Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol. 14, 2000, Fasc. 1-2

Extension of Group-Valued Function and Density of Sets in Topological Group

S.K. Kundu

Presented by P. Kenderov

1. Introduction

This paper is about the extension of a topological group-valued function $m : S \to \hat{G}$ satisfying some given conditions, where G and $\hat{G}$ are topological groups - the former locally compact Hausdorff and the latter complete, commutative, and normally preordered (see Def.2.2) and S is the σ-ring generated by compact subsets of G. We obtain an extensiion m^* of m on $P(G)$ - the power class of G, and study the properties of m^* which are in tune with the order-structure of $\hat{G}$. Certain real-valued functions f are associated with $\hat{G}$ in a natural manner (see Th.2.3). We also study these functions.

In the concluding section we utilize m^* and f to define density of set in G. The notion of density of sets is rooted in the classical analysis. It has been studied extensively in the context of metric space [2], measure space [7], Romanovsky space [9] and topological group [6]. Bhakat and Kundu [1] have considered the idea in a uniform space with respect to a positive outer measure μ which besides satisfying a number of conditions { [1], §2.8 }, has been made to satisfy the Vitaly axiom and regularity conditions { [1], §§2.6 and 2.7 }.

We have made m^* to satisfy the Vitali axiom which is an adaptation from that in [1]. We have found a necessary and sufficient condition for the equality $D^*(E,x) = D_*(E,x)$ to hold, where $E \subset G, X \in G$.

In our work the order-structure of $\hat{G}$ plays a crucial role, and as such, we have been based on [8] for some results and definitions which we mention in the following section for ready reference.

2. Definitions and known results

A pre-order $'\leq'$ on a set E is a reflexive and transitive relayion; if, in addition $'\leq'$ is antisymmetric, it is called an order. A set E equipped with a preorder (order) is called a preordered (ordered) set. We write $b \geq a$ iff $a \leq b$.

Let E be a preoredered set and $F \subset E$; F is called increasing or decraesing according as $(b \geq a, a \in F) \Rightarrow b \in F$ or $(c \leq d, d \in F) \Rightarrow c \in F$. It is easy to see that $\cup_\alpha F_\alpha$, $\alpha \in \Gamma$ is increasing or decreasing for every α. The same is true for $\cap_\alpha F_\alpha$, $\alpha \in \Gamma$.

Given a set $H \subset E$, there exists a unique increasing set $i(H)$ and a unique decreasing set $d(H)$ containing H. It is, in fact, the smallest increasing or decreasing set containing H.

An element $x \in i(H)[x \in d(H)]$ iff it is possible to choose $y \in H$ such that $x \geq y[x \leq y]$.

Let E be a topological space equipped with a preorder; the preorder is called closed if its graph in E^2 is a closed subset. In fact, if E is Hausdorff and we define $x \leq y$ iff $x = y$, then $' \leq'$ is a closed preorder on E.

Theorem 2.1. {[8], Ch.1, Prop.1}. *The preorder of a topological space E is closed if and only if for every two points $a, b \in E$ such that $a \leq b$ is false, it is possible to determine an increasing neighbourhood V of $'a'$ and a decreasing neighbourhood W of $'b'$ which are disjoint. If the preorder of E is closed, then for every point $a \in E$, the sets $d(a)$ and $i(a)$ are closed.*

Theorem 2.2. {[8], Ch.1, Prop.2}. *Every topological space E equipped with a closed preorder is a Hausdorff space.*

Definition 2.1. {[8], Ch.1, §2}. A topological space E is said to be normally preordered if, for every two disjoint closed sets F_0 and F_1 of E, F_0 being decreasing and F_1 increasing, there exists two disjoint open sets $A_0 \supset F_0, A_1 \supset F_1$ such that A_0 is decreasing and A_1 increasing.

Theorem 2.3. {[8], Ch.1, Th.1}. *A topological space E equipped with a preorder is normally preordered, if and only if for every pair of disjoint closed sets F_0 and F_1 of E, F_0 decreasing and F_1 increasing, there exists on E a continues, increasing real-valued function f such that $f(x) = 0$ for $x \in F_0, f(x) = 1$ for $x \in F_1$ and $0 \leq f(x) \leq 1$ for $x \in E$.*

Definition 2.2. Let E be a nonempty preordered set; E is called a normally preordered topological group if:

i) E is an additive topological equipped with a closed preorder;

ii) E is normally preordered; and

iii) for any pair of elements $a, b \in E (a \leq b) \rightarrow (a + c \leq b + c)$ for every $c \in E$.

For instance, $\mathbf{R}^n$ is a normally preordered topological group, if one defines the closed preorder as $(x_i) \leq (y_i)$ iff $x_i = y_i$, $1 \leq i \leq n$. We call $E^+ = \{x \in$

$E \mid x \geq 0\}$ the positive cone of E; E^+ is called generating {[12], Ch. 12, §89, p.96}, if every element $s \in E$ can be expressed as $s = u - \nu$, $u, \nu \in E^+$ in at least one way.

