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An Estimation of the Best Monotone Spline Approximation with the Averaged Moduli of Smoothness

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In this paper a Jackson-type estimation for the approximation of a monotone nondecreasing funcion f by monotone nondecreasing splines with equally spaced knots in the $L_p[0,1]$-norm $(1 \leq p \leq \infty)$ is obtained. The estimation involves the high order Sendov-Popov averaged moduli of smoothness of the derivative of f and is obtained for function f with a bounded and measurable derivative. The techniques of Chui, Smith and Ward are used. The result is a generalization of the results in [2].

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1. Introduction

For $1 \leq p < \infty$ let $L_p[0,1]$ denote the space of measurable functions whose p-th power is integrable and let $L_\infty[0,1]$ denote the space of bounded and measurable functions. Given $f \in L_p[0,1]$, define its r-th L_p-modulus of smoothness by

$$\omega_r(f,h)_{p[0,1]} \stackrel{\text{def}}{=} \sup\left\{ \|\Delta^r_{t,[0,1]} f(\cdot)\|_{p[0,1]} \ ; \ 0 \leq t \leq h \right\},$$

where

$$\Delta^r_{t,[0,1]} f(x) \stackrel{\text{def}}{=} \begin{cases} \sum_{i=0}^r (-1)^{r-i} \binom{r}{i} f(x+it) & if\ \ x, x+rt \in [0,1]; \\ 0 & \text{otherwise.} \end{cases}$$

Let $S(r,n)(r \geq 1)$ denote the space of all splines of order r on the $n+1$ equally spaced knots $\left\{\frac{i}{n}\right\}_{i=0}^n$, i.e. $s \in S(r,n)$, if s is a polynomial of degree $\leq r-1$ in each interval $\left[\frac{i}{n}, \frac{i+1}{n}\right]$ and $s^{(r-2)}$ is continuous in $[0,1]$. For $r=1$, s is a piecewise constant function without continuity at the knots.

If $f \in L_p[0,1]$ is monotone nondecreasing, denote

$$E_n^{\uparrow}(f,r)_{p[0,1]} \stackrel{\text{def}}{=} \inf\left\{\|f-s\|_{p[0,1]}\ ;\ s\in S(r,n)\,,\ s \text{ nondecreasing}\right\}.$$

The following two theorems were proved by Leviatan-Mhaskar [2].

Theorem 1. *If f possesses a continuous nonnegative derivative f' on [0,1], then there is a constant $c(r)$ depending only on $r \geq 2$ such that*

$$E_n^{\uparrow}(f,r)_{\infty[0,1]} \leq c(r)n^{-1}\omega_{r-1}(f',n^{-1})_{\infty[0,1]}\,.$$

Theorem 2. *Let $1 \leq p < \infty$. If f is the second primitive of $f'' \in L_p[0,1]$ and f is nondecreasing, then there is a constant $c(r)$ depending only on $r \geq 3$ such that*

$$E_n^{\uparrow}(f,r)_{p[0,1]} \leq c(r)n^{-2}\omega_{r-2}(f'',n^{-1})_{p[0,1]}\,.$$

For a function f bounded on [0,1] the local modulus of smoothness of order r at the point $x \in [0,1]$ is the function (see Definition 1.4 of [3]):

$$\omega_r(f,x;\delta) \stackrel{\text{def}}{=} \sup\left\{|\Delta^r_{h,[0,1]}f(t)|\ ;\ t,t+rh\in\left[x-\frac{r\delta}{2},x+\frac{r\delta}{2}\right]\right\}.$$

For $1 \leq p \leq \infty$ the r-th order averaged Sendov-Popov modulus of smoothness of a function f bounded and measurable on [0,1] is (see Definition 1.5 of [3])

$$\tau_r(f,\delta)_{p[0,1]} \stackrel{\text{def}}{=} \|\omega_r(f,\cdot;\delta)\|_{p[0,1]}\,.$$

