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Fixed Coefficients for a Class of Univalent Functions with Negative Coefficients

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In this paper we consider the class $Q_{n,c}(\alpha)$ consisting of analytic and univalent functions with negative coefficients and fixed second coefficient. The object of the present paper is to show coefficients estimates, convex linear combinations, some distortion theorems and radii of starlikeness and convexity for $f(z)$ in the class $Q_{n,c}(\alpha)$. The results are generalized to families with finitely many fixed coefficients.

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1. Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. Given two functions $f, g \in S$, where $f(z)$ is given by (1.1) and $g(z)$ is defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.2}$$

the *Hadamar product* or *convolution* $f * g(z)$ is defined by

$$f * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in U. \tag{1.3}$$

By using the Hadamar product, Ruscheweyh [3] defined

$$D^{\beta}f(z) = \frac{z}{(1-z)^{\beta+1}} * f(z), \quad (\beta \geq -1). \tag{1.4}$$

Ruscheweyh [3] observed that

$$D^{n}f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!} \tag{1.5}$$

when $n = \beta \in N_0 = N \cup \{0\}$, where $N = \{1, 2, \ldots\}$. This symbol $D^n f(z)$ $(n \in N_0)$ was called the ***n-th order Ruscheweyh derivative*** of $f(z)$ by Al-Amiri [1]. We note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$. It is easy to see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} \delta(n,k) a_k z^k, \tag{1.6}$$

where

$$\delta(n,k) = \binom{n+k-1}{n}. \tag{1.7}$$

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0). \tag{1.8}$$

Further, let $Q_n(\alpha)$ $(0 \leq \alpha < 1, n \in N_0)$ denote the subclass of T whose members satisfy

$$\Re(D^n f(z)) > \alpha, \quad z \in U. \tag{1.9}$$

The class $Q_n(\alpha)$ was studied by Uralegaddi and Sarangi [6]. We note that for $n = 0$ the class $Q_n(\alpha)$ was studied by Sarangi and Uralegaddi [4] and Al-Amiri [2].

For the class $Q_n(\alpha)$, Uralegaddi and Sarangi [6] proved the following lemma.

Lemma 1. *A function $f(z)$ defined by (1.8) is in the class $Q_n(\alpha)$ if and only if*

$$\sum_{k=2}^{\infty} k\delta(n,k) a_k \leq 1 - \alpha. \tag{1.10}$$

The result is sharp.

In view of Lemma 1, we can see that the function $f(z)$ defined by (1.8) in the class $Q_n(\alpha)$ satisfy

$$a_2 \leq \frac{1-\alpha}{2(n+1)}. \tag{1.11}$$

Let $Q_{n,c}(\alpha)$ denote the class of functions $f(z)$ in $Q_n(\alpha)$ of the form

$$f(z) = z - \frac{c(1-\alpha)}{2(n+1)}z^2 - \sum_{k=3}^{\infty} a_k z^k \quad (a_k \geq 0), \tag{1.12}$$

where $0 \leq c \leq 1$.

2. Coefficient estimates

Theorem 1. *Let the function $f(z)$ be defined by (1.12). Then $f(z) \in Q_{n,c}(\alpha)$ if and only if*

$$\sum_{k=3}^{\infty} k\delta(n,k)a_k \leq (1-c)(1-\alpha). \tag{2.1}$$

The result is sharp.

Proof. Putting

$$a_2 = \frac{c(1-\alpha)}{2(n+1)}, \quad 0 \leq c \leq 1, \tag{2.2}$$

in (1.10) and simplifying we get the result. The result is sharp for the function

$$f(z) = z - \frac{c(1-\alpha)}{2(n+1)}z^2 - \frac{(1-c)(1-\alpha)}{k\delta(n,k)}z^k \quad (k \geq 3). \tag{2.3}$$

■

Corollary 1. *Let the function $f(z)$ defined by (1.12) be in the class $Q_{n,c}(\alpha)$. Then*

$$a_k \leq \frac{(1-c)(1-\alpha)}{k\delta(n,k)} \quad (k \geq 3). \tag{2.4}$$

The result is sharp for the function $f(z)$ given by (2.3).