Thus, in view of Theorem 2.2, $\hat{G}$ is Hausdorff and the sets $d(a)$ and $i(a)$ are closed sets because of Theorem 2.1, $a \in \hat{G}$. We note that $d(\hat{0}) = \{x \in \hat{G} \mid x \leq \hat{0}\}$.

Lemma 2.1. *Let $D \subset \hat{G}^+ - \{\hat{0}\}$ be a finite set; then $i(D)$ is a closed subset of $\hat{G}$ and $i(D) \cap d(\hat{0}) = \emptyset, \hat{0}$ being the zero-element of $\hat{G}$.*

The proof is omitted.

Let $\xi_D(\hat{G})$ be the class of all continuous, increasing real-valued functions f on $\hat{G}$, relative to D, such that $f(x) = 0$ if $x \in d(\hat{0})$, $f(x) = 1$ if $x \in i(D)$ and $0 \leq f(x) \leq 1$ for all $x \in \hat{G}$. $\xi_D(\hat{G})$ is nonempty (see Theorem 2.3).

Lemma 2.2. *Let U be an increasing (resp. decreasing) neighbourhood of $\hat{0}$ in $\hat{G}$; then for any $x \in \hat{G}, x + U$ is an increasing (resp. decreasing) neighbourhood of x. Further, $A + U$ is an increasing (resp. decreasing) neighbourhood of $A \subset \hat{G}$.*

The proof is omitted.

Lemma 2.3. *Let decreasing neighbourhoods form a neighbourhood basis at $\hat{0}$, and $x_2 \in \hat{G}^+$; then for any $f \in \xi_D(\hat{G})$ and $x_1 \in \hat{G}$,*

$$f(x_1 + x_2) = f(x_1) + f(x_2).$$

Proof. Choose $\epsilon > 0$ and decreasing neighbourhoods U, U_1 and U_2 of $\hat{0}$ such that

$$(2.1) \qquad |f(x_1 + x_2) - f(y)| < \frac{\epsilon}{3} \quad \text{whenever} \quad y \in x_1 + x_2 + U,$$

$$(2.2) \qquad |f(x_1) - f(y)| < \frac{\epsilon}{3} \quad \text{whenever} \quad y \in x_1 + U_1,$$

and

$$(2.3) \qquad |f(x_2) - f(y)| < \frac{\epsilon}{3} \quad \text{whenever} \quad y \in x_2 + U_2.$$

By Lemma 2.2, $x_2 + U_2$ is a decreasing neighbourhood of x_2, and since $\hat{0} \leq x_2 \in x_2 + U_2$ it follows that

$$(2.4) \qquad \hat{0} \in x_2 + U_2.$$

Set $W = U \cap U_1$; then W is decreasing and since $x_1 \leq x_1 + x_2 \in x_1 + x_2 + W \subset x_1 + x_2 + U$ it follows that $x_1 \in x_1 + x_2 + U$. Accordingly,

$$(x_1 + U_1) \cap (x_1 + x_2 + U) \neq \emptyset.$$

Choose $y \in (x_1 + U_1) \cap (x_1 + x_2 + U)$. Now

$$f(x_1+x_2)-f(x_1)-f(x_2)| \leq |f(x_1+x_2)-f(y)|+|f(y)-f(x_1)|$$
$$+|f(x_2)-f(\hat{0})|+|f(\hat{0})|<\epsilon,$$

by (2.1)-(2.4). Since $\epsilon > 0$ is arbitrary,

$$f(x_1+x_2)=f(x_1)+f(x_2). \quad \blacksquare$$

Corollary 2.1. *If $x, y, \in \hat{G}^+, f \in \xi_D(\hat{G})$, then*

$$f(x+y)=f(x)+f(y).$$

Corollary 2.2. *If $\hat{G}^+$ is generating, then*

$$f(x+y)=f(x)+f(y), \quad x,y \in \hat{G}^+, \; f \in \xi_D(\hat{G}).$$

This follows from Lemma 2.3.

Corollary 2.3. *If $\hat{G}^+$ is generating, then*

$$f(x-y)=f(x)-f(y), \quad x,y \in \hat{G}^+, \; f \in \xi_D(\hat{G}).$$

3. Extension of m

Let G be an additive locally compact Hausdorff topological group, s be the σ-ring generated by the compact subsets of G and $K \subset S$ be the class of open sets. For $E \subset G$, define

$$S(E)=\{F|E \subset F \in K\}.$$

Clearly, $S(E)$ is a directed set - $F_1 \geq F_2$ iff $F_1 \subset F_2$ where $F_1, F_2 \in S(E)$.

