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A Generalization of Obreshkoff-Ehrlich Method for Multiple Roots of Algebraic, Trigonometric and Exponential Equations

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In this paper methods for simultaneous finding all roots of generalized polynomials are developed. These methods are related to the case when the roots are multiple. They possess cubic rate of convergence and they are as labour-consuming as the known methods related to the case of polynomials with simple roots only.

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1. Introduction

After 1960 the question of simultaneous finding all roots (SFAR) of polynomials became very actual and it is considered by many authors. The reason of this interest is the better behaviour of the methods for SFAR with respect to the methods for individual search of the roots. Also these methods are very convenient for application on the computers with parallel processors. Methods for SFAR have a wider region of convergence and they are more stable. In several survey publications [1,2,3] this question is considered in details. The first methods for SFAR are related to the case when the roots are simple. The well-known method of Dochev [4] is for SFAR of algebraic polynomial with real and simple roots. The developments of this same method for the case of nonalgebraic polynomials (trigonometric, exponential and generalized) are performed in [5,6,7]. The classical method of Obreshkoff-Ehrlich [8] possessing cubic rate of convergence is also generalized [9]. Using the approach base on the divided differences with multiple knots, Semerdzhiev [10] generalized the method of Dochev to the case when the roots have arbitrary, but given multiplicities. The same question for the case of trigonometric and exponential polynomials is solved in [11,3]. The new methods preserve their quadratic rate of convergence. The method of Obreshkoff-Ehrlich is also generalized to the most general case [12,3] of polynomials upon some Chebyshev system, having multiple roots with given multiplicities. The rate of convergence is cubic but, unfortunately, this generalization requires at each iteration to calculate determinants which is a labour-consuming operation.

In this paper we develop a new method which is a generalization of the Obreshkoff-Ehrlich method for the cases of algebraic, trigonometric and exponential polynomials. This method has a cubic rate of convergence. It is efficient from the computational point of view and can be used for SFAR if the roots have known multiplicities. This new method in spite of the arbitrariness of multiplicities is of the same complexity as the methods for SFAR of simple roots. We do not use divided differences with multiple knots and this fact does not lead to calculation of derivatives of the given polynomial of higher order, but only of first ones. The results of this paper are published in shortened form as a preliminary communication in [13].

2. Algebraic polynomials

Let the algebraic polynomial

$$A_n(x) = x^n + a_1 x^{n-1} + \ldots + a_n \tag{1}$$

be given and let $x_1, x_2, \ldots, x_m$ be its roots with given multiplicities $\alpha_1, \alpha_2, \ldots, \alpha_m$ respectively, $(\alpha_1 + \alpha_2 + \ldots + \alpha_m = n)$. For SFAR of (1) the Ehrlich formula

$$x_i^{[k+1]} = x_i^{[k]} - A_n\left(x_i^{[k]}\right)\left[A_n'\left(x_i^{[k]}\right) - A_n\left(x_i^{[k]}\right)\sum_{j=1\,,j\neq i}^{n}\left(x_i^{[k]} - x_j^{[k]}\right)^{-1}\right]^{-1},$$

$$i = \overline{1,n}\,, k = 0,1,2,\ldots, \tag{2}$$

is well known. Formula (2) can be written in the form

$$x_i^{[k+1]} = x_i^{[k]} - A_n\left(x_i^{[k]}\right)\left[A_n'\left(x_i^{[k]}\right) - A_n\left(x_i^{[k]}\right)Q''^{[k]}\left(x_i^{[k]}\right)\left[2\,Q'^{[k]}\left(x_i^{[k]}\right)\right]^{-1}\right]^{-1},$$

$$i = \overline{1,n}\,, k = 0,1,2,\ldots, \tag{3}$$

where

$$Q^{[k]}(x) = \prod_{j=1}^{n}\left(x - x_j^{[k]}\right). \tag{4}$$

We define

$$x_i^{[k+1]} = x_i^{[k]} - \alpha_i A_n\left(x_i^{[k]}\right)\left[A_n'\left(x_i^{[k]}\right) - A_n\left(x_i^{[k]}\right) Q_i'^{[k]}\left(x_i^{[k]}\right)/Q_i^{[k]}\left(x_i^{[k]}\right)\right]^{-1},$$

$$i = \overline{1,m}\,, k = 0,1,2\ldots,$$

where

$$Q_i^{[k]}(x) = \prod_{j=1\,,j\neq\,i}^{m}\left(x - x_j^{[k]}\right)^{\alpha\,j}. \tag{6}$$