3. Closure theorems

Theorem 2. *The class $Q_{n,c}(\alpha)$ is closed under convex linear combinations.*

Proof. Let the function $f(z)$ be defined by (1.12). Define the function $g(z)$ by

(3.1) $$g(z) = z - \frac{c(1-\alpha)}{2(n+1)}z^2 - \sum_{k=3}^{\infty} b_k z^k \quad (b_k \geq 0).$$

Assuming that $f(z)$ and $g(z)$ are in the class $Q_{n,c}(\alpha)$, it is sufficient to prove that the function $H(z)$ defined by

(3.2) $$H(z) = \lambda f(z) + (1-\lambda)g(z) \quad (0 \leq \lambda \leq 1)$$

is also in the class $Q_{n,c}(\alpha)$.

Since

(3.3) $$H(z) = z - \frac{c(1-\alpha)}{2(n+1)}z^2 - \sum_{k=3}^{\infty}\{\lambda a_k + (1-\lambda)b_k\}z^k,$$

we observe that

(3.4) $$\sum_{k=3}^{\infty} k\delta(n,k)\{\lambda a_k + (1-\lambda)b_k\} \leq (1-c)(1-\alpha).$$

with the aid of Theorem 1. Hence $H(z) \in Q_{n,c}(\alpha)$. This completes the proof of Theorem 2. ■

Theorem 3. *Let the functions*

(3.5) $$f_j(z) = z - \frac{c(1-\alpha)}{2(n+1)}z^2 - \sum_{k=3}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0)$$

] be in the class $Q_{n,c}(\alpha)$ for every $j = 1, 2, \ldots, m$. Then the function $F(z)$ defined by

(3.6) $$F(z) = \sum_{j=1}^{m} d_j f_j(z)$$

is also in the same class $Q_{n,c}(\alpha)$, where

(3.7) $$d_j \geq 0, \; j = 1, 2 \ldots, m; \quad \sum_{j=1}^{m} d_j = 1.$$

P r o o f. Combining the definitions (3.5) and (3.6), we have

(3.8) $$F(z) = z - \frac{c(1-\alpha)}{2(n+1)}z^2 - \sum_{k=3}^{\infty}(\sum_{j=1}^{m} d_j a_{k,j})z^k,$$

where we have also used the relationship (3.7). Since $f_j(z) \in Q_{n,c}(\alpha)$ for every $j = 1, 2, \ldots, m$. Theorem 1 yields

(3.9) $$\sum_{k=3}^{\infty} k\delta(n,k) a_{k,j} \leq (1-c)(1-\alpha)$$

for $j = 1, 2, \ldots, m$. Thus we obtain

$$\sum_{k=3}^{\infty} k\delta(n,k)(\sum_{j=1}^{m} d_j a_{k,j}) = \sum_{j=1}^{m} d_j(\sum_{k=3}^{\infty} k\delta(n,k)a_{k,j}) \leq (1-c)(1-\alpha)$$

which (in view of Theorem 1) implies that $f(z) \in Q_{n,c}(\alpha)$. ■

Theorem 4. *Let*

$$f_2(z) = z - \frac{c(1-\alpha)}{2(n+1)} z^2 \tag{3.10}$$

and

$$f_k(z) = z - \frac{c(1-\alpha)}{2(n+1)} z^2 - \frac{(1-c)(1-\alpha)}{k\delta(n,k)} z^k \tag{3.11}$$

for $k = 3, 4, \ldots$. Then $f(z)$ is in the class $Q_{n,c}(\alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=2}^{\infty} \lambda_k f_k(z), \tag{3.12}$$

where $\lambda_k \geq 0$ and

$$\sum_{k=2}^{\infty} \lambda_k = 1.$$

Proof. We suppose that $f(z)$ can be expresed in the form (3.12). Then we have

$$f(z) = z - \frac{c(1-\alpha)}{2(n+1)} z^2 - \sum_{k=2}^{\infty} \frac{(1-c)(1-\alpha)\lambda_k}{k\delta(n,k)} z^k. \tag{3.13}$$

Since

$$\sum_{k=3}^{\infty} \frac{(1-c)(1-\alpha)\lambda_k}{k\delta(n,k)} k\delta(n,k)$$

$$= (1-c)(1-\alpha)\sum_{k=3}^{\infty} \lambda_k = (1-c)(1-\alpha)(1-\lambda_2) \leq (1-c)(1-\alpha). \tag{3.14}$$

it follows from (2.1) that $f(z)$ is in the class $Q_{n,c}(\alpha)$.