Let $m : S \to \hat{G}$ be a monotone increasing function satisfying the following conditions {[5], §2 }:

i) $m(\emptyset)=\hat{0}$;

ii) $m(a+F) = m(a)+m(F)$ for every compact set F [we write $m(a)$ for $m(\{a\})$]

iii) Given a neighbourhood U of $\hat{0}$ and $A \in S$, $m(B)-m(A) \in U$ whenever $B \subset A$ and $B \in S$.

Definition 3.1. {[10], §0.2} For any index set $I, x : I \to \hat{G}$ and $y \in \hat{G}, y = \sum x_l$ iff for every neighbourhood U of y there exists a finite set $J \subset I$ such that $'$ is finite and $J \subset J' \subset I \Rightarrow \sum_{l \in J} x_l \in U$.

Definition 3.2. {[11]} For any $E \subset G$ we define

$$m^*(E) = \lim m(D), D \in S(E).$$

Since $\hat{G}$ is complete, the limit exists uniquely.

Definition 3.3. {[10], §2.1} A set E is m^*-measurable iff $E \subset G$ and, for every $T \subset G, m^*(T) = m^*(T \cap E) + M^*(T \setminus E)$.

Let $T(m)$ denote the set of all m^*-measurable sets.

Definition 3.4. {[5], §3} Two sets E_1 and E_2 are m^*-separated iff $E_1 \subset G$, $E_2 \subset G$ and given any neighbourhood U of $\hat{0}$ there exist open sets A_1 and A_2 in S, $A_1 \supset E_1 supset A_2 \supset E_2$ such that $m^*(A_1 \cup A_2) \in U$.

Lemma 3.1. *For any $A \in S, m^*(A) = m(A)$, and in particular, $m^*(\emptyset) = 0$.*

Proof. Let U be an arbitrary neighbourhood of $\hat{0}$ in $\hat{G}$; choose symmetric neighbourhoods U_1 and U_2 of $\hat{0}$ such that $U_1 + U_2 \subset U$. There exists $D_0 \in S(A)$ such that

$$m(D) \in m^*(A) + U_1.$$

Whenever $A \subset D \subset D_0$. Further, in view of axiom (iii),

$$m(D) - m(A) \in U_2.$$

So, $m^*(A) - m(A) = m^*(A) - m(D) + m(D) - m(A) \in U_1 + U_2 \subset U$. Since $\hat{G}$ is Hausdorff, it follows that

$$m^*(A) = m(A)$$

and when $A = \emptyset, m^*(\emptyset) = 0$. ■

Lemma 3.2. *m^* is monotone increasing on $P(G)$.*

Proof. Let $E, F \in \mathbf{P}(G)$ and $E \subset F$; suppose $m^*(E) \leq m^*(F)$ is false. By Theorem 2.1, there exists an increasing neighbourhood U of $M^*(E)$ and a decreasing neighbourhood V of $m^*(F)$ such that

$$U \cap V = \emptyset \tag{3.1}$$

However,

$$m^*(E) = \lim m(D),\ D \in S(E)$$

and

$$m^*(F) = \lim m(D),\ D \in S(F).$$

Accordingly, there exists $\bar{D} \in S(E), \tilde{D} \in S(F)$ such that

$$\begin{cases} m(D_1) \in U & \text{whenever} \quad E \subset D_1 \subset \bar{D} \\ m(D_2) \in V & \text{whenever} \quad F \subset D_2 \subset \tilde{D}. \end{cases} \tag{3.2}$$

Since $E \subset D_1 \cup D_2 \subset \bar{D} \cap \tilde{D} \subset \bar{D}($ or$\tilde{D})$ and $\bar{D} \cap \tilde{D} \in S(E)$ it follows, in view of (3.2), that $m(D_1 \cap D_2) \in U$. Further, m being monotone, U increasing and

$$m(D_2) \geq m(D_2 \cap D_2) \in U$$

$$\Rightarrow m(D_2) \in U \Rightarrow m(D_2) \in U \cap V.$$

This contradicts (3.1). Hence the lemma is proved. ■

Remark 3.2. $A \subset G \Rightarrow m^*(A) \geq 0$.

Lemma 3.3. *m^* is countably subadditive on $\mathbf{P}(G)$.*

Proof. Let $\{E_n\} \subset \mathbf{P}(G)$ be an arbitrary sequence with $E = \bigcup_{n=1}^{\infty} E_n$. Suppose

$$m^*(E) \leq \sum_{i=1}^{\infty} m^*(E_i) \tag{3.3}$$

is false. There exist, in view of Theorem 2.1, an increasing neighbourhood U of $m^*(E)$ and a decreasing neighbourhood V of $y = \sum_{i=1}^{\infty} m^*(E_i)$ such that

$$U \cap V = \emptyset. \tag{3.4}$$

Since

$$m^*(E) = \lim m(D),\ D \in S(E),$$

there exists $D_0 \in S(E)$ such that

$$m(D) \in U. \tag{3.5}$$

Whenever $E \subset D \subset D_0$, $D \in S(E)$.