Theorem 1. *Let q, c and $d \overset{\text{def}}{=} \min_{i\neq j}|x_i - x_j|$ be real constants such that the following inequalities*

$$\begin{array}{l} 1 > q > 0, \quad c > 0, \quad d - 2c > 0, \\ 0 < c^2(n - 3\alpha_i) + (n + (3d-1)\alpha_i)c < d^2\alpha_i \;\; i = \overline{1,m} \end{array} \tag{7}$$

are satisfied. If the initial approximations $x_1^{[0]}, \ldots, x_m^{[0]}$ to the exact roots $x_1, \ldots, x_m$ of (1) are chosen so that the inequalities $\left|x_i^{[0]} - x_i\right| \leq cq$, $i = \overline{1,m}$ hold true, then for every natural k the inequalities

$$\left|x_i^{[k]} - x_i\right| \leq cq^{3^k}, \quad i = \overline{1,m} \tag{8}$$

also hold true.

Proof. We prove Theorem 1 by means of induction with respect to the number of the iterations. From the assumptions, we have that (8) are fulfiled for $k = 0$. Suppose that (8) hold true for some $k > 0$.

From (5) we obtain, for $i = \overline{1,m}$,

$$x_i^{[k+1]} - x_i = x_i^{[k]} - x_i - \alpha_i\left[A_n'\left(x_i^{[k]}\right)/A_n\left(x_i^{[k]}\right) - Q_i'^{[k]}\left(x_i^{[k]}\right)/Q_i^{[k]}\left(x_i^{[k]}\right)\right]^{-1}. \tag{9}$$

Using the representation

$$Q_i'^{[k]}\left(x_i^{[k]}\right)/Q_i^{[k]}\left(x_i^{[k]}\right) = \sum_{j=1\,,j\neq i}^{m}\alpha_j/\left(x - x_j^{[k]}\right),\; i = \overline{1,m},$$

from (9) we receive ($i = \overline{1,m}$)

$$x_i^{[k+1]} - x_i = x_i^{[k]} - x_i - \alpha_i\left[\sum_{j=1}^{m}\alpha_j/\left(x_i^{[k]} - x_j\right) - \sum_{j=1\,,j\neq i}^{m}\alpha_j/\left(x_i^{[k]} - x_j^{[k]}\right)\right]^{-1}.$$

Further transformations lead to

$$x_i^{[k+1]} - x_i = \left(x_i^{[k]} - x_i\right)\left[1 - \alpha_i\left[\alpha_i + \left(x_i^{[k]} - x_i\right)\sum_{j=1\,,j\neq i}^{m} \alpha_j\left(x_j - x_j^{[k]}\right)\right.\right.$$

$$\left.\left.\times\left[\left(x_i^{[k]} - x_j\right)\left(x_i^{[k]} - x_j^{[k]}\right)\right]^{-1}\right]^{-1}\right] = \left(x_i^{[k]} - x_i\right)^2\left[\alpha_i + \left(x_i^{[k]} - x_i\right)\right. \tag{10}$$

$$\left.\times\sum_{j=1\,,j\neq i}^{m} \alpha_j\left(x_j - x_j^{[k]}\right)\left[\left(x_i^{[k]} - x_j\right)\left(x_i^{[k]} - x_j^{[k]}\right)\right]^{-1}\right]^{-1}$$

$$\times\sum_{j=1\,,j\neq i}^{m} \alpha_j\left(x_j - x_j^{[k]}\right)\left[\left(x_i^{[k]} - x_j\right)\left(x_i^{[k]} - x_j^{[k]}\right)\right]^{-1}, \quad i = \overline{1,m}\,.$$