Conversely, we suppose that $f(z)$ defined by (1.12) is in the class $Q_{n,c}(\alpha)$. Then by using (2.4), we get

$$a_k \leq \frac{(1-c)(1-\alpha)}{k\delta(n,k)} \quad (k \geq 3). \tag{3.15}$$

Setting

(3.16) $$\lambda_k = \frac{k\delta(n,k)}{(1-c)(1-\alpha)}a_k \quad (k \geq 3)$$

and

(3.17) $$\lambda_2 = 1 - \sum_{k=3}^{\infty} \lambda_k,$$

we have (3.12). This completes the proof of the theorem. ■

Corollary 2. *The extreme points of the class $Q_{n,c}(\alpha)$ are the functions $f_k(z)$ $(k \geq 2)$ given by Theorem 4.*

4. Distortion theorems

First we need the following lemmas.

Lemma 2. *Let the function $f_3(z)$ be defined by*

(4.1) $$f_3(z) = z - \frac{c(1-\alpha)}{2(n+1)}z^2 - \frac{2(1-c)(1-\alpha)}{3(n+1)(n+2)}z^3.$$

Then, for $0 \leq r < 1$ and $0 \leq c \leq 1$,

(4.2) $$|f(re^{i\theta})| \geq r - \frac{c(1-\alpha)}{2(n+1)}r^2 - \frac{2(1-c)(1-\alpha)}{3(n+1)(n+2)}r^3$$

with equality for $\theta = 0$. For either $0 \leq c < c_0$ and $0 \leq r \leq r_0$ or $c_0 \leq c \leq 1$,

(4.3) $$|f_3(re^{i\theta})| \leq r + \frac{c(1-\alpha)}{2(n+1)}r^2 - \frac{2(1-c)(1-\alpha)}{3(n+1)(n+2)}r^3$$

with equality for $\theta = \pi$. Further, for $0 \leq c < c_0$ and $r_0 \leq r < 1$,

(4.4) $$\begin{aligned} |f_3(re^{i\theta})| \leq r\{[1 + &\frac{3c^2(1-\alpha)(n+2)}{32(1-c)(n+1)}] \\ &+(1-\alpha)[\frac{c^2(1-\alpha)}{8(n+1)^2} + \frac{4(1-c)}{3(n+1)(n+2)}]r^2 \\ &+\frac{(1-c)(1-\alpha)^2}{3(n+1)(n+2)}[\frac{4(1-c)}{3(n+1)(n+2)} + \frac{c^2(1-\alpha)}{8(n+1)^2}]r^4\}^{\frac{1}{2}} \end{aligned}$$

with equality for

(4.5) $$\theta = \cos^{-1}\left(\frac{2c(1-c)(1-\alpha)r^2 - 3c(n+1)(n+2)}{16(1-c)(n+1)r}\right),$$

where

$$c_0 = \frac{1}{4(1-\alpha)}\{-[16(n+1)+3(n+1)(n+2)-2(1-\alpha)]$$

$$\text{(4.6)} \quad +\sqrt{[16(n+1)+3(n+1)(n+2)-2(1-\alpha)]^2+128(1-\alpha)(n+1)}\}$$

and

$$r_0 = \frac{1}{2c(1-c)(1-\alpha)}\{-8(1-c)(n+1)$$

$$\text{(4.7)} \quad +\sqrt{64(1-c)^2(n+1)^2+6c^2(1-c)(1-\alpha)(n+1)(n+2)}\}.$$

Proof. We employ the same technique as used by Silverman and Silvia [5]. Since

$$\frac{\partial|f_3(re^{i\theta})|^2}{\partial\theta} = \frac{(1-\alpha)r^3\sin\theta}{(n+1)}$$

$$\text{(4.8)} \quad \times\{c+\frac{16(1-c)}{3(n+2)}r\cos\theta - \frac{2c(1-c)(1-\alpha)}{3(n+1)(n+2)}r^2\},$$

we can see that

$$\text{(4.9)} \quad \frac{\partial|f_3(re^{i\theta})|^2}{\partial\theta} = 0$$

for $\theta_1 = 0,\ \theta_2 = \pi$, and

$$\text{(4.10)} \quad \theta_3 = \cos^{-1}(\frac{2c(1-c)(1-\alpha)r^2 - 3c(n+1)(n+2)}{16(1-c)(n+1)r}).$$

