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On the Asymptotics of a Function Related to Tricomi's Confluent Hypergeometric Function [1]

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Let $\Psi(a,c;z)$ be the main branch of the Tricomi confluent hypergeometric function with paramaters a, c. In the paper the asymptotics of the function

$$\Psi_a^{(\alpha)}(z) = z^\alpha\Gamma(a+\alpha+1)\Psi(a+\alpha+1,\alpha+1;z)$$

is studied when a tends to infinity and z remains bounded. More precisely, let $a>0, \alpha \in \mathbb{C}\setminus(-\infty,-1]), z\in\mathbb{C}\setminus(-\infty,0]$ and define

$$\mu_a^{(\alpha)}(z) = \left\{E_a^{(\alpha)}(z)\right\}^{-1}\Psi_a^{(\alpha)}(z) - 1,$$

where

$$E_a^{(\alpha)}(z) = \sqrt{\pi}\exp(z/2)z^{\alpha/2-1/4}a^{\alpha/2-1/4}\exp(-2z^{1/2}a^{1/2}).$$

It is proved, that $\lim\limits_{a\to\infty}\mu_a^{(\alpha)}(z) = 0$ uniformly with respect to z on any compact subset of the region $\mathbb{C}\setminus(-\infty,0]$.

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1. If $\Re a>0$ and $\Re z>0$, then the main branch of the Tricomi function is the Laplace transform of the function $(\Gamma(a))^{-1}t^{a-1}(1+t)^{c-a-1}(0<t<\infty)$ [1, 6.5, (2)], i.e.

$$\Psi(a,c;z) = \frac{1}{\Gamma(a)}\int_0^\infty \frac{t^{a-1}\exp(-zt)}{(1+t)^{a-c+1}}\,dt. \tag{1}$$

Let $-\pi/2\le\theta\le\pi/2$ and $\gamma(\theta)$ be the ray $\{\zeta\in\mathbb{C}:\zeta=\exp i\theta.t, 0\le t<\infty\}$. Then the analytic continuation of the function (1) in the half-plane

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$H_\theta := \{z \in \mathbb{C} : |\arg(\exp(-i\theta)z)| < \pi/2$ can be realised by means of the equality [1, 6.5, (3)]

$$\Psi(a,c;z) = \frac{1}{\Gamma(a)} \int_{\gamma(\theta)} \frac{\zeta^{a-1}\exp(-z\zeta)}{(1+\zeta)^{a-c+1}}\, d\zeta. \tag{2}$$

Indeed, as it is easily proved by means of the Cauchy integral theorem, the right-hand sides of (1) and (2) coincide when $z \in H_\theta \bigcap H_0$.

Now we will see that if $\Re a > 0$, then the main branch of the Tricomi function has the following integral representation in the region $\mathbb{C} \setminus (-\infty, 0]$, namely:

$$\Psi(a,c;z) = \frac{z^{1-c}}{\Gamma(a)} \int_0^\infty \frac{t^{a-1}\exp(-t)}{(z+t)^{a-c+1}}\, dt. \tag{3}$$

Let $z = x > 0$ and change the variable t by t/x in the integral on the right-hand side of the equality

$$\Psi(a,c;x) = \frac{1}{\Gamma(a)} \int_0^\infty \frac{t^{a-1}\exp(-xt)}{(1+t)^{a-c+1}}\, dt.$$

As a result, we obtain that

$$\Psi(a,c;x) = \frac{x^{1-c}}{\Gamma(a)} \int_0^\infty \frac{t^{a-1}\exp(-t)}{(x+t)^{a-c+1}}\, dt. \tag{4}$$

But the integral on the right-hand side of (3) is uniformly convergent on every compact subset of the region $\mathbb{C} \setminus (-\infty, 0]$, i.e. it defines a holomorphic function there. Because of the equality (4), the right-hand sides of (1) and (3) coincide when $z = x > 0$ and by the identity theorem for holomorphic functions they coincide in the whole half-plane H_0.