Obviously, we have

$$\left|x_i^{[k]} - x_j\right| \geq |x_i - x_j| - \left|x_i - x_i^{[k]}\right| \geq d - cq^{3^k} > d - c,\ i,j = \overline{1,m},\ i \neq j\,. \tag{11}$$

On the other hand,

$$\left|x_i^{[k]} - x_j^{[k]}\right| \geq \left|x_i^{[k]} - x_j\right| - \left|x_j - x_j^{[k]}\right| \geq d - 2cq^{3^k} > d - 2c,\ i,j = \overline{1,m},\ i \neq j\,. \tag{12}$$

Using (10)-(12) we find as a final result,

$$\left|x_i^{[k+1]} - x_i\right| \leq \left|x_i^{[k]} - x_i\right|^2\left[\alpha_i - \left|x_i^{[k]} - x_i\right|\sum_{j=1\,,j\neq i}^{m} \alpha_j\left|x_j - x_j^{[k]}\right|\left[\left|x_i^{[k]} - x_j\right|\right.\right.$$

$$\left.\left.\times\left|x_i^{[k]} - x_j^{[k]}\right|\right]^{-1}\right]^{-1}\sum_{j=1\,,j\neq i}^{m} \alpha_j\left|x_j - x_j^{[k]}\right|\left[\left|x_i^{[k]} - x_j\right|\left|x_i^{[k]} - x_j^{[k]}\right|\right]^{-1}$$

$$\leq c^3\left(q^{3^k}\right)^3\left[\alpha_i - c\sum_{j=1\,,j\neq i}^{m} \alpha_j\left[(d-c)(d-2c)\right]^{-1}\right]^{-1}\sum_{j=1\,,j\neq i}^{m} \alpha_j\left[(d-c)(d-2c)\right]^{-1}$$

$$= cq^{3^{k+1}}c^2\left[\alpha_i - c(n-\alpha_i)\left[(d-c)(d-2c)\right]^{-1}\right]^{-1}(n-\alpha_i)\left[(d-c)(d-2c)\right]^{-1}$$

$$< cq^{3^{k+1}}, \quad i = \overline{1,m},$$

which proves the theorem completely. ∎

Proposition 1. *In the case when* $\alpha_1 = \alpha_2 = ... = \alpha_m = 1$, *the corelations*

$$Q''^{[k]}\left(x_i^{[k]}\right)/Q'^{[k]}\left(x_i^{[k]}\right) = 2Q_i'^{[k]}\left(x_i^{[k]}\right)/Q_i^{[k]}\left(x_i^{[k]}\right), \quad i = \overline{1,m} \tag{13}$$

hold true.

Proof. In this case (6) reduces to (4) and we have

$$Q^{[k]}(x) = Q_i^{[k]}(x)\left(x - x_i^{[k]}\right), \quad i = \overline{1,m}. \tag{14}$$

Differentiating (14), we obtain

$$\begin{aligned} Q'^{[k]}(x) &= Q_i'^{[k]}(x)\left(x - x_i^{[k]}\right) + Q_i^{[k]}(x) \\ Q''^{[k]}(x) &= Q_i''^{[k]}(x)\left(x - x_i^{[k]}\right) + 2Q_i'^{[k]}(x), \quad i = \overline{1,m}. \end{aligned} \tag{15}$$

From (15) we receive (13). ∎

Proposition 1 shows that method (5) coincides with the method (3) and consequently with the method of Ehrlich in the case when $\alpha_i = 1$, $i = \overline{1,m}$.

Example 1. For the equation $A_6(x) = (x+2)^2(x-1)(x-3)^3 = 0$ at the initial approximations $x_1^{[0]} = -3$, $x_2^{[0]} = 0.1$ and $x_3^{[0]} = 4$ using the formula (5), we receive the roots with 18 decimal digits after only 4 iterations.