The following properties of τ_r are used (see Theorem 1.5 and Property 5 of [3]). Let $1 \leq p \leq \infty$ and f is the primitive of $f' \in L_p[0,1]$, then there is a constant $c(r)$ which depends only on $r \geq 2$ such that

(1) $$\tau_r(f,\delta)_{p[0,1]} \leq c(r)\delta\omega_{r-1}(f',\delta)_{p[0,1]}\,.$$

Let f be measurable on [0,1] and k is integer. Then,

(2) $$\tau_r(f,k\delta)_{p[0,1]} \leq k^{r+1}\tau_r(f,\delta)_{p[0,1]}.$$

The main result of this paper is the following stronger estimation of the best monotone spline approximation.

Theorem 3. *Let $1 \leq p \leq \infty$. If f is the primitive of a bounded and measurable on [0,1] function f' and f is nondecreasing, then there is a constant $c(r)$ which depends only of $r \geq 2$ such that*

$$E_n^{\uparrow}(f,r)_{p[0,1]} \leq c(r)n^{-1}\tau_{r-1}(f',n^{-1})_{p[0,1]} \, .$$

Remark 1. For $p = \infty$ Theorem 3 coincides with Theorem 1 because of $\tau_r(f,\delta)_{\infty[0,1]} \equiv \omega_r(f,\delta)_{\infty[0,1]}$.

Remark 2. Theorem 2 follows from Theorem 3 because of (1).

In order to prove the main result we use some statements from [2].

Lemma 1. *Let f be continuously differentiable on [-1,1] and nondecreasing there. Then there is nondecreasing polynomial P on [-1,1] of degree $\leq r$ ($r \geq 1$) which interpolates f at 0 and 1 and such that*

$$\|f - P\|_{\infty[-1,1]} \leq c(r)\, \omega_r(f',1)_{\infty[-1,1]}.$$

This is Lemma 3.2(i) from [2].

Remark 3. This statement is valid for a nondecreasing function f which is the primitive of a bounded and measurable function f' (see the proof of Lemma 3.2(i) from [2]).

Lemma 2. *Let f be a nondecreasing function which is the primitive of a bounded and measurable on $[-1,1]$ function f'. For $r \geq 1$ there exists a nondecreasing continuous function g on [-1,1] such that g interpolates f at -1, 0 and 1 and has the properties:*

(*i*) *The restrictions of g to $[-1,0]$ and $[0,1]$ are polynomials of* deg $\leq r$;

(*ii*) $\|f - g\|_{\infty[-1,1]} \leq c(r)\, \omega_r(f',1)_{\infty[-1,1]}$;

(*iii*) $\displaystyle\sum_{k=1}^{r} |g^{(k)}(0+) - g^{(k)}(0-)| \leq c(r)\, \omega_r(f',1)_{\infty[-1,1]}$.

This is Theorem 3.1(i) from [2], according to Remark 3.

Lemma 3. *Let f be a nondecreasing function which is the primitive of a bounded and measurable on $[-2,2]$ function f' and let g_1 and g_1 be the piecewise polynomials guaranteed by Lemma 2 for the intervals $I = [-2,0]$ and $I = [0,2]$, respectively. Then,*

$$\sum_{k=1}^{r} |g_2^{(k)}(0+) - g_1^{(k)}(0-)| \leq c(r)\, \omega_r(f',1)_{\infty[-2,2]} \, .$$

This is Theorem 3.2(i) from [2], according to Remark 3.

The next lemma is similar to Lemma 2 and the proof runs along the lines of that of Lemma 2.

Lemma 4. *Let f be a nondecreasing function which is the primitive of a bounded and measurable on $[-m,l]$ (m and l natural) function f'. For $r \geq 1$ there exists a nondecreasing continuous function g on $[-m,l]$ such that g interpolates f at $-m, 0$ and l and has the properties:*

(*i*) *The restrictions of g to $[-m,0]$ and $[0,l]$ are polynomials of* deg $\leq r$;

(*ii*) $\|f-g\|_{\infty[-m,l]} \leq c(r)\,\omega_r(f',1)_{\infty[-\max\{m,l\},\max\{m,l\}]}$;

(*iii*) $\displaystyle\sum_{k=1}^{r} |g^{(k)}(0+)-g^{(k)}(0-)| \leq c(r)\,\omega_r(f',1)_{\infty[-\min\{m,l\},-\min\{m,l\}]}$.