Let $\Re a > 0, \Re\alpha > -1, z \in \mathbb{C} \setminus (-\infty, 0]$ and define

$$\Psi_a^{(\alpha)}(z) = z^\alpha \Gamma(a+\alpha+1)\Psi(a+\alpha+1; z).$$

Then as a corollary of (3) we obtain the representation

$$\Psi_a^{(\alpha)}(z) = \int_0^\infty \frac{t^{a+\alpha}\exp(-t)}{(z+t)^{a+1}}\, dt. \tag{5}$$

After integration by parts we come to the relation $(a+\alpha+1)\Psi_a^{(\alpha)}(z) = \Psi_a^{(\alpha+1)}(z) + \Psi_{a+1}^{(\alpha)}(z)$, which gives the possibility to define the function $\Psi_a^{(\alpha)}(z)$ for each $\alpha \in \mathbb{C} \setminus (-\infty, -1]$ provided that $\Re a > 0$.

2. Let $a > 0, \alpha \in \mathbb{C} \setminus (-\infty, -1], z \in \mathbb{C} \setminus (-\infty, 0]$ and define

(6) $$\mu_a^{(\alpha)}(z) = \{E_a^{(\alpha)}(z)\}^{-1} \Psi_a^{(\alpha)}(z) - 1,$$

where

$$E_a^{(\alpha)}(z) = \sqrt{\pi} \exp(z/2) z^{\alpha/2-1/4} a^{\alpha/2-1/4} \exp(-2z^{1/2} a^{1/2}).$$

The main result in this paper is the following statement.

Main Result. *If $\alpha \in \mathbb{C} \setminus (-\infty, -1]$, then*

(7) $$\lim_{a \to \infty} \mu_a^{(\alpha)}(z) = 0$$

uniformly with respect to z on any compact susbset of the region $\mathbb{C} \setminus (-\infty, 0]$.

The proof is based on several lemmas. In order to formulate and prove them we need a preliminary preparation.

Let $l(z)$ be the ray $\{\zeta \in \mathbb{C} : \zeta = -z - t, 0 \leq t < \infty\}$ and define the function $P_a^{(\alpha)}(z; \zeta)$ in the region $\mathbb{C} \setminus \{l(z) \bigcup (-\infty, 0]\}$ by the equality

$$P_a^{(\alpha)}(z; \zeta) = \frac{\zeta^{a+\alpha} \exp(-\zeta)}{(z + \zeta)^{a+1}} .$$

It is clear that this function is holomorphic in the region just indicated and moreover that the equality $\partial P_a^{(\alpha)}(z; \zeta)/\partial \zeta = 0$ is equivalent to the quadratic equation

(7) $$\zeta^2 + (z - \alpha + 1)\zeta - (a + \alpha)z = 0.$$

Let $\alpha \in \mathbb{C} \setminus (-\infty, -1]$ and the compact set $K \subset \mathbb{C} \setminus (-\infty, 0]$ be fixed. The first auxiliary statement we need is the following lemma.

Lemma 1. *There exists $a_0 = a_0(K, \alpha) > 0$ such that if $z \in K$, and $a \geq a_0$, then:*

(i) $(a + \alpha)z + (z - \alpha + 1)^2/4 \in \mathbb{C} \setminus (-\infty, 0]$;

(ii) $\tau_a^{(\alpha)}(z) := \{(a + \alpha)z + (z - \alpha + 1)/4\}^{1/2} - (z - \alpha + 1)/2$

is that root of the equation (8) *for which $\Re\{\tau_a^{(\alpha)}(z)\} > 0$;*

(iii) *the asymptotic representations:*

(9) $$\tau_a^{(\alpha)}(z) = (za)^{1/2} - (z - \alpha + 1)/2 + O(a^{-1/2})$$

and

(10) $$z\{\tau_a^{(\alpha)}(z)\}^{-1} = (z/a)^{1/2} + (z - \alpha + 1)/2a + O(a^{-3/2})$$

hold uniformly with respect to $z \in K$ when a tends to infinity.

Proof. (i) is a corollary of the equality $(a+\alpha)+(z-\alpha+1)^2/4 = za\{1+\alpha z/a+(z_\alpha)^2/4az\}$ since $\lim_{a\to\infty}\{1+\alpha z/a+(z-\alpha+1)^2/4az\} = 1$ uniformly with respect $z \in K$. (ii) is a corollary of the equality $\lim_{a\to\infty}(za)^{-1/2}\{\{(a+\alpha)z+(z-\alpha+1)^2/4\}^{1/2}-(z-\alpha+1)/2\} = 1$, which holds uniformly on K. The asymptotic formula (9) is a corollary of the equality $(1+\alpha z/a+(z-\alpha+1^2)/4a)^{1/2} = 1+O(a^{-1})$, which holds uniformly on K when a tends to infinity. The representation (10) is a corollary of (9). ■