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$
0	−3.000000000000000000	0.100000000000000000	4.000000000000000000
1	−1.999423631123919310	1.035328192685374560	3.039859320046893320
2	−2.000000001433040880	0.999961906975802837	2.999995399844032900
3	−2.000000000000000000	1.000000000000005010	3.000000000000000070
4	−2.000000000000000000	1.000000000000000000	3.000000000000000000

3. Trigonometric Polynomials

For the trigonometric polynomial

$$T_n(x) = a_0/2 + \sum_{l=1}^{n}\left(a_l \cos(lx) + b_l \sin(lx)\right)$$

we suppose that at least one of the leading coefficients a_n and b_n is not zero and that it has real roots $x_1, ..., x_m$ with given multiplicities $\alpha_1, \alpha_2, ..., \alpha_m$ $(\alpha_1+\alpha_2+... +\alpha_m = 2n)$. Analoguously to (5), we can use the iteration method

$$x_i^{[k+1]} = x_i^{[k]} - \alpha_i T_n\left(x_i^{[k]}\right)\left[T_n'\left(x_i^{[k]}\right) - T_n\left(x_i^{[k]}\right) Q_i'^{[k]}\left(x_i^{[k]}\right)/Q_i^{[k]}\left(x_i^{[k]}\right)\right]^{-1}, \tag{16}$$

$$i = \overline{1,m},\ k = 0,1,2,..., \quad \text{where} \quad Q_i^{[k]}(x) = \prod_{j\neq i\,,\,j=1}^{m} \sin^{\alpha_j}\left(\left(x - x_j^{[k]}\right)/2\right).$$

Formula (16) for $\alpha_1 = \alpha_2 = ... = \alpha_m = 1$ coincides with the analogue of the Obreshkoff-Ehrlich formula [9] for trigonometric polynomials.

Theorem 2. *Let us denote* $d \stackrel{\text{def}}{=} \min_{i\neq j} |x_i - x_j|$. *Let* c, q *and* ξ *be positive real numbers so that* $q < 1$, $2c < \xi$, $d - 2c > 0$ *and* $\max_{i\neq j} |x_i - x_j| < 2\pi - 2\xi$. *Denote the expression* $\min\{|\sin \xi/2|\ ,\ |\sin(d/2 - c)|\}$ *by* A. *If* $c^2\left(4n + \alpha_i\ (9A^2/8 - 2)\right) < A^2\alpha_i$, $i = \overline{1,m}$ *and initial approximations* $x_i^{[0]}$, $i = \overline{1,m}$ *are chosen so that* $\left|x_i^{[0]} - x_i\right| \leq cq$, $i = \overline{1,m}$, *then for every natural* k *the inequalities* $\left|x_i^{[k]} - x_i\right| \leq cq^{3^k}$, $i = \overline{1,m}$ *also hold true.*

P r o o f. We divide the numerator and denominator of second summand in the right side of (16) by $T_n\left(x_i^{[k]}\right)$ and obtain $(i = \overline{1,m})$

$$x_i^{[k+1]} - x_i = x_i^{[k]} - x_i - \alpha_i\left[T_n'\left(x_i^{[k]}\right)/T_n\left(x_i^{[k]}\right) - Q_i'^{[k]}\left(x_i^{[k]}\right)/Q_i^{[k]}\left(x_i^{[k]}\right)\right]^{-1}. \tag{17}$$

On the other hand, we have

$$T_n'\left(x_i^{[k]}\right)/T_n\left(x_i^{[k]}\right) = 2^{-1}\sum_{j=1}^{m}\alpha_j \operatorname{cotg}\left(\left(x_i^{[k]} - x_j\right)/2\right), \quad i = \overline{1,m} \tag{18}$$

and

$$Q_i'^{[k]}\left(x_i^{[k]}\right)/Q_i^{[k]}\left(x_i^{[k]}\right) = 2^{-1}\sum_{j=1\,,\,j\neq i}^{m}\alpha_j \operatorname{cotg}\left(\left(x_i^{[k]} - x_j^{[k]}\right)/2\right), \quad i = \overline{1,m}. \tag{19}$$

