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Bounds and Numerical Methods for the Unique Positive Root of a Polynomial [1]

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In this paper we obtain some lower and upper bounds for the unique positive root of the algebraic equation $t^n - \sum_{i=1}^{n} a_i t^{n-i} = 0$. Two-sided methods for calculating of positive root of this equation are considered. Such results are important for the determination of the R-order of convergence of iterative processes in numerical analysis, also in the financial mathematics.

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1. Introduction

Let IP denote an iterative process that produces a sequence of approximations $\{t^{(k)}\}$ with the limit point t^*. For the errors

$$\epsilon^{(k)} = \|t^* - t^{(k)}\|,$$

it is often possible to derive a difference inequality like

$$(1)\quad \epsilon^{(k+1)} \le \alpha \prod_{i=0}^{n-1} \left(\epsilon^{(k-i)}\right)^{a_{i+1}}, \; a_i \ge 0, \; i = 1, 2, \ldots, n, \; \alpha > 0, \; k \ge n-1.$$

According to Schmidt [6], the recurrence (1) has the R-order of convergence $O_R(IP, t^*)$ of at least σ, where σ is the unique positive root of the equation

$$(2)\qquad p(t) := t^n - \sum_{i=1}^{n} a_i t^{n-i} = 0, \; a_i \ge 0, \; i = 1, 2, \ldots, n.$$

Various estimations for σ can be found in Herzberger [2], Kyurkchiev [4], M. Petkovic and L. Petkovic [5]. In this paper we give another estimations.

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The following theorem by Deutsch [1] is more often applicable: *Let* $A = (a_{ij})$ *be a non-negative irreducible* $n \times n$ *matrix and let* $x = (x_1, x_2, \ldots, x_n)^T$ *and* $y = (y_1, y_2, \ldots, y_n)^T$ *be positive vectors satisfying*

$$Ax = Dx,$$

$$A^T y = Dy$$

for some positive diagonal matrix

$$D = \mathrm{diag}(d_1, d_2, \ldots, d_n).$$

If x *is not the Perron vector of* A, *then*

$$\rho(A) \geq \prod_{i=1}^{n} d_i^{\frac{d_i x_i y_i}{y^T D x}} \geq \frac{y^T D x}{y^T x}, \tag{3}$$

where $\rho(A)$ *is the spectral radius of the matrix* A.

2. Main results

The first estimations for positive root σ based on Deutsch's theorem with $x = (1, 1, \ldots, 1)^T$ can be found in [5]. We will use the same theorem for arbitrary positive vector x.

Theorem 1. *For arbitrary positive* $x_1, x_2, \ldots, x_n$, *for the positive root* σ *of the polynomial* $p(t)$ *the following estimations hold:*

$$\sigma \geq \left(\sum_{i=1}^{n} a_i x_i\right)^{\sum_{i=1}^{n} a_i x_i / \sum_{i=1}^{n} i a_i x_i} \exp\left(-\frac{\sum_{i=1}^{n} a_i x_i \ln x_i}{\sum_{i=1}^{n} i a_i x_i}\right) \tag{4}$$

and

$$\sigma \geq \frac{\sum_{i=1}^{n} i a_i x_i}{x_1 + \sum_{i=2}^{n} \mu_i a_i x_i}, \tag{5}$$

where

$$\mu_i = \sum_{k=1}^{i-1} \frac{x_{k+1}}{x_k}, \quad i = 2, 3, \ldots, n.$$

Proof. Let us associate with the polynomial $p(t)$ the corresponding matrix

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

with $\det(A - tI) = (-1)^n p(t)$. The matrix A is non-negative and irreducible. By the Perron-Frobenius theorem, this implies that A has a positive eigenvalue equal to its spectral radius $\rho(A)$, i.e. $\rho(A) = \sigma$. From the relation $Ax = Dx$ we find

$$d_1 = \frac{1}{x_1}\sum_{i=1}^{n} a_i x_i, \quad d_k = \frac{x_{k-1}}{x_k}, \quad k = 2, 3, \dots, n.$$

Then the system

$$(A^T - D)y = 0$$

yields

$$y_1 = 1, \quad y_k = \frac{1}{x_{k-1}}\sum_{i=k}^{n} a_i x_i, \quad k = 2, 3, \dots, n.$$