Lemma 2. *If $z \in K$ and $a \geq a_0$, then*

$$\Psi_a^{(\alpha)}(z) = \{\tau_a^{(\alpha)}(z)\}^\alpha \int \frac{t^{a+\alpha}\exp(-\tau_a^{(\alpha)}(z)t)}{(t+\sigma_a^{(\alpha)}(z))^{a+1}}\,dt, \tag{11}$$

where

$$\sigma_a^{(\alpha)}(z) = z\{\tau_a^{(\alpha)}(z)\}^{-1}. \tag{12}$$

Proof. We denote by $l_a^{(\alpha)}(z)$ the ray $\{\zeta \in \mathbf{C} : \zeta = \tau_a^{(\alpha)}(z).t, 0 \leq t < \infty\}$. It is easy to see that except the point $\zeta = 0$ this ray lies in the region $H_0 \setminus l(z)$. Then the Cauchy integral theorem gives that

$$\int_0^\infty \frac{t^{a+\alpha}\exp(-t)}{(z+t)^{a+1}}\,dt = \int_{l_a^{(\alpha)}(z)} \frac{\zeta^{a+\alpha}\exp(-\zeta)}{(z+\zeta)^{a+1}}\,d\zeta.$$

By setting $\zeta = \tau_a^{(\alpha)}(z)t, (0 \leq t < \infty)$ in the integral on the right-hand side of the above equality we obtain the representation (1).

Remark. It is clear that a_0 is chosen so that $\Re(a+\alpha) > -1$ when $a \geq a_0$.

Further we change t by $1+a^{-1/4}t$ in the integral on the right-hand side of (10) and in this way we come to the representation $(z \in K, a \geq a_0)$

$$\Psi_a^{(\alpha)}(z) = \frac{a^{-1/4}\{\tau_a^{(\alpha)}(z)\}^\alpha \exp(-\tau_a^{(\alpha)}(z))}{(1+\sigma_a^{(\alpha)}(z))^{a+1}}\{A_a^{(\alpha)}(z)+R_a^{(\alpha)}(z)\}, \tag{13}$$

where

$$A_a^{(\alpha)}(z) = \int_{-a^{-1/4}}^{a^{1/4}} F_a^{(\alpha)}(z;a^{-1/4}t)\,dt,$$

$$R_a^{(\alpha)}(z) = \int_{a^{1/4}}^{\infty} F_a^{(\alpha)}(z;a^{-1/4}t)\,dt \tag{14}$$

and

$$F_a^{(\alpha)}(z;u) = \frac{(1+u)^{a+\alpha}\exp(-\tau_a^{(\alpha)}(z)u)}{(1+(1+\sigma_a^{(\alpha)}(z))^{-1}u)^{a+1}}, \quad -1 < u < \infty.$$

Let us define

(15) $$\varphi_k^{(\alpha)}(a;z) = a + \alpha - (a+1)(1+\sigma_a^{(\alpha)}(z))^{-k}, \quad k = 1,2,3,\ldots$$

provided that $z \in K$ and $a \geq a_0$.

Lemma 3. *There exist positive constants $B = B(K,\alpha)$ and $C = C(K,\alpha)$ so that the inequality*

(16) $$|\varphi_k^{(\alpha)}(a;z)| \leq (B + (k-1)C)a^{1/2}$$

holds when $z \in K, a \geq a_0$ and $k = 1,2,3,\ldots$.

Proof. A simple calculation based on the definition of $\sigma_a^{(\alpha)}(z)$ by (12), as well as on the fact that $\tau_a^{(\alpha)}(z)$ is a root of the equation (8), gives that $\varphi_1^{(\alpha)}(a;z) = \tau_a^{(\alpha)}(z)$. Therefore if we define $B(K,\alpha) = \sup\{a^{-1/2}|\tau_a^{(\alpha)}(z)| : z \in K, a \geq a_0\}$, then the asymptotic formula (9) yields that the inequality (16) will be satisfied when $z \in K, a \geq a_0$ and $k = 1$. ∎

As a corollary of (15) we have that

(17) $$\varphi_{k+1}^{(\alpha)}(a;z) = \varphi_k^{(\alpha)}(a;z) + \frac{(a+1)\sigma_a^{(\alpha)}(z)}{(1+\sigma_a^{(\alpha)}(z))^{k+1}}, \quad k = 1,2,3,\ldots.$$

From the asymptotic formula (10) it follows, that $\Re\{\sigma_a^{(\alpha)}(z)\} > 0$ and therefore $|1 + \sigma_a^{(\alpha)}(z)| > 1$ when $z \in K$ and $a \geq a_0$.