Using (18) and (19), we transform (17) into the form $(i = \overline{1,m})$:

$$x_i^{[k+1]} - x_i = x_i^{[k]} - x_i - 2\alpha_i\Big[\sum_{j=1}^{m}\alpha_j \operatorname{cotg}\left(\left(x_i^{[k]} - x_j\right)/2\right)$$

$$- \sum_{j=1\ ,\ j\neq i}^{m} \alpha_j \text{cotg}\left(\left(x_i^{[k]} - x_j^{[k]}\right)/2\right)\Big]^{-1}. \qquad (20)$$

If we multiply the numerator and denominator of the second summand in the right side of (20) with $\sin\left(\left(x_i^{[k]} - x_i\right)/2\right)$, then we obtain

$$x_i^{[k+1]} - x_i = x_i^{[k]} - x_i - 2\alpha_i\Big[\alpha_i \cos\left(\left(x_i^{[k]} - x_i\right)/2\right) + \sin\left(\left(x_i^{[k]} - x_i\right)/2\right)$$

$$\times \sum_{j=1,j\neq i}^{m} \alpha_j \left[\text{cotg}\left(\left(x_i^{[k]} - x_j\right)/2\right) - \text{cotg}\left(\left(x_i^{[k]} - x_j^{[k]}\right)/2\right)\right]\Big]^{-1} \sin\left(\left(x_i^{[k]} - x_i\right)/2\right)$$

$$= \left(x_i^{[k]} - x_i\right)\Big[\alpha_i\left(\cos\left(\left(x_i^{[k]} - x_i\right)/2\right) - 2\left(x_i^{[k]} - x_i\right)^{-1} \sin\left(\left(x_i^{[k]} - x_i\right)/2\right)\right)$$

$$+ \sum_{j=1\ ,\ j\neq i}^{m} \alpha_j \left[\text{cotg}\left(\left(x_i^{[k]} - x_j\right)/2\right) - \text{cotg}\left(\left(x_i^{[k]} - x_j^{[k]}\right)/2\right)\right]\Big]$$

$$\times \Big[\alpha_i \cos\left(\left(x_i^{[k]} - x_i\right)/2\right) + \sin\left(\left(x_i^{[k]} - x_i\right)/2\right)$$

$$\times \sum_{j=1\ ,\ j\neq i}^{m} \alpha_j \left[\text{cotg}\left(\left(x_i^{[k]} - x_j\right)/2\right) - \text{cotg}\left(\left(x_i^{[k]} - x_j^{[k]}\right)/2\right)\right]\Big]^{-1},\ i = \overline{1,m}.$$

Further, the difference $\left[\text{cotg}\left(\left(x_i^{[k]} - x_j\right)/2\right) - \text{cotg}\left(\left(x_i^{[k]} - x_j^{[k]}\right)/2\right)\right]$ can be transformated as follows $(i, j = \overline{1,m}, i \neq j)$:

$$\text{cotg}\left(\left(x_i^{[k]} - x_j\right)/2\right) - \text{cotg}\left(\left(x_i^{[k]} - x_j^{[k]}\right)/2\right) = \Big[\sin\left(\left(x_i^{[k]} - x_j\right)/2\right)$$

$$\times \sin\left(\left(x_i^{[k]} - x_j^{[k]}\right)/2\right)\Big]^{-1}\Big[\cos\left(\left(x_i^{[k]} - x_j\right)/2\right)\sin\left(\left(x_i^{[k]} - x_j^{[k]}\right)/2\right)$$

$$- \cos\left(\left(x_i^{[k]} - x_j^{[k]}\right)/2\right)\sin\left(\left(x_i^{[k]} - x_j\right)/2\right)\Big]$$

$$= \sin\left(\left(x_j - x_j^{[k]}\right)/2\right)\left[\sin\left(\left(x_i^{[k]} - x_j\right)/2\right)\sin\left(\left(x_i^{[k]} - x_j^{[k]}\right)/2\right)\right]^{-1}.$$