Now, we have

$$y^T Dx = \sum_{i=1}^{n} i a_i x_i, \tag{6}$$

$$d_k x_k y_k = \sum_{i=k}^{n} a_i x_i, \quad k = 1, 2, \dots, n, \tag{7}$$

$$y^T x = x_1 + \sum_{i=2}^{n} \mu_i a_i x_i. \tag{8}$$

According to (7), we have

$$\prod_{i=1}^{n} d_i^{d_i x_i y_i} = \left(\sum_{i=1}^{n} a_i x_i\right)^{\sum_{i=1}^{n} a_i x_i} \left(\frac{1}{x_1}\right)^{\sum_{i=1}^{n} a_i x_i} \left(\frac{x_1}{x_2}\right)^{\sum_{i=2}^{n} a_i x_i} \cdots \left(\frac{x_{n-1}}{x_n}\right)^{a_n x_n}$$
$$= \left(\sum_{i=1}^{n} a_i x_i\right)^{\sum_{i=1}^{n} a_i x_i} \prod_{i=1}^{n} \left(\frac{1}{x_i}\right)^{a_i x_i}$$
$$= \left(\sum_{i=1}^{n} a_i x_i\right)^{\sum_{i=1}^{n} a_i x_i} \exp\left(-\sum_{i=1}^{n} a_i x_i \ln x_i\right).$$

From the last expression, (6) and from the left inequality in (3) we get (4). From (6), (8) and inequality (3) we have (5). The proof is complete. ■

3. Special choise of the parameters

The vector $x = (x_1, x_2, \ldots, x_n)^T > 0$ can be taken in arbitrary way. First, if we choose $x_i = \lambda^i$, $i = 1, 2, \ldots, n$, $\lambda > 0$ we get the following result.

Theorem 2. *For arbitrary positive λ for the positive root σ of the polynomial $p(t)$ the following estimations hold:*

(9) $$\sigma \geq \frac{1}{\lambda}(q(\lambda))^{\frac{q(\lambda)}{\lambda q'(\lambda)}},$$

and

(10) $$\sigma \geq \frac{q'(\lambda)}{q'(\lambda)\lambda - q(\lambda) + 1},$$

where $q(\lambda) = \sum_{i=1}^{n} a_i \lambda^i$.

Proof. Substituting $x_i = \lambda^i$, $i = 1, 2, \ldots, n$, for the first multiplier in the right hand side of (4) we have

$$\left(\sum_{i=1}^{n} a_i x_i\right)^{\sum_{i=1}^{n} a_i x_i / \sum_{i=1}^{n} i a_i x_i} = q(\lambda)^{q(\lambda)/\sum_{i=1}^{n} i a_i \lambda^i} = q(\lambda)^{q(\lambda)/\lambda q'(\lambda)}$$

and the second multiplier in the right hand side of (4) gets the form

$$\exp\left(-\frac{\sum_{i=1}^{n} a_i x_i \ln x_i}{\sum_{i=1}^{n} i a_i x_i}\right) = \exp\left(-\frac{\sum_{i=1}^{n} a_i \lambda^i i \ln\lambda}{\sum_{i=1}^{n} i a_i \lambda^i}\right) = \exp(-\ln\lambda) = \frac{1}{\lambda},$$

and so we get (9).

Analogously, substituting in (5) $x_i = \lambda^i$ we find

$$\frac{\sum_{i=1}^{n} i a_i x_i}{x_1 + \sum_{i=2}^{n} \mu_i a_i x_i} = \frac{\lambda q'(\lambda)}{\lambda + \sum_{i=2}^{n}(i-1)a_i\lambda^{i+1}} = \frac{\lambda q'(\lambda)}{\lambda + \left(\frac{q(\lambda)}{\lambda}\right)' \lambda^3}$$

$$= \frac{\lambda q'(\lambda)}{q'(\lambda)\lambda^2 - q(\lambda)\lambda + \lambda} = \frac{q'(\lambda)}{q'(\lambda)\lambda - q(\lambda) + 1},$$

which gives estimation (10). The proof is complete. ∎

Remark 1. We can prove that

(11) $$q\left(\frac{1}{\sigma}\right) = 1.$$

Evidently,

$$0 = p(\sigma) = \sigma^n - \sum_{i=1}^{n} a_i \sigma^{n-i} = \sigma^n \left(1 - q\left(\frac{1}{\sigma}\right)\right),$$

i.e. $q\left(\frac{1}{\sigma}\right) = 1$, because $\sigma > 0$. Using (11) we see that equality in (9) and (10) is attained for $\lambda = \frac{1}{\sigma}$. Consequently, if we choose λ to be near $\frac{1}{\sigma}$, then the estimations (9) and (10) will be more precisely.