Further, there exists a positive integer N such that

$$M^*(E_N) \leq \sum_{i=1}^{n} m^*(E_i) \in V,\ n \geq N.$$

This gives, as V is decreasing,

$$m^*(E_n) \in V,\ n \geq N.$$

Accordingly, there exists $D^* \in S(E_n)$ such that

$$m(D) \in V. \tag{3.6}$$

Whenever $E_n \subset D \subset D^*$, $n \geq N$.

Choose open sets $D_1, D_2 \in K$ such that

$$E \subset D_1 \subset D_0$$
$$E \subset D_2 \subset D^*$$

So,

$$E \subset D_1 \cap D_2 \subset D_0 \cap D^* \subset D_0$$

(3.7)

$$E_n \subset E \subset D_1 \cap D_2 \subset D_2 \subset D^*.$$

Now

(3.8) $$m^*(D_1 \cap D_2) = m(D_1 \cap D_2) \in U$$

by (3.5), (3.7) and Lemma 3.1.

Also,

(3.9) $$m^*(D_1 \cap D_2) = m(D_1 \cap D_2) \in V$$

by (3.6) and (3.7). Therefore,

$$m^*(D_1 \cap D_2) \in U \cap V$$

which contradicts (3.4).

This completes the proof. ■

Theorem 3.1. *If $\hat{G}$ is ordered, then m^* is countably additive on $T(m)$.*

Proof of this, being routine, is omitted.

Theorem 3.2. *Let $\hat{G}$ be such that the neighbourhood system $\hat{0}$ has a countable base consisting of decreasing neighbourhoods whose intersection is $\{\hat{0}\}$. Then for any $E \in \mathbf{P}(G)$ there exists $D \in S$, $E \subset D$ such that $m(D) = m^*(E)$.*

Proof. Let $\{U_n\}$ be a countable base at $\hat{0}$ consisting of decreasing neighbourhoods with
bigcap$_{n=1}^{\infty} U_n = \{\hat{0}\}$. So, for every n, there exists $F_n \in S(E)$ such that $m(E_n) \in m^*(E) + U_n$ whenever

$$E \subset E_n \subset F_n,\ E_n \in S(E),\ n = 1, 2, \ldots.$$

Set

$$D = \bigcap_{n=1}^{\infty} E_n,$$

then $D \in S$ and $e \subset D$. Therefore,

$$m(E) \leq m(D) \leq m(E_n) \in m^*(E) + U_n, \quad n = 1, 2, \ldots.$$

Since $m^*(E) + U_n$ is decreasing by Lemma 2.2 for every n, it follows that

$$m(D) \in m^*(E) + U_n, \quad n = 1, 2, \ldots.$$

Consequently,

$$m(D) - m^*(E) \in U_n, \quad n = 1, 2, \ldots,$$

and as such

$$m(D) - m^*(E) \in \bigcap_{n=1}^{\infty} U_n = \{\hat{0}\}.$$

Therefore, $m(D) = m^*(E)$. ■

4. Density of sets

In this section we assume the following:

i) The class of decreasing open neighbourhoods at $\hat{0}$ forms a neighbourhood basis at $\hat{0}$;

ii) $\hat{G}$ is ordered and $\hat{G}^+$ is generating;

iii) m^* satisfies the Vitali axiom which follows.

We consider the topological group G to be a uniform space $(G,\mathcal{U})$, where the uniformity $\mathcal{U}$ is generated by sets of the form

$$R_U = \{(x,y) \in G \times G | y - x \in U\},$$

U being an arbitrary neighbourhood of the identity element in G. Let $\mathcal{V}$ be a base of $\mathcal{U}$ consisting of closed and symmetric members of $\mathcal{U}$.

Definition 4.1. {[1], Def.2.1} Let $A \subset G$; if $A \times A \subset U$ for some $U \in \mathcal{V}$, we say that the diameter of A is less than U and write $\delta(A) < U$.

Definition 4.2. {[1], Def.2.2} Let $\{A_\alpha, \alpha \in \Gamma, \geq\}$ be a net of subset of G; if for every $U \in \mathcal{V}$ there exists $\alpha_0 \in \Gamma$ such that $\delta(A_\alpha) < U$ for all $\alpha \in \Gamma, \alpha \geq \alpha_0$, we say that the diameter of A_α tends to zero and write $\delta(A_\alpha) \to 0$.

Definition 4.3. {[1], Def.2.3} A net $\{A_\alpha, \alpha \in \gamma, \geq\}$ of subsets of G is said to converge to $x \in G$, if $x \in \bigcap_\alpha A_\alpha$ and $\delta(A_\alpha) \to 0$.