Consequently, for the deviation of $x_i^{[k+1]}$ from x_i, we receive the expression

$$x_i^{[k+1]} - x_i = \left[\alpha_i \left[\left(x_i^{[k]} - x_i\right)\cos\left(\left(x_i^{[k]} - x_i\right)/2\right) - 2\sin\left(\left(x_i^{[k]} - x_i\right)/2\right)\right]\right.$$

$$\left. + \left(x_i^{[k]} - x_i\right) Y_i^{[k]}\left(x_i^{[k]}\right)\right]\left[\alpha_i \cos\left(\left(x_i^{[k]} - x_i\right)/2\right) + Y_i^{[k]}\left(x_i^{[k]}\right)\right]^{-1}, \qquad (21)$$

$$Y_i^{[k]}\left(x_i^{[k]}\right) = \sin\left(\left(x_i^{[k]} - x_i\right)/2\right) \sum_{j=1\,,\,j\neq i}^{m} \alpha_j \sin\left(\left(x_j - x_j^{[k]}\right)/2\right)$$

$$\times\left[\sin\left(\left(x_i^{[k]} - x_j\right)/2\right)\sin\left(\left(x_i^{[k]} - x_j^{[k]}\right)/2\right)\right]^{-1}, \quad i = \overline{1,m}.$$

In order to find an estimate for the expressions

$$\left(x_i^{[k]} - x_i\right)\cos\left(\left(x_i^{[k]} - x_i\right)/2\right) - 2\sin\left(\left(x_i^{[k]} - x_i\right)/2\right), \quad i = \overline{1,m},$$

we consider the auxiliary function $F(x) = x\cos(x/2) - 2\sin(x/2)$ and its MacLaurent expansion, until the remainder term with third derivative of $F(x)$. In this way we obtain

$$F\left(x_i^{[k]} - x_i\right) = \left(-2^{-1}\cos\left(\zeta_i^{[k]}/2\right) + \left(\zeta_i^{[k]}/8\right)\sin\left(\zeta_i^{[k]}/2\right)\right)\left(x_i^{[k]} - x_i\right)^3/6$$

$$\left(\zeta_i^{[k]} = \theta_i^{[k]}\left(x_i^{[k]} - x_i\right), \; 0 < \theta_i^{[k]} < 1, \quad i = \overline{1,m}\right)$$

and, therefore, the following estimate $(i = \overline{1,m})$

$$\left|F\left(x_i^{[k]} - x_i\right)\right| \le (1/12 + 2\pi/48)\left|x_i^{[k]} - x_i\right|^3 \le \left|x_i^{[k]} - x_i\right|^3/4 \le \left|x_i^{[k]} - x_i\right|^3.$$

On the other hand, the inequalities (11) and (12) hold true. Because of the fact that all the roots are in an interval with a length 2π, i.e. $|x_i - x_j| < 2\pi$, $i,j = \overline{1,m}, i \neq j$, it exists a positive number ξ such that $|x_i - x_j| < 2\pi - 2\xi$, $i,j = \overline{1,m}$, $i \neq j$. We now obtain the inequalities

$$\left|x_i^{[k]} - x_i\right| \le \left|x_i^{[k]} - x_i\right| + \left|x_j^{[k]} - x_j\right| + |x_i - x_j| < 2\pi - 2\xi + 2cq^{3^k}, \; i,j = \overline{1,m}, i \neq j,$$

$$\left|x_i^{[k]} - x_j\right| \le \left|x_i^{[k]} - x_i\right| + |x_i - x_j|, \; i,j = \overline{1,m}, i \neq j.$$

From the suppositions of the theorem, it follows that $2c < \xi$. Then

$$\left|x_i^{[k]} - x_j^{[k]}\right| < 2\pi - \xi,\ i,j = \overline{1,m},\ i \neq j$$

$$\left|x_i^{[k]} - x_j\right| < 2\pi - 2\xi + \xi/2 < 2\pi - \xi,\ i,j = \overline{1,m},\ i \neq j.$$