We observe that in the case $\lambda = 1$ the estimation (9) has the following simple form

(12) $$\sigma \geq \left(\sum_{i=1}^{n} a_i\right)^{\sum_{i=1}^{n} a_i / \sum_{i=1}^{n} i a_i}.$$

Let the coefficients of the polynomial $p(t)$ be such that for $i = i_1, i_2, \ldots, i_m$, $a_i > 0$, and for $i \neq i_1, i_2, \ldots, i_m$, $a_i = 0$. In that case we give another choise for the values of $x_1, x_2, \ldots, x_n$ in (4), but in such manner that $\sum_{i=1}^{n} a_i x_i = 1$.

Theorem 3. *For the positive root σ of the polynomial $p(t) = t^n - \sum_{k=1}^{m} a_{i_k} t^{n-i_k}$ the following estimations hold:*

$$\sigma \geq m^{m/\sum_{k=1}^{m} i_k} (a_{i_1} a_{i_2} \ldots a_{i_m})^{1/\sum_{k=1}^{m} i_k}. \tag{13}$$

Proof. Let

$$x_i = \frac{1}{ma_i}, \quad i = i_1, i_2, \ldots, i_m$$

and $0 < x_i$ be arbitrary for $i \neq i_1, i_2, \ldots, i_m$. The first multiplier in the right hand side of (4) is equal to 1, and for the second one we have

$$\exp\left(-\frac{\sum_{i=1}^{n} a_i x_i \ln x_i}{\sum_{i=1}^{n} i a_i x_i}\right) = \exp\left(\frac{\frac{1}{m}\sum_{k=1}^{m} \ln(m a_{i_k})}{\frac{1}{m}\sum_{k=1}^{m} i_k}\right) = \exp\left(\frac{\ln\left(m\,(a_{i_1} a_{i_2} \ldots a_{i_m})^{\frac{1}{m}}\right)}{\frac{1}{m}\sum_{k=1}^{m} i_k}\right)$$

$$= m^{m/\sum_{k=1}^{m} i_k} (a_{i_1} a_{i_2} \ldots a_{i_m})^{1/\sum_{k=1}^{m} i_k}.$$

In the special case $m = 1$, $i_1 = k$ the estimation (13) can be rewritten as

$$\sigma \geq a_k^{\frac{1}{k}},$$

i.e. the equality in (13) is attained, because for $p(t) := t^n - a_k t^{n-k} = 0$ we have $\sigma = a_k^{\frac{1}{k}}$. The proof is complete. ■

Another special choose can be the following

$$x_{i_k} = \frac{a_{i_k}^{\nu_k - 1}}{a_{i_1}^{\nu_1} + a_{i_2}^{\nu_2} + \ldots + a_{i_m}^{\nu_m}}, \quad k = 1, 2, \ldots, m,$$

where $0 < x_i$ are arbitrary for $i \neq i_1, i_2, \ldots, i_m$, and $\nu_1, \nu_2, \ldots, \nu_m$ are completely arbitrary numbers. We see that the equality $\sum_{i=1}^{n} a_i x_i = 1$ is true.