Definition 4.4. {[1], Def.2.4} For every $V \in \mathcal{V}$ and $x \in G$ we define

$$V^x = \{y \in G \,|\, (y,x) \in V\}$$

and call V^* a closed ball with center x and radius V.

Let $U, V \in \mathcal{V}$; we define $U \geq V$ iff $U \subset V$. It is easy to check that $(\mathcal{V}, \geq)$ is a directed set. For $V_0 \in \mathcal{V}$, we write

$$\mathcal{V}(V_0) = \{V \in \mathcal{V} \,|\, V \subset V_0\}.$$

Definition 4.5. {[1], Def.2.5} Let $E \subset U$ and let $\mathcal{F}$ be a family of closed balls of G. We say that $\mathcal{F}$ covers E in the sense of Vitali, if for every $x \in E$ there is a net $\{F_\alpha\}, \alpha \in \Gamma$ of closed balls such that $F_\alpha \to x$.

Vitali axiom. {[1], §2.6} *Let $\mathcal{F}$ be a family of closed balls in G which covers $E \subset G$ in the sense of Vitali; then for every neighbourhood U of $\hat{0}$ in $\hat{G}$ there is a countable family of pairwise disjoint closed balls $\{F_i\} \subset \mathcal{F}$ such that*

$$\sum_{i=1}^{\infty} m^*(F_i) + m^*(E) \in U.$$

Theorem 4.1. *Let $E \subset G$ and $\mathcal{F}$ be a family of closed balls in G which covers E in the sense of Vitali. Then for a neighbourhood U of $\hat{0}$ in $\hat{G}$ there is a finite family of pairwise disjoint closed balls $\{F_{j_i} : 1 \leq i \leq n\}$ in $\mathcal{F}$ such that*

$$\sum_{i=1}^{\infty} m^*(F_{j_i}) + m^*(E) \in U.$$

Proof. Choose neighbourhoods $V_i, 1 \leq i \leq 3$ of $\hat{0}$ such that

$$\sum_{i=1}^{3} V_i \subset U. \tag{4.1}$$

By the Vitali axiom, there exists a countable family of pairwise disjoint closed balls $\{F_i\}, i = 1, 2, 3, \dots$ such that

$$\sum_{i=1}^{\alpha} m^*(F_i) + m^*(E) \in V_1. \tag{4.2}$$

Let $y = \sum_{i=1}^{\infty} m^*(F_i)$; then for V_2 there exists [cf. Def.3.1] a finite set $J \subset N$ such that $J\prime$ is finite, and

$$J \subset J\prime \subset N \Rightarrow \sum_{i=1}^{r} m^*(F_{j_i}) \in y + V_2, \tag{4.3}$$

where we suppose $J\prime = \{J_i | i = 1, 2, \dots r\}$ and N is the set of natural numbers.

Choose decreasing open neighbourhood $W_i, 1 \leq i \leq r$ such that

$$\sum_{i=1}^{r} W_i \in V_3. \tag{4.4}$$

Now,

$$m^*(E \cap F_{j_i}) \leq m^*(Fji) \in m^*(Fji) + W_i, \quad 1 \leq i \leq r,$$

by Lemma 3.2; however $m^*(Fji) + W_i \quad 1 \leq i \leq r$ are decreasing (cf. Lemma 2.2) and so,

$$m^*(E \cap F_{j_i}) - m^*(F_{j_i}) \in W_i, \quad 1 \leq i \leq r.$$

Summing over i, we get

$$\sum_{i=1}^{r} m^*(E \cap F_{j_i}) \in \sum_{i=1}^{r} m^*(F_{j_i}) \sum_{i=1}^{r} W_i \subset -m^*(E) + U,$$

by using (4.1)-(4.4). Therefore,

$$\sum_{i=1}^{r} m^*(E \cap F_{ji}) + m^*(E) \subset U.$$

This proves the theorem. ∎

Definition 4.6. {[1], Def.4.1} Let $E \subset G$, $c \in G$, $V \in \mathcal{V}$ and $f \in \xi_D(\hat{G})$. Write $\Delta(x,v) = \{W^x | W \in \mathcal{V}(v)\}$. Define

$$D^*(E,x;v) = \sup\left\{ \frac{f[m^*(E \cap W^*)]}{f[m^*(W^*)]} \middle| \; W^x \in \Delta(x,v) \right\}$$

and

$$D_*(E,x;v) = \inf\left\{ \frac{f[m^*(E \cap W^*)]}{f[m^*(W^*)]} \middle| \; W^x \in \Delta(x,v) \right\}$$

$$\left[\text{If} \quad f[m^*(W^*)] = 0, \quad \text{we take} \frac{f[m^*(E \cap W^*)]}{f[m^*(W^*)]} = 0 \right].$$

Also, define

$$D^*(E,x) = \inf\{D^*(E,x;V) \quad | \quad V \in \mathcal{V}$$

and

$$D_*(E,x) = \sup\{D^*(E,x;V) \quad | \quad V \in \mathcal{V}.$$

$D^*(E,x)$ and $D_*(E,x)$ are called respectively the upper and lower density of E at x.