If we define $C(K,\alpha) = \sup\{a^{1/2}(1 + a^{-1})|\sigma_a^{(\alpha)}(z)| : z \in K, a \geq a_0\}$, then (17) gives that $|\varphi_{k+1}^{(\alpha)}(a;z)| \leq |\varphi_k^{(\alpha)}(a;z)| + Ca^{1/2}, (z \in K, a \geq a_0)$ and the validity of the inequality (15) when $z \in K, a \geq a_0$ and $k = 1,2,3,\ldots$ can be proved by induction.

Lemma 4. *Let us define*

(18) $$L_a^{(\alpha)}(z;u) = \log F_a^{(\alpha)}(z;u) + \varphi_2^{(\alpha)}(a;z)u^2/2.$$

Then the inequality

(19) $$|L_a^{(\alpha)}(z;u)| \leq (B+C)a^{1/2}|u|^3(1-|u|)^{-1}$$

holds when $z \in K, a \geq a_0$ and $|u| < 1$.

Proof. Since $|u| < 1$ and moreover, $|1 + \sigma_a^{(\alpha)}(z)| > 1$ when $z \in K$ and $a \geq a_0$, we have the expansion

$$\log F_a^{(\alpha)}(z;u) = (a+\alpha)\log(1+u) - \tau_a^{(\alpha)}(z)u - (a+1)\log(1+(1+\sigma_a^{(\alpha)}(z)^{-1}u)$$

$$= (a+\alpha)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}u^k - \tau_a^{(\alpha)}(z)u - (a+1)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}u^k}{k(1+\sigma_a^{(\alpha)}(z))^k}.$$

Since $\tau_a^{(\alpha)}(z)$ is a root of the equation (8), the coefficient of u is equal to zero and in view of (15) we have that

$$\log F_a^{(\alpha)}(z)(z;u) = -\varphi_2^{(\alpha)}(a;z)u^2/2 + \sum_{k=3}^{\infty}\frac{(-1)^{k-1}}{k}\varphi_k^{(\alpha)}(a;z)u^k. \tag{20}$$

Then the inequalities (15) give that

$$|L_a^{(\alpha)}(z;u)| \leq \sum_{k=3}^{\infty}\frac{|\varphi_k^{(\alpha)}(a;z)|}{k}|u|^k$$

$$\leq \sum_{k=3}^{\infty}k^{-1}(B+(k-1)C)a^{1/2}|u|^k \leq (B+C)a^{1/2}|u|^3\sum_{k=0}^{\infty}|u|^k.$$

■

Lemma 5. *If $\Re\alpha \geq 1$, then*

$$F_a^{(\alpha)}(z;u) = O\{\exp(-z^{1/2}a^{1/2}u^2/2)\} \tag{22}$$

uniformly with respect to $z \in K$ and $|u| < 1$ when a tends to infinity.

Proof. It was mentioned, that $|1+\sigma_a^{(\alpha)}(z)| > 1$ when $z \in K$ and $a \geq a_0$. Therefore $\Re\{(1+\sigma_a^{(\alpha)}(z))^{-k}\} < 1$ when $z \in K$, $a \geq a_0$ and $k = 1,2,3,\ldots$. Since $\Re\alpha \geq 1$, from (15) it follows that $\Re\{\varphi_k^{(\alpha)}(a;z)\} > 0$ when $z \in K, a \geq a_0$ and $k = 1,2,3,\ldots$.

