classes F_X and F_T, respectively. Let $K(x,t)$ be a kernel such that $K(x,t)\in F_X$ and $K(x,t)\in F_T$ for each fixed t and x in the corresponding domains of the functions from F_T and F_X, respectively. Let $\varphi_1(x),\ldots,\varphi_n(x)$ be a given system of functions from F_X. Introduce the corresponding functions from F_T by

$$\Psi_i(t):=X[K(x,t)\varphi_i(x)],\quad i=1,\ldots,n.$$

Assume that for a certain linear functionals $L_1,\ldots,L_m$ defined on F_T and a function $g\in F_T$, we have

$$T[\Psi_i(t)g(t)]=\sum_{j=1}^m a_jL_j[\Psi_i]\quad \text{for } i=1,\ldots,n.$$

Finally, assume that X commutes with T and $L_1,\ldots,L_m$. Then the approximation

$$\sigma(x):=\sum_{j=1}^m a_jL_j[K(x_i,\cdot)]$$

to the function

$$f(x):=T[K(X,\cdot)g(\cdot)]$$

preserves the "moments"

$$\mu_i(f):=X[f\varphi_i],\quad i=1,\ldots,n,$$

of f.

The proof is very simple. Using the assumptions imposed on the functionals above, we obtain:

$$\begin{aligned}\mu_i(f) &= X[T[K(x,t)g(t)]\varphi_i(x)]\\ &= T[X[K(x,t)\varphi_i(x)g(t)]\end{aligned}$$

$$\begin{aligned}
&= T[\Psi_i(t)g(t)] = \sum_{j=1}^{m} L_j[\Psi_i] = \sum_{j=1}^{m} a_j L_j[X[K(x,t)\varphi_i(x)]] \\
&= X\Big[\Big(\sum_{j=1}^{m} a_j L_j[K(x,t)]\Big)\varphi_i(x)\Big] \\
&= X[\sigma(x)\varphi_i(x)] = \mu_i(\sigma).
\end{aligned}$$

4. Spline approximation

We shall illustrate here the general approach outlined in the previous sections to the particular and most studied case (see [4], [8], [6], [7], [9]) of constructing a spline that preserves a maximal number of algebraic moments of a given function f.

Recall that a *spline function* of degree $r-1$ with knots $t_1 < \cdots < t_n$ is any expression of the form

$$s(x) = \sum_{j=1}^{r} \alpha_j x^{j-1} + \sum_{k=1}^{n} c_k (t_k - x)_+^{r-1},$$

where $\{\alpha_j\}$ and $\{c_k\}$ are real coefficients and $u_+ := \max\{u, 0\}$. Clearly the functions

$$\begin{aligned}
\psi_j(x) &:= \frac{(b-x)^{j-1}}{(j-1)!}, \quad j = 1, \ldots, r, \\
\psi_{r+k}(x) &:= \frac{(t_k - x)_+^{r-1}}{(r-1)!}, \quad k = 1, \ldots, n
\end{aligned}$$

form a basis in the linear space of splines of degree $r-1$ with knots at $t_1, \ldots, t_n$.

Assume that $f \in C^r[a,b]$. Consider the problem of constructing a spline $s \in S_{rn}$ (that is, a spline of degree $r-1$ with n knots) such that

$$\int_a^b s(x) x^{j-1}\, dx = \int_a^b f(x) x^{j-1}\, dx =: \mu_j(f) \quad \text{for} \quad j = 1, \ldots, r+2n.$$

In order to do this, we shall follow the algorithm described in Theorem 1. For any given set of distinct points $\mathbf{t} := \{t_1, \ldots, t_n\}$ in $[a,b]$, we extend the system $\{\psi_i\}_1^{r+n}$ to the system

$$\psi_{\mathbf{t}} := \{\psi_1(x), \ldots, \psi_{r+n}(x), \psi_{r+n+1}(x), \ldots, \psi_{r+2n}(x)\}$$

adding the functions

$$\psi_{r+n+k}(x) := \frac{(t_k - x)_+^{r-2}}{(r-2)!}, \quad k = 1, \ldots, n.$$

Next we find the algebraic system $\{\hat{\varphi}_j(x)\}_1^{r+2n}$ which is biorthogonal to ψ_t. Introduce the algebraic polynomials $\{P_i(t)\}_{i=1}^{r+2n}$ of degree $2r+2n-1$ defined by the interpolation conditions:

$$P_i(a) = P_i'(a) = \cdots = P_i^{(r-1)}(a) = 0 \quad \text{for all } \ i;$$

If $i \in \{1, \ldots, r\}$, then $P_i^{(r-j)}(b) = \delta_{ij}$ for $j = 0, \ldots, r-1$ and

$$P_i(t_k) = P_i'(t_k) = 0 \quad \text{for } \ k = 1, \ldots, n;$$

If $i = r+1, \ldots r+2n$, then

$$P_i^{(j)}(a) = P_i^{(j)}(b) = 0 \quad \text{for } \ j = 0, \ldots, r-1,$$

and in addition

$$\begin{aligned} P_i(t_k) &= \delta_{ik}, \quad P_i'(t_k) = 0, \quad k = 1, \ldots, n, \quad \text{for } r < i \le r+n \\ P_i'(t_k) &= \delta_{ik}, \quad P_i(t_k) = 0, \quad k = 1, \ldots, n, \quad \text{for } r+n < i \le r+2n. \end{aligned}$$