Consequenly, $d/2 - c < \left|x_i^{[k]} - x_j\right|/2 < \pi - \xi/2$, $i,j = \overline{1,m},\ i \neq j$ and $d/2 - c < \left|x_i^{[k]} - x_j^{[k]}\right|/2 < \pi - \xi/2$, $i,j = \overline{1,m},\ i \neq j$. It is easy to find that both expressions $\left|\sin\left(\left(x_i^{[k]} - x_j^{[k]}\right)/2\right)\right|$ and $\left|\sin\left(\left(x_i^{[k]} - x_j\right)/2\right)\right|$ are greater than A, $A \stackrel{\text{def}}{=} \min\{|\sin(\xi/2)|, |\sin(d/2 - c)|\}$. From (21) we estimate the absolute value of $x_i^{[k+1]} - x_i$, $i = \overline{1,m}$, i.e.

$$\left|x_i^{[k+1]} - x_i\right| \leq \left[\alpha_i \left|x_i^{[k]} - x_i\right|^3/4 + \left|x_i^{[k]} - x_i\right| Z_i^{[k]}\left(x_i^{[k]}\right)\right]$$

$$\times \left[\alpha_i \left|\cos\left(\left(x_i^{[k]} - x_i\right)/2\right)\right| - Z_i^{[k]}\left(x_i^{[k]}\right)\right]^{-1},$$

$$Z_i^{[k]}\left(x_i^{[k]}\right) = \left|\sin\left(\left(x_i^{[k]} - x_i\right)/2\right)\right| \sum_{j=1\ ,\ j\neq i}^{m} \alpha_j \left|\sin\left(\left(x_j - x_j^{[k]}\right)/2\right)\right| \tag{22}$$

$$\times \left[\left|\sin\left(\left(x_i^{[k]} - x_j\right)/2\right)\right| \left|\sin\left(\left(x_i^{[k]} - x_j^{[k]}\right)/2\right)\right|\right]^{-1}, i = \overline{1,m}.$$

Because of the presentation $\sin\left(\left(x_i^{[k]} - x_i\right)/2\right) = \left(\left(x_i^{[k]} - x_i\right)/2\right)\cos\varsigma_i^{[k]}$, $i = \overline{1,m}$, where $\varsigma_i^{[k]} = \theta_i^{[k]}\left(\left(x_i^{[k]} - x_i\right)/2\right)$, $0 < \theta_i^{[k]} < 1$, $i = \overline{1,m}$, the estimates $\left|\sin\left(\left(x_i^{[k]} - x_i\right)/2\right)\right| \leq \left|x_i^{[k]} - x_i\right|/2$, $i = \overline{1,m}$ hold true. Then from (22), we obtain

$$\left|x_i^{[k+1]} - x_i\right| \leq c^3\left(q^{3^k}\right)^3 \left[\alpha_i + A^{-2}\sum_{j=1,j\neq i}^{m} \alpha_j\right]$$

$$\times, \left[\alpha_i \left|\cos\left(\left(x_i^{[k]} - x_i\right)/2\right)\right| - (c/A)^2 \sum_{j=1\ ,\ j\neq i}^{m} \alpha_j\right]^{-1},\quad i = \overline{1,m}.$$

From the inequality $\left|\left(x_i^{[k]} - x_i\right)/2\right| < c/2$, $i = \overline{1,m}$ and the presentation $\cos\left(\left(x_i^{[k]} - x_i\right)/2\right) = 1 - (1/8)\left(x_i^{[k]} - x_i\right)^2 \cos\zeta_i^{[k]}$, $i = \overline{1,m}$, it follows that $\left|\cos\left(\left(x_i^{[k]} - x_i\right)/2\right)\right| > 1 - c^2/8$, $i = \overline{1,m}$ for sufficiently small c. Finally, we receive

$$\left|x_i^{[k+1]} - x_i\right| \le cq^{3^{k+1}} c^2 \left[\alpha_i + (2n - \alpha_i)/A^2\right]$$

$$\times \left[\alpha_i\left(1 - c^2/8\right) - (2n - \alpha_i)(c/A)^2\right]^{-1} < cq^{3^{k+1}}, \quad i = \overline{1,m}$$

for a small enough c. Thus the theorem is proved completely. ∎

Example 2. For the trigonometric polynomial

$$T_3(x) = \sin^3((x-1)/2)\sin^2((x-2)/2)\sin((x-2.5)/2)$$

at initial approximations $x_1^{[0]} = 0.2$, $x_2^{[0]} = 1.7$ and $x_3^{[0]} = 3$, we reach the roots of $T_3(x)$ with an accuracy of 18 digits at the 5$^{\text{th}}$ iteration.