It is clear that

$$0 \leq D_*(E,x) \leq D^*(E,x) \leq 1.$$

If $D_*(E,x) = D^*(E,x) = 1$, then x is called density point of E; on the other hand, if $D_*(E,x) = D^*(E,x) = 0$, x is called a dispersion point of E.

Theorem 4.2. *The functions $D_*(E,x)$ and $D^*(E,x)$ are monotone increasing and finitely subadditive for any fixed $x \in G$ and $E \subset G$.*

Proof. Let $E, F \subset G$, $E \subset F$ and $x \in G$. Since m^* is monotone increasing, $m^*(E \cap W^x) \leq m^*(F \cap W^x), W^x \in \Delta(x,v), V \in \mathcal{V}$. Consequently,

$$\frac{f[m^*(E \cap W^*)]}{f[m^*(W^*)]} \leq \frac{f[m^*(F \cap W^*)]}{f[m^*(W^*)]} \leq D^*(E,x;v)$$

for every $V \in \mathcal{V}$, $f \in \xi_D(\hat{G})$, and hence

$$D^*(E,x;v) \leq D^*(F,x;v)$$

for every $V \in \mathcal{V}$. This gives

$$D^*(E,x) \leq D^*(F,x).$$

Likewise,

$$D_*(E,x) \leq D_*(F,x).$$

Further, m^* is subadditive and so,

$$f[m^*(E \cup F) \cap W^x] \leq f[m^*(E \cap W^x)] + f[m^*(F \cap W^x)],$$

by Corollary 2.2, $E, F \subset G, \quad W^x \in \Delta(x,v), \quad V \in \mathcal{V}$.

It follows, taking sup over $W^x \in \Delta(x,v)$, that

$$D^*(E \cup F, x; v) \leq D^*(E,x;v) + D^*(F,x;v), \quad V \in \mathcal{V}.$$

This leads to

$$D^*(E \cup F, x) \leq D^*(E,x) + D^*(F,x),$$

on taking inf over $V \in \mathcal{V}$.

Similarly,

$$D_*(E \cup F, x) \leq D_*(E,x) + D^*(F,x).$$

This proves the theorem. ■

Lemma 4.1. *Let $V_1, V_2 \in \mathcal{V}$ and $V_1 \subset V_2$; then*

i) $D^*(E,x;v_1) \leq D^*(E,x;v_2)$

ii) $D_*(E,x;v_1) \leq D_*(E,x;v_2)$

for any $x \in G$ and $E \subset G$.

The proof is omitted.

Theorem 4.3. *A necessary and sufficient condition that $D^*(E,x) = D_*(E,x)$ for any $x \in G$ and $E \subset G$ is that given $\epsilon > 0$ there exists W in $\mathcal{V}$ such that*

$$D^*(E,x;W) - D_*(E,x;W) < \epsilon.$$

Proof. <u>Sufficiency:</u> Let the condition hold. Then,

$$D^*(E,x) \leq D^*(E,x;W) < D_*(E,x;W) + \epsilon \leq D_*(E,x) + \epsilon$$

$$\Rightarrow 0 \leq D^*(E,x) - D_*(E,x) < \epsilon.$$

Since ϵ is arbitrary,

$$D^*(E,x) = D_*(E,x).$$

<u>**Necessity:**</u> Choose $\epsilon > 0$, and suppose that $D^*(E,x) = D_*(E,x)$. There exists $V_1, V_2 \in \mathcal{V}$ such that

$$D^*(E,x) + \epsilon/2 > D^*(E,x;V1),$$

$$D_*(E,x) - \epsilon/2 < D_*(E,x;V2),$$

choose $W \subset V_1 \cap V_2, \quad W \in \mathcal{V}$.

By the preceding lemma,

$$D^*(E,x) + \epsilon/2 > D^*(E,x;V1) \geq D^*(E,x;W),$$

$$D_*(E,x) - \epsilon/2 < D_*(E,x;V2) \leq D^*(E,x;W).$$

Therefore,

$$D_*(E,x) - \epsilon/2 < D_*(E,x;W) \leq D^*(E,x;W) < D^*(E,x) + \epsilon/2$$

$$\Rightarrow \qquad D^*(E,x;W) - D_*(E,x;W) < \epsilon.$$

This completes the proof. ■

Lemma 4.2. *Let $E \subset G$, $x \in G$ and $0 < \lambda < 1$. If $D_*(E,x) < \lambda$, then there is a net $\{F_V^x \mid V \in \mathcal{V}(V_0)\}$ of closed balls with centre x which converges to x, and*

$$f[m^*(E \cap F_V^x)] < \lambda f[m^*(F_V^x)]$$

for all $V \in \mathcal{V}(V_0)$, where V_0 is a fixed member of $\mathcal{V}$.