Very often the following polynomial equations arise by the determination of the rate of convergence of IP:

$$p(t) := t^n - (p+1)\sum_{i=1}^{n} r^{i-1} t^{n-i} = 0, \quad p \geq 0, \quad r > 0. \tag{14}$$

There exist explicit upper and lower bounds for the positive root σ of (14) (see [2]):

$$(15) \qquad \frac{n}{n+1}(p+r+1) \le \sigma \le p+r+1,$$

$$(16) \quad p+r+1-\frac{(p+1)r^n}{(p+r+1)^n}\left(1+\frac{1}{n}\right)^n \le \sigma \le p+r+1-\frac{(p+1)r^n}{(p+r+1)^n}\,.$$

We observe that the coressponding polynomial $q(\lambda)$ to the equation (14) is of the form

$$q(\lambda) = \frac{p+1}{r}\sum_{i=1}^{n}(r\lambda)^i = \frac{(p+1)\lambda}{1-r\lambda}(1-(r\lambda)^n).$$

In spite of Remark 1, we find that for $\lambda_0 := \frac{1}{1+p+r}$ such that $\frac{(p+1)\lambda_0}{1-r\lambda_0} = 1$ and $q(\lambda_0) = 1-(r\lambda_0)^n \approx 1$. Then for the positive root of the equation (14) the estimations (9) and (10) with $\lambda = \lambda_0 := \frac{1}{1+p+r}$ can be rewritten respectively as

$$\sigma \ge \frac{1}{\lambda_0}(1-(r\lambda_0)^n)^{\psi_{p,r,n}} = (1+p+r)\left(1-\left(\frac{r}{1+p+r}\right)^n\right)^{\psi_{p,r,n}},$$

where

$$\psi_{p,r,n} = (1+p)\lambda_0\frac{1-(r\lambda_0)^n}{1-(r\lambda_0)^{n+1}-(n+1)(p+1)\lambda_0(r\lambda_0)^n}$$

and

$$\sigma \ge \lambda_0^{-1}-(p+1)\frac{(r\lambda_0)^n}{1-(r\lambda_0)^{n+1}-n(p+1)\lambda_0(r\lambda_0)^n}\,.$$

Comparing with (15) and (16), we see that the last estimations have a good order of exactness.

Remark 2. Several problems in the classical financial mathematics lead to a class of polynomial equation [3],

$$P(x) = Cx^n - \sum_{j=1}^{n-1}B_jx^{n-j} - A = 0$$

with only one single positive root (bound for the effective rate of an annuity with geometrically growing payments), where A is the purchase price of the bond paid to the issuer; $B_j \equiv B$ is a periodic payment paid according to the contract rate of the bond; $C = K+B$, where K is the purchase rate of the bond when sold at the bond market; n is the term of the bond (usually number of full years or months).

The explicit bounds for the unique positive root in the term of finance can be obtained using the approach given in this paper.

4. Numerical methods for solving the positive root of the equation (2)

From (11) we see that the equation $F(t) := q(t) - 1 = 0$ has the unique positive root equal to $\frac{1}{\sigma}$. Evidently for the function $F(t)$ we have $F'(t) > 0$, $F''(t) > 0$ for all positive t. The estimations (9) and (10) gives the following estimations for the positive zero of the equation $F(t) = 0$

$$\frac{1}{\sigma} \leq \lambda(q(\lambda))^{-\frac{q(\lambda)}{\lambda q'(\lambda)}}, \tag{17}$$

$$\frac{1}{\sigma} \leq \lambda - \frac{q(\lambda) - 1}{q'(\lambda)}, \tag{18}$$

for every $\lambda > 0$. Let mention that equality in (17) and (18) is attained for $\lambda = \frac{1}{\sigma}$ and

$$\frac{1}{\sigma} \leq \lambda(q(\lambda))^{-\frac{q(\lambda)}{\lambda q'(\lambda)}} \leq \lambda - \frac{q(\lambda) - 1}{q'(\lambda)}. \tag{19}$$

We define for arbitrary $\lambda_0 > 0$ the following iteration formulae

$$\mu_{k+1} = \mu_k(q(\mu_k))^{-\frac{q(\mu_k)}{\mu_k q'(\mu_k)}}, \quad k = 0, 1, 2, \ldots, \quad \mu_0 = \lambda_0; \tag{20}$$

$$\lambda_{k+1} = \lambda_k - \frac{q(\lambda_k) - 1}{q'(\lambda_k)}, \quad k = 0, 1, 2, \ldots. \tag{21}$$

We remarked that the iteration (21) is Newton's formulae for the equation $q(t) - 1 = 0$ and from the positivity of q' and q'' the iteration is convergent with quadratically order for every initial point $\lambda_0 > 0$. Therefore, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \frac{1}{\sigma}$, i.e. the process is monotonical and one-sided.