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$
0	0.200000000000000000	1.70000000000000000	3.00000000000000000
1	1.08093197781206681	2.13081574593339511	2.68530050098035859
2	0.999087999636487434	1.98917328088624173	2.46587439388854078
3	1.00000001182848523	2.00000867262537340	2.50012119040535689
4	1.00000000000000000	1.99999999999998133	2.49999999999881136
5	1.00000000000000000	2.00000000000000000	2.50000000000000000

4. Exponential polynomials

Let us now consider the polynomial

$$E_n(x) = a_0/2 + \sum_{l=1}^{n}\left(a_l \mathrm{ch}(lx) + b_l \mathrm{sh}(lx)\right) = a_0/2 + \sum_{l=1}^{n}\left(a\prime_l e^{lx} + b\prime_l e^{-lx}\right). \quad (23)$$

We suppose that at least one of the leading coefficients a_n or b_n is not zero and that $E_n(x)$ has real roots $x_1, x_2, ..., x_m$ with known multiplicities $\alpha_1, \alpha_2, ..., \alpha_m$ $(\alpha_1 + \alpha_2 + ... + \alpha_m = 2n)$, correspondingly. The roots of (23) can be refined simultaneously by the help of the computational scheme

$$x_i^{[k+1]} = x_i^{[k]} - \alpha_i E_n\left(x_i^{[k]}\right)\left[E_n'\left(x_i^{[k]}\right) - E_n\left(x_i^{[k]}\right) Q_i^{\prime[k]}\left(x_i^{[k]}\right)/Q_i^{[k]}\left(x_i^{[k]}\right)\right]^{-1},$$

$i = \overline{1,m}$, $k = 0, 1, 2, \ldots$, where

$$Q_i^{[k]}(x) = \prod_{j \neq i\,,\, j=1}^{m} \mathrm{sh}^{\alpha_j}\left(\left(x - x_j^{[k]}\right)/2\right).$$

Theorem 3. *Denote* $\min_{i \neq j} |x_i - x_j|$ *by* d. *Let* q *and* c *be real numbers such that* $1 > q > 0$, $c > 0$, $d - 2c > 0$, $c^2\left(4n + \left(S^2 - 2\right)\alpha_i\right) < S^2\alpha_i$, $i = \overline{1,m}$, *where the expression* $sh((d-2c)/2)$ *is denoted by* S. *If the initial approximations* $x_i^{[0]}$, $i = \overline{1,m}$ *are taken such that* $\left|x_i^{[0]} - x_i\right| \leq cq$, $i = \overline{1,m}$, *then for every* $k \in N$ *the inequalities* $\left|x_i^{[k]} - x_i\right| \leq cq^{3^k}$, $i = \overline{1,m}$ *hold true.*

The proof of Theorem 3 can be carried out by the similar manner as in Theorem 2 with corresponding changes, related to the properties of the hyperbolic functions.

Example 3. The iteration method (24) was applied for SFAR of the exponential polynomial $E_2(x) = \mathrm{sh}^2((x+2)/2)\,\mathrm{sh}^2((x-3)/2)$. Using initial approximations $x_1^{[0]} = -1$ and $x_2^{[0]} = 4$, by formula (24) we receive the roots with 18 decimal digits after only 4 iterations.

k	$x_1^{[k]}$	$x_2^{[k]}$
0	−1.00000000000000000	4.00000000000000000
1	−1.93448948248966207	3.07207901269406155
2	−1.99997875689833755	3.00002895806496640
3	−1.99999999999999929	3.00000000000000190
4	−2.00000000000000000	3.00000000000000000

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