We use also the following fundamental lemma of Chui, Smith and Ward (see [1]).

Lemma CSW. *Let $r \geq 2$ and $d = 4r^2$ and let g be a nondecreasing continuous function on $[-3d,3d]$, the restriction of which to $[-3d,0]$ and to $[0,3d]$ polynomials of degree $\leq r-1$. Then there is a nondecreasing spline s of order r and knots at the integers such that*

$$\|s-g\|_{p[-3d,3d]} = \|s-g\|_{p[-d,d]} \leq c(r)\sum_{k=1}^{r-1} |g^{(k)}(0+)-g^{(k)}(0-)|.$$

2. Main result

Proof of Theorem 3. It suffices to prove the theorem for $n > 12d$, where $d = 4r^2$. Let $F(t) = f\left(\frac{t}{n}\right)$, $t \in [0,n]$, and let $m = 2\left[\frac{n}{6d}\right]$ (where [.] denotes the integral part). Denote $I_1 = [0,3d]$, $I_2 = [3d,6d], \dots, I_{m-1} = [3(m-2)d, 3(m-1)d]$ and $I_m = [3(m-1)d, n$. By Lemma 2, for each pair of intervals $I_{2j-1} \cup I_{2j}$, $j = 1,2,\dots,\frac{m}{2}-1$, there exists a monotone nondecreasing continuous function G_j interpolating F at $6(j-1)d$, $(6j-3)d$ and $6jd$, such that G_j is a polynomial of degree $\leq r-1$ on I_{2j-1} and on I_{2j}. Also,

$$\|F-G_j\|_{\infty(I_{2j-1}\cup I_{2j})} \leq c(r)\,\omega_{r-1}(F',1)_{\infty(I_{2j-1}\cup I_{2j})}$$

and

$$\sum_{k=1}^{r-1} |G_j^{(k)}((6j-3)d+) - G_j^{(k)}((6j-3)d-)| \leq c(r)\,\omega_{r-1}(F',1)_{\infty(I_{2j-1}\cup I_{2j})}.$$

Let us note that the constants in the inequalities are independent on the intervals. We note also that the length of I_m may be $> 3d$. This is the reason for the use of Lemma 4. By Lemma 4 for the last pair of intervals $I_{m-1} \cup I_m$, there exists a monotone nondecreasing continuous function $G_{\frac{m}{2}}$ interpolating F at $3(m-2)d$, $3(m-1)d$ and n, such that $G_{\frac{m}{2}}$ is a polynomial of degree $\le r-1$ on I_{m-1} and on I_m. Also,

$$(3) \qquad \|F - G_{\frac{m}{2}}\|_{\infty(I_{m-1}\cup I_m)} \le c(r)\,\omega_{r-1}(F',1)_{\infty(I_{m-2}\cup I_{m-1}\cup I_m)}.$$

and

$$(4) \quad \sum_{k=1}^{r-1} |G_{\frac{m}{2}}^{(k)}(3(m-1)d+) - G_{\frac{m}{2}}^{(k)}(3(m-1)d-)| \le c(r)\,\omega_{r-1}(F',1)_{\infty(I_{m-1}\cup I_m)}.$$

In the right hand side of (3) and (4) we use that $3d \le n - 3(m-1)d \le 6d$ and I_{m-2} exists because $m \ge 2$ $(n > 12d)$.