Proof. Choose $U \in \mathcal{V}$; then there exists $V_0 \in \mathcal{V}$ such that $V_0 \cdot V_0 \subset U$ {[4], Ch.6, Th.2}. Take $V \in \mathcal{V}_{\prime}$; consequently,

$$D_*(E,x;V) \leq D_*(E,x) < \lambda.$$

So, there exist closed balls $F_V^x \in \Delta(x,V), V \in \mathcal{V}$ with x as centre such that

$$f[m^*(E \cap F_V^x)] < \lambda f[m^*(F_V^x)].$$

Now we consider the net of the closed balles $\{F_V^x\}$ each with centre x and $V \in \mathcal{V}(V_0)$. This net has the desired property. For, $F_V^x \subset V^x \Rightarrow F_V^x \times F_V^x \subset V * x \times V^x \subset V \cdot v \subset V_0 \cdot V_0 \subset U$, and so,

$$\delta(F_V^x) < U$$

and $x \in F_V^x$.

This proves the lemma. ■

Lemma 4.3. *Let $E \subset G$, $x \in G$ and $0 < \lambda < 1$. If $D^*(E,x) > \lambda$, then there is a net $\{F_V^x \mid V \in \mathcal{V}(V_0)\}$ of closed balls with centre x which converges to x, and*

$$f[m^*(E \cap F_V^x)] > \lambda f[m^*(F_V^x)]$$

for all $V \in \mathcal{V}(V_0)$, where V_0 is a fixed member of $\mathcal{V}$.

The proof is identical to that of Lemma 4.1.

Theorem 4.4. *Let $E \subset G$; then almost all points of E are density points of E.*

Proof. Let W be an arbitrary neighbourhood of $\hat{0}$ in $\hat{G}$; choose a neighbourhood V of $\hat{0}$ such that

$$V - V \in W.$$

Choose a decreasing neighbourhood U of $\hat{0}$ such that

$$U \subset V.$$

Let $\{\lambda_n\}$ be a sequence of positive real numbers such that $\lambda_n \to 1$ as $n \to \infty$. Set

$$A_n = \{x \in E \mid D_*(E,x) < \lambda_n\},$$

and choose $x \in A_n$; in view of Lemma 4.2, there exsists a net of closed balls $\{F_V^x \mid V \in \mathcal{V}(V_0)\}$ with centre x which converges to x. We, therefore, obtain a family of closed balls $\{F_V^x\}$, $V \in \mathcal{V}(V_0)$ corresponding to all points $x \in A_n$ which covers A_n in the sensen of vitali. Accordingly, by theorem 4.1 there exists a finite pairwise disjoint sequence of closed bals

$$\{F_{V_1}^{x_1}, F_{V_2}^{x_2}, \ldots F_{V_n}^{x_n}\} \subset \{F_V^x\}, \quad x_i, x \in A_n, \quad 1 \le i \le n$$

such that

$$\sum_{i=1}^{n} m^*(F_{V_i}^{x_i}) + m^*(A_n) \in U$$

and as such

$$\sum_{i=1}^{n} m^*(A_n \cap F_{V_i}^{x_i}) - m^*(A_n) \in U.$$

For some $u_1, u_2 \in U$ we obtain, therefore,

$$\sum_{i=1}^{n} m^*(F_{V_i}^{x_i}) = -m^*(A_n) + u_1 \tag{4.5}$$

$$(4.6) \qquad \sum_{i=1}^{n} m^*(A_n \cap F_{V_i}^{x_i}) = m^*(A_n) + u_2.$$

Because of monotonicity of m^* we have from (4.5) and (4.6)

$$\begin{aligned} m^*(A_n) = u_2 &= \sum_{i=1}^{n} m^*(A_n \cap F_{V_i}^{x_i}) \leq \sum_{i=1}^{n} m^*(F_{V_i}^{x_i}) = -m^*(A_n) + u_1 \\ &\Rightarrow \ m^*(A_n) \leq 2m^*(A_n) \leq u_1 - u_2 \\ &\Rightarrow \ m^*(A_n) \in U - U \subset V - V \subset W. \end{aligned}$$

Since $\mathcal{G}$ is Hausdorff, it follows that

$$m^*(A_n) = 0.$$

If A is the set of points in E at which the lower density is less than unity, then

$$A = \bigcup_{n=1}^{\infty} A_n.$$

So,

$$\begin{aligned} & m^*(A) \leq \textstyle\sum_{n=1}^{\infty} m^*(A_n) \\ \Rightarrow \ & m^*(A) \leq 0 \\ \Rightarrow \ & m^*(A) = 0. \end{aligned}$$

This proves the theorem. ■

Theorem 4.5. *Let E_1 and E_2 be two subsets of G, then almost all points of E_2 are dispersition points of E_1 and vice versa.*

Proof. Let W be an arbitrary neighbourhood of $\hat{0}$ in $\hat{G}$; choose a decreasing neighbourhood U of $\hat{0}$ such that $U + U \subset W$.