Suppose that $-1 < u \leq 0$. Then (20) yields that

$$\Re\{\log F_a^{(\alpha)}(z;u) + \varphi_2^{(\alpha)}(a;z)u^2/2\} = -\sum_{k=3}^{\infty}\frac{(-u)^k}{k}\Re\{\varphi_k^{(\alpha)}(a;z)\} \leq 0$$

provided that $z \in K$ and $a \geq a_0$. Therefore,

$$F_a^{(\alpha)}(z;u) = O\{\exp(-\varphi_2^{(\alpha)}(a;z)u^2/2)\} \tag{22}$$

uniformly with respect to z and $u \in (-1,0]$ when a tends to infinity. But the asymptotic formula (10) gives that

$$\varphi_2^{(\alpha)}(a;z) = -2z^{1/2}a^{1/2} + O(1) \tag{23}$$

uniformly with respect to $z \in K$ when a tends to infinity and thus (21) is verified when $-1 < u \leq 0$.

Let $0 \leq u < 1$. Then as a corollary of the equality

$$F_a^{(\alpha)}(z;u) = \frac{(1+u)^{\alpha-1}\exp(-\tau_a^{(\alpha)}(z)u)}{(1-\sigma_a^{(\alpha)}(z)(1+\sigma_a^{(\alpha)}(z))^{-1}u(1+u)^{-1})^{a+1}}$$

we obtain that the representation

$$\log F_a^{(\alpha)}(z;u) = (\alpha-1)\log(1+u) - \tau_a^{(\alpha)}(z)u$$
$$+(a+1)\log(1-\sigma_a^{(\alpha)}(z)(1+\sigma_a^{(\alpha)}(z))^{-1}u(1+u)^{-1})$$
$$= (\alpha-1)\log(1+u) - \tau_a^{(\alpha)}(z)u + (a+1)\sigma_a^{(\alpha)}(z)(1+\sigma_a^{(\alpha)}(z))^{-1}u(1+u)^{-1} + O(1)$$

holds uniformly when $z \in K, a \geq a_0$ and $0 \leq u < 1$. Then the asymptotic formulas (9) and (10) give that $\log F_a^{(\alpha)}(z;u) = -z^{1/2}a^{1/2} + z^{1/2}a^{1/2}u(1+u)^{-1} + O(1) = -z^{1/2}a^{1/2}u^2(1+u)^{-1} + O(1) = O(-z^{1/2}a^{1/2}u^2/2)$ uniformly with respect to $z \in K$ and $u \in [0,1)$ when a tends to infinity.

3. Now we are ready to prove the valididty of (7). At first we will show that under the assumption $\Re\alpha \geq 1$ we have already made, the following equality

$$\lim_{a\to\infty} A_a^{(\alpha)}(z) = \sqrt{\pi}z^{-1/4} \tag{24}$$

holds uniformly with respect to z on any compact subset of the region $\mathbb{C} \setminus (-\infty, 0]$. This will be the fact if we show that

$$\lim_{a\to\infty} A_a^{(\alpha)}(z) = 2\int_0^\infty \exp(-z^{1/2}t^2)\,dt \tag{25}$$

uniformly on K. Indeed, if $z = x > 0$, then

$$\int_0^\infty \exp(-x^{1/2}t^2)\,dt = x^{-1/4}\int_0^\infty \exp(-t^2)\,dt = \frac{\sqrt{\pi}}{2}x^{-1/4}.$$

But the integral on the right-hand side of (25) is uniformly convergent on each compact subset of the region $\mathbb{C} \setminus (-\infty, 0]$ i.e. it defines a holomorphic function there. Therefore the validity of the equality (24) is a corollary of the identity theorem for holomorphic functions.

Let $\varepsilon > 0$ and $T_0 = T_0(K, \varepsilon) > 0$ be choosen so that

$$(26) \qquad \int_{|t|\geq T_0} |\exp(-z^{1/2}t^2/2)|\, dt < \varepsilon$$

Further, whatever $T > 0$ be, we have that $\lim_{a\to\infty} F_a^{(\alpha)}(z; a^{-1/4}t) = \exp(-z^{1/2}t^2)$ uniformly with respect to $z \in K$ and $|t| \leq T$. This is a corollary of (18). Indeed, from one hand the asymptotic formula (23) gives that

$$\lim_{a\to\infty} \varphi_2^{(\alpha)}(a; z)(a^{-1/4}t)^2/2 = -z^{1/2}t^2$$

uniformly with respect to $z \in K$ and $|t| \leq T$. From the other hand the inequality (19) yields that $|L_a^{(\alpha)}(z; a^{-1/4}t)| \leq (B + C)T^3(1 - a_0^{-1/4}T)a^{-1/4}$ uniformly when $z \in K, |t| \leq T$ and $a \geq a_0$ provided that a_0 is choosen so that $a_0^{-1/4}T < 1$.

Let a_0 be so large that the inequality

$$(27) \qquad |F_a^{(\alpha)}(z; a^{-1/4}t) - \exp(-z^{1/2}t^2)| < \frac{\varepsilon}{2T_0}$$

holds when $z \in K, |t| \leq T_0$ and $a \geq a_0$. Further Lemma 5 gives that there existst a constants $Q = Q(K, \alpha) > 0$ such that the inequality

$$(28) \qquad |F_a^{(\alpha)}(z; a^{-1/4}t)| \leq Q|\exp(-z^{1/2}t^2/2)|$$

holds uniformly when $z \in K, |t| \leq a^{1/4}$ and $a \geq a_0$.

As a corollary of (26), (27) and (28) we have that

$$\left|A_a^{(\alpha)}(z) - \int_{-\infty}^{\infty} \exp(-z^{1/2}t^2)\, dt\right| \leq \int_{|t|\leq T_0} |F_a^{(\alpha)}(z; a^{-1/4}t) - \exp(-z^{1/2}t^2)|\, dt$$

$$+ \int_{T_0\leq|t|\leq a^{1/4}} |F_a^{(\alpha)}(z; a^{-1/4}t)|\, dt + \int_{|t|\geq T_0} |\exp(-z^{1/2}t^2)|\, dt$$

$$\leq \varepsilon + Q\int_{|t|\geq T_0} |\exp(-z^{1/2}t^2/2)|\, dt + \int_{|t|\geq T_0} |\exp(-z^{1/2}t^2)|\, dt \leq (Q+2)\varepsilon.$$

Now we are going to prove that if $Re\alpha \geq 1$, then

$$\lim R_a^{(\alpha)}(z) = \lim_{a\to\infty} \int_{a^{1/4}}^{\infty} F_a^{(\alpha)}(z; a^{-1/4}t)\, dt = 0$$

uniformly with respect to z on any compact subset of the region $\mathbb{C}\setminus(-\infty, 0]$. To that end we will use the following representation of $R_a^{(\alpha)}(z)$, which is a corollary of (14), namely

$$(29) \qquad R_a^{(\alpha)}(z) = a^{1/4}\exp\{\tau_a^{(\alpha)}(z)\}(1 + \sigma_a^{(\alpha)}(z))^{a+1}I_a^{(\alpha)}(z),$$

where

$$I_a^{(\alpha)}(z) = \int_2^\infty \frac{t^{a+\alpha} \exp(-\tau_a^{(\alpha)}(z)t)}{(t+\sigma_a^{(\alpha)}(z))^{a+1}}\,dt. \tag{30}$$

Since $\Re\{\sigma_a^{(\alpha)}(z)\} > 0$ when $z \in K$ and $a \geq a_0$, we have that

$$\left|I_a^{(\alpha)}(z)\right| \leq \int_2^\infty t^{\Re\alpha - 1} \exp(-\xi^{(\alpha)}(z)t)\,dt,$$

where $\xi_a^{(\alpha)}(z) = \Re\{\tau_a^{(\alpha)}(z)\}$.

We change t by $(\xi_a^{(\alpha)}(z))^{-1}t$ and thus obtain the inequality

$$\left|I_a^{(\alpha)}(z)\right| \leq \{\xi_a^{(\alpha)}(z)\}^{-\Re\alpha} \int_{2\xi_a^{(\alpha)}(z)} t^{\Re\alpha - 1} \exp(-t)\,dt. \tag{31}$$

If k is a positive integer such that $k > \Re\alpha$, then after integration by parts we come to the equality

$$\int_{2\xi_a^{(\alpha)}(z)}^\infty t^{\Re\alpha-1} \exp(-t)\,dt = \exp(-2\xi_a^{(\alpha)}(z)) \sum_{s=1}^k C_s^{(\alpha)}\{\xi_a^{(\alpha)}(z)\}^{\Re\alpha - s}$$
$$+ \int_{2\xi_a^{(\alpha)}(z)}^\infty t^{\Re\alpha-k} \exp(-t)\,dt,$$

where $C_s^{(\alpha)}(s = 1,2,3,\ldots,k)$ are constants not depending on a and z.

From the above equality it follows that

$$\int_{2\xi_a^{(\alpha)}(z)}^\infty t^{\Re\alpha-1} \exp(-t)\,dt = O\left\{(\xi_a^{(\alpha)}(z))^{\Re\alpha-1} \exp(-2\xi_a^{(\alpha)}(z))\right\} \tag{32}$$

uniformly with respect to z on K when a tends to infinity.

There exists a positive constant $D = D(K,\alpha)$ such that the inequality $\xi_a^{(\alpha)}(z) \geq Da^{1/2}$ holds when $z \in K$ and $a \geq a_0$. Then as a corollary of (31) and (32) we obtain that

$$I_a^{(\alpha)}(z) = O(a^{-1/2} \exp(-2z^{1/2}a^{1/2})) \tag{33}$$

uniformly on K when a tends to infinity.

Since $\exp(\tau_a^{(\alpha)}(z))(1+\sigma_a^{(\alpha)}(z))^{a+1} = O(\exp(2z^{1/2}a^{1/2}))$ uniformly on K when a tends to infinity, from (29), (30) and (33) it follows that

$$R_a^{(\alpha)}(z) = O(a^{-1/4}) \tag{34}$$

uniformly on K when a tends to infinity .

In view of (24) and (34) we can write that

$$A_a^{(\alpha)}(z) + R_a^{(\alpha)}(z) = \sqrt{\pi} z^{-1/4}\{1 + \rho_a^{(\alpha)}(z)\}, \tag{35}$$

where $\lim_{a\to\infty} \rho_a^{(\alpha)}(z) = 0$ uniformly on K.

Further (9) and (10) give that if $z \in K$ and $a \geq a_0$, then the following representations

$$\{\tau_a^{(\alpha)}(z)\}^\alpha = z^{\alpha/2} a^{\alpha/2}\{1 + \omega_a^{(\alpha)}(z)\} \tag{36}$$

and

$$\exp(-\tau_a^{(\alpha)}(z))(1 + \sigma_a^{(\alpha)}(z))^{-a-1} = \exp(z/2 - 2z^{1/2}a^{1/2})\{1 + \eta_a^{(\alpha)}(z)\} \tag{37}$$

hold and moreover $\lim_{a\to\infty} \omega_a^{(\alpha)}(z) = \lim_{a\to\infty} \eta_a^{(\alpha)}(z) = 0$ uniformly with respect to $z \in K$.

Then as a corollary of (6), (13), (35), (36) and (37) we have that the equality $\mu_a^{(\alpha)}(z) = 1+\omega_a^{(\alpha)}(z)+\eta_a^{(\alpha)}(z)+\rho_a^{(\alpha)}(z)+\omega_a^{(\alpha)}(z)\eta_a^{(\alpha)}(z)+\eta_a^{(\alpha)}(z)\rho_a^{(\alpha)}(z)+ \omega_a^{(\alpha)}(z)\rho_a^{(\alpha)}(z)+\omega_a^{(\alpha)}(z)\eta_a^{(\alpha)}(z)\rho_a^{(\alpha)}(z)$ holds when $z \in K$ and $a \geq a_0$ and therefore $\lim_{a\to\infty} \mu_a^{(\alpha)}(z) = 0$ uniformly on K.

So far, the main statement of the paper, namely the validity of equality (7) uniformly on every compact subset of the region $\mathbb{C}\setminus(-\infty, 0]$ is proved when $\Re\alpha \geq 1$. In order to prove that (6) holds for each $\alpha \in \mathbb{C}\setminus(-\infty,-1]$ we use the relation $(a+\alpha+1)\Psi_a^{(\alpha)}(z) = \Psi_a^{(\alpha+1)}(z) - \{\Psi_a^{(\alpha+1)}(z)\}'$, which is a corollary of (5). Indeed,

$$\{\Psi_a^{(\alpha+1)}(z)\}' = -(a+1)\int_0^\infty \frac{t^{a+\alpha+1}\exp(-t)}{(z+t)^{a+2}}\,dt$$

$$= \int_0^\infty t^{a+\alpha+1}\exp(-t)d\{(z+t)^{-a-1}\} = -(a+\alpha+1)\Psi_a^{(\alpha)}(z) + \Psi_a^{(\alpha+1)}(z).$$

References

[1] H. B a t e m a n, A. E r d é l y i (Ed-s). *Higher Transcendental Functions*, vol-s **I, II, III**, Mc Graw-Hill BC, New York, 1953.

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