Since θ_3 is a valid root only when $-1 \le \cos\theta_3 \le 1$, we have a third root if and only if $r_0 \le r < 1$ and $0 \le c \le c_0$. Thus the results of the theorem follow from comparing the extremal values $|f_3(re^{i\theta_k})|$ $(k = 1,2,3)$ on the appropriate intervals. ■

Lemma 3. *Let the functions $f_k(z)$ be defined by (3.11) and $k \ge 4$. Then*

$$\text{(4.11)} \quad |f_k(re^{i\theta})| \le |f_4(-r)|.$$

Proof. Since

$$f_k(z) = z - \frac{c(1-\alpha)}{2(n+1)}z^2 - \frac{(1-c)(1-\alpha)}{k\delta(n,k)}z^k$$

and

$$\frac{(1-c)(1-\alpha)}{k\delta(n,k)}r^k$$

is a decreasing function of k, we have

$$|f_k(re^{i\theta})| \leq r + \frac{c(1-\alpha)}{2(n+1)}r^2 + \frac{3(1-c)(1-\alpha)}{2(n+1)(n+2)(n+3)}r^4 = -f_4(-r)$$

which shows (4.11). ∎

Theorem 5. *Let the function $f(z)$ defined by (1.12) belongs to the class $Q_{n,c}(\alpha)$. Then for $0 \leq r < 1$,*

$$|f(re^{i\theta})| \geq r - \frac{c(1-\alpha)}{2(n+1)}r^2 - \frac{2(1-c)(1-\alpha)}{3(n+1)(n+2)}r^3 \tag{4.12}$$

with equality for $f_3(z)$ at $z = r$, and

$$|f(re^{i\theta})| \leq \max\{\max_\theta |f_3(re^{i\theta})|, -f_4(-r)\}, \tag{4.13}$$

where $\max_\theta |f_3(re^{i\theta})|$ is given by Lemma 2.

The proof of Theorem 5 is obtained by comparing the bounds of Lemma 2 and Lemma 3.

Remark . Putting $c = 1$ in Theorem 5 we obtain the following result.

Corollary 3. *Let the function $f(z)$ defined by (1.8) be in the class $Q_{n,c}(\alpha)$. Then for $|z| = r < 1$, we have*

$$r - \frac{(1-\alpha)}{2(n+1)}r^2 \leq |f(z)| \leq r + \frac{(1-\alpha)}{2(n+1)}r^2. \tag{4.14}$$

The result is sharp.

Lemma 4. *Let the function $f_3(z)$ be defined by (4.1). Then, for $0 \leq r < 1$ and $0 \leq c \leq 1$,*

$$|f_3'(re^{i\theta})| \geq 1 - \frac{c(1-\alpha)}{(n+1)}r - \frac{2(1-c)(1-\alpha)}{(n+1)(n+2)}r^2 \tag{4.15}$$

with equality for $\theta = 0$. For either $0 \leq c < c_1$ and $0 \leq r \leq r_1$ or $c_1 \leq c \leq 1$,

$$|f_3'(re^{i\theta})| \leq 1 + \frac{c(1-\alpha)}{(n+1)}r - \frac{2(1-c)(1-\alpha)}{(n+1)(n+2)}r^2 \tag{4.16}$$

with equality for $\theta = \pi$. Further, for $0 \leq c < c_1$ and $r_1 \leq r < 1$,

$$|f_3'(re^{i\theta})| \leq \{[1 + \frac{c^2(1-\alpha)(n+2)}{8(1-c)(n+1)}] + \frac{(1-\alpha)}{(n