Since, by Taylor's formula, every polynomial P_i can be presented in the form

$$P_i(t) = \int_a^b \frac{(t-x)_+^{r-1}}{(r-1)!} P_i^{(r)}(x)\, dx,$$

it is clear from the conditions above that the system $\{P_i^{(r)}(t)\}_1^{r+2n}$ is a basis in the space of polynomials of degree $r+2n-1$ and it is biorthogonal to ψ_t. Set

$$\hat{\mu}_k(f) := \int_a^b f(x) P_k^{(r)}(x)\, dx, \quad k = 1, \ldots, r+2n.$$

Then, according to Theorem 1, the approximation

$$\sigma(x) = \sum_{j=1}^{r} \hat{\mu}_j(f) \frac{(b-x)^{j-1}}{(j-1)!} + \sum_{k=1}^{n} \hat{\mu}_{r+k}(f) \frac{(t_k - x)_+^{r-1}}{(r-1)!} \tag{7}$$

will preserve the moments $\mu_1(f), \ldots, \mu_{r+2n}(f)$ provided

$$\hat{\mu}_k(f) = 0 \quad \text{for } \ k = r+n+1, \ldots, r+2n.$$

But an integration by parts yields

$$\hat{\mu}_k(f) = \sum_{j=0}^{r-1} (-1)^j f^{(j)}(b) P_k^{(r-j-1)}(b) + (-1)^r \int_a^b f^{(r)}(x) P_k(x)\, dx \tag{8}$$

for each P_k, $k = 1, \ldots, r+2n$ (since $P_k^{(i)}(a) = 0$ for $i = 0, \ldots, r-1$). In particular, for $k = r+n+1, \ldots, r+2n$, we have $P_k^{(i)}(b) = 0$, $i = 0, \ldots, r-1$, and thus, from (8),

$$\hat{\mu}_k(f) = (-1)^r \int_a^b f^{(r)}(x) P_k(x)\, dx. \tag{9}$$

The last integral is just the coefficient B_j (with $j = k-r-n$) of the interpolatory quadrature formula

$$\int_a^b \omega(x)F(x)\,dx \approx \sum_{j=1}^{r} a_j F^{(j-1)}(a) + \sum_{k=1}^{n} A_k F(t_k) + \sum_{k=1}^{n} B_k F'(t_k) + \sum_{j=1}^{r} b_j F^{(j-1)}(b)$$

with a weight $\omega(x) := (-1)^r f^{(r)}(x)$. Therefore, if $f^{(r)}(x)$ does not change sign in $[a,b]$, there exists a unique set of nodes $t_1 < \cdots < t_n$, namely the Gaussian nodes corresponding to the weight ω, which annihilate the coefficients $\{B_j\}$. Consequently, there exists a unique moment preserving approximation to f of double precision. Moreover, the spline approximation is given in the form (7). In order to find the coefficients $\{\hat{\mu}_k(f)\}$ of this spline we first observe that (9) holds also for $k = r+1, \ldots, r+n$ and applying the corresponding Gauss-Lobatto quadrature formula to evaluate the integral in (9), we see that $\hat{\mu}_k(f) = A_{k-r}$, the latter being the coefficients of this quadrature. Similarly, using (8) and the fact that $P_k^{(r-j)}(b) = \delta_{kj}$ for $k = 1, \ldots, r$, we get

$$\hat{\mu}_k(f) = (-1)^{k-1} f^{(k-1)}(b) + b_k \quad \text{for } k = 1, \ldots, r.$$

Thus the best moment preserving spline approximation to f is given by

$$\sigma(x) = \sum_{j=1}^{r} [b_j + (-1)^{j-1} f^{(j-1)}(b)] \frac{(b-x)^{j-1}}{(j-1)!} + \sum_{k=1}^{n} A_k \frac{(t_k - x)_+^{r-1}}{(r-1)!},$$

where $\{b_j\}$ and $\{A_k\}$ are the coefficients of the Gaussian quadrature formula of the form

$$\int_a^b \omega(x)F(x)\,dx \approx \sum_{j=1}^{r} a_j F^{(j-1)}(a) + \sum_{k=1}^{n} A_k F(t_k) + \sum_{j=1}^{r} b_j F^{(j-1)}(b)$$

with a weight $\omega(x) = (-1)^r f^{(r)}(x)$ in the space of algebraic polynomials.

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