Analogously for the iterations (20) the iteration function is of the form

$$\phi(t) := t(q(t))^{-\frac{q(t)}{tq'(t)}} = t \exp\left(-\frac{q(t)\ln q(t)}{tq'(t)}\right).$$

We have

$$\phi\left(\frac{1}{\sigma}\right) = \frac{1}{\sigma},$$

$$\phi'(t) = q(t)^{-\frac{q(t)}{tq'(t)}} \frac{tq(t) + q(t) - tq'(t)}{tq'(t)} \ln q(t).$$

From the last equation we have $\phi'(\frac{1}{\sigma}) = 0$, i.e. the iteration (20) has at least a quadratical order of convergence. From (19) and from the strong increasing of the function

$$\lambda - \frac{q(\lambda) - 1}{q'(\lambda)}$$

for $\lambda > \frac{1}{\sigma}$ we have

$$\frac{1}{\sigma} \le \mu_k \le \lambda_k, \quad k = 1, 2, \ldots$$

Starting from $x_0 = y_0 < \frac{1}{\sigma}$, i.e. $q(x_0) < 1$ the following two-sided iteration method can be constructed using procedures (20) or (21) with combination of the following *regula falsi* iterations

(22) $$x_{k+1} = x_k - (q(x_k) - 1)\frac{\lambda_{k+1} - x_k}{q(\lambda_{k+1}) - q(x_k)}, \quad k = 0, 1, \ldots,$$

or

(23) $$y_{k+1} = y_k - (q(y_k) - 1)\frac{\mu_{k+1} - y_k}{q(\mu_{k+1}) - q(y_k)}, \quad k = 0, 1, \ldots.$$

The following relations hold:

$$x_k \le x_{k+1} \le \ldots \le \frac{1}{\sigma} \le \ldots \le \lambda_{k+1} \le \lambda_k, \quad k = 1, 2, \ldots$$

$$y_k \le y_{k+1} \le \ldots \le \frac{1}{\sigma} \le \ldots \le \mu_{k+1} \le \mu_k, \quad k = 1, 2, \ldots$$

$$\lambda_k^{-1} \le \lambda_{k+1}^{-1} \le \ldots \le \sigma \le \ldots \le x_{k+1}^{-1} \le x_k^{-1}, \quad k = 1, 2, \ldots.$$

$$\mu_k^{-1} \le \mu_{k+1}^{-1} \le \ldots \le \sigma \le \ldots \le y_{k+1}^{-1} \le y_k^{-1}, \quad k = 1, 2, \ldots$$

If we have that $q(x_0) < 1$, then from (22) and (23) we see that x_k and y_k for arbitrary k give lower bounds for $\frac{1}{\sigma}$. Respectively, $\frac{1}{x_k}$ and $\frac{1}{y_k}$ are upper bounds for σ, where k is arbitrary.

5. Numerical examples

For the zero $\sigma \approx 5.1$ of the polynomial

$$p(t) = t^6 - 5t^5 - 0.5t^4 - 0.05t^3 - 0.005t^2 - 0.0005t - 0.00005$$

from (12) we get the following bound: $\sigma \ge 5.55555^{0.9000048} \approx 4.68$. From (9) and (10) with $\lambda = \frac{1}{5}$ we find respectively $\sigma \ge 5.0999797$ and $\sigma \ge 5.0999592$.

For the zero $\sigma \approx 1.992$ of the polynomial

$$p(t) = t^7 - t^6 - t^5 - t^4 - t^3 - t^2 - t - 1$$

from (9) and (10) with $\lambda = \frac{1}{2}$ we find respectively $\sigma \geq 1.99195508$ and $\sigma \geq 1.9919355$.

For the same polynomial using iterations (21) and (22), we generate the two-side method for $\frac{1}{\sigma}$ which gives

$$\lambda_0 = x_0 = 0.5 < x_1 = 0.5020170292323 < \frac{1}{\sigma}$$

$$< \lambda_2 = 0.5020170552711 < \lambda_1 = 0.5020241291498.$$

Using iterations (20) and (23) we generate another two-side method for $\frac{1}{\sigma}$ and find

$$\mu_0 = y_0 = 0.5 < y_1 = 0.5020170430612 < \frac{1}{\sigma} < \mu_1 = 0.5020204346287.$$

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