Let

$$A_n = \{x \in E_2 \mid D^*(E_1, x) > \frac{1}{n}, \quad n = 1, 2, \ldots.$$

If $x \in A$, then by Lemma 4.3, there exists a net of closed balls $\{F_V^x \mid V \in \mathcal{V}(V_0)\}$ with centre x which converges to x. Consequently, we obtain, as we did in the proof of the last theorem, a finite sequence of pairwise disjoint closed balls

$$\{F_{V_1}^{x_1}, F_{V_2}^{x_2}, \ldots F_{V_n}^{x_n}\} \subset \{F_V^x\}, \quad x_i, x \in A_n, \quad 1 \leq i \leq n$$

such that

$$\sum_{i=1}^{n} m^*(F_{V_i}^{x_i}) + m^*(A_n) \in U$$

and so

$$\sum_{i=1}^{n} m^*(A_n \cap F_{V_i}^{x_i}) - m^*(A_n) \in U.$$

Proceeding analogously as in the proof of the preceding theorem, we obtain

$$m^*(A_n) = 0$$

for every n.

Let

$$A = \{x \in E_2 \mid D^*(E_1, x) > 0\}.$$

Then

$$\begin{aligned} & A = \bigcup_{n=1}^{\infty} A_n \\ \Rightarrow\ & m^*(A) \leq \sum_{n=1}^{\infty} m^*(A_n) = 0 \quad \text{[Lemma 3.2]} \\ \Rightarrow\ & m^*(A) = 0. \end{aligned}$$

Therefore, the upper density of E_1 is zero almost everywhere in E_2; accordingly, lower density of E_1 is also zero almost everywhere in E_2. So, almost all points of E_2 are dispersion points of E_1. Similarly, one can show that almost all points of E_1 are dispersion points of E_2. This proves the theorem. ■

Theorem 4.6. *Let $E \subset G$ be arbitrary; then almost all points of G are either density points or dispersion points of E.*

Proof. Suppose $E_1 \subset E$ is the set of density points of E; by Theorem 4.4, $m^*(e \setminus E_1) = 0$. If $E_2 \subset E^c$ be the set of dispersion points of E, then because of Theorem 4.5,

$$m^*(E^c \setminus E_2) = 0.$$

Therefore,

$$\begin{aligned} G = E \cup E^c &= E_1 \cup (E \setminus E_1) \cup (E^c \setminus E_2) \cup E_2 \\ &= E_1 \cup E_2 \cup (E \setminus E_1) \cup (E^c \setminus E_2), \end{aligned}$$

where $m^*((E \setminus E_1) \cup (E^c \setminus E_2)) - 0$ and $E_1 \cup E_2$ is the set of either a density or a dispersion point. Hence the theorem is proved. ■

References

[1] P. C. Bhakta, S. K. Kundu, On μ-separated sets and density of sets in uniform space, *J. Indian Inst. Sci.*, **65(B)** (1984), 255-262.

[2] W. Eames, A local property of measurable sets, *Canad. J. Math.*, **12** (1960), 632.

[3] R. L. J e f f e r y, *Theory of Functions of a Real Variable*, Toronto (1953)

[4] J. L. K e l l e y, *General Topology*, D. Van Nostrand Inc., Princeton, New Jersy (1967).

[5] S. K. K u n d u, K. N. B h a u m i k, On some results in connection with topological group-valued functions, *Indian J. Pure Appl. Math.*, **22(5)** (1991), 379-387.

[6] B. K. L a h i r i, Density and approximation continuity in topological groups, *J. Ind. Math. Soc.*, **41** (1977), 129-141.

[7] N. F. G. M a r t i n, A topology for certain measure spaces, *Trans. Amer. Math. Soc.*, **112** (1964), 1.

[8] L. N a c h b i n, *Topology and Order*, D. Van Nostrand Inc., Princeton, New Jersy (1965).

[9] D. W. S o l m o n, On separation in measure and metric density in Romanovski space, *Duke Math. J.*, **36** (1969), 81.

[10] M. S i o n, *Theory of Semy-group Valued Measures*, Ser. Lecture Notes on Mathematics, Springer Verlag (1973).

[11] M. S i o n, Outer measures with values in a topological group, *Proc. Lond. Math. Soc.*, **XIX**, Part 1 (Jan. 1969), 89-106.

[12] A. C. Z a n n e n, *Riesz Spaces, II*, North-Holland Publ. Comp. (1983).

Dept of Mathematics, Jadavpur University
Calcutta - 700 032, INDIA

Received: 10.06.1996