Now by Lemma 3, we may define a continuous nondecreasing function $G = G_j$ on $I_{2j-1} \cup I_{2j}$, $j = 1, 2, ..., \frac{m}{2}$ such that

$$(5) \qquad \begin{array}{ll} \|F - G\|_{\infty(I_{2j-1}\cup I_{2j})} \le c(r)\,\omega_{r-1}(F',1)_{\infty(I_{2j-1}\cup I_{2j})}, & j < \frac{m}{2}; \\ \|F - G\|_{\infty(I_{m-1}\cup I_m)} \le c(r)\,\omega_{r-1}(F',1)_{\infty(I_{m-2}\cup I_{m-1}\cup I_m)}, & j = \frac{m}{2}. \end{array}$$

and for $i = 1, 2, ..., m-1$

$$(6) \qquad \sum_{k=1}^{r-1} |G^{(k)}(3id+) - G^{(k)}(3id-)| \le c(r)\,\omega_{r-1}(F',1)_{\infty(I_i\cup I_{i+1})}.$$

Applying Lemma CSW to each pair of intervals $I_i \cup I_{i+1}$, we have a spline S_i on $I_i \cup I_{i+1}$ such that $S_i = G$ outside $[(3i-1)d, (3i+1)d]$ and by (6),

$$(7) \qquad \|S_i - G\|_{\infty[(3i-1)d,(3i+1)d]} \le c(r)\,\omega_{r-1}(F',1)_{\infty(I_i\cup I_{i+1})}.$$

We define the spline

$$S(t) \stackrel{\text{def}}{=} \begin{cases} S_i(t) & if\ t \in [(3i-1)d, (3i+1)d]\,,\ i = 1, 2, ..., m-1; \\ G(t) & \text{otherwise.} \end{cases}$$

Now we let $s(t) = S(nt)$, $0 \le t \le 1$. Then $s \in S(r,n)$, s is monotone nondecreasing and using (5), (7) and (2), we obtain

$$\|f - s\|^p_{p[0,1]} = \frac{1}{n}\|F - S\|^p_{p[0,n]} \le \frac{2^p}{n}\left(\|F - G\|^p_{p[0,n]} + \|G - S\|^p_{p[0,n]}\right)$$
$$\le \frac{2^p}{n}\left(\sum_{j=1}^{\frac{m}{2}-1}\|F - G\|^p_{p(I_{2j-1}\cup I_{2j})} + \|F - G\|^p_{p(I_{m-1}\cup I_m)} + \sum_{i=1}^{m-1}\|G - S\|^p_{p(I_i\cup I_{i+1})}\right)$$
$$\le \frac{2^p}{n}c(r)\left(\sum_{j=1}^{\frac{m}{2}-1}\int_{I_{2j-1}\cup I_{2j}} \omega^p_{r-1}(F', 1)_{\infty(I_{2j-1}\cup I_{2j})}dt\right.$$
$$\left. + \int_{I_{m-1}\cup I_m} \omega^p_{r-1}(F', 1)_{\infty(I_{m-2}\cup I_{m-1}\cup I_m)}dt + \sum_{i=1}^{m-1}\int_{I_i\cup I_{i+1}} \omega^p_{r-1}(F', 1)_{\infty(I_i\cup I_{i+1})}dt\right)$$
$$\le \frac{2^p}{n}c(r)\left(\sum_{j=1}^{\frac{m}{2}-1}\int_{I_{2j-1}\cup I_{2j}} \omega^p_{r-1}(F', t; c(r))dt + \int_{I_{m-1}\cup I_m} \omega^p_{r-1}(F', t; c(r))dt\right.$$
$$\left. + \sum_{i=1}^{m-1}\int_{I_i\cup I_{i+1}} \omega^p_{r-1}(F', t; c(r))dt\right) \le \frac{2^p}{n}c(r)\tau^p_{r-1}(F', c(r))_{p[0,n]}$$
$$= c(r)\left(n^{-1}\tau_{r-1}(f', c(r)n^{-1})_{p[0,1]}\right)^p \le c(r)\left(c(r)n^{-1}\tau_{r-1}(f', n^{-1})_{p[0,1]}\right)^p .$$

Therefore,

$$\|f - s\|_{p[0,1]} \le c(r)n^{-1}\tau_{r-1}(f', n^{-1})_{p[0,1]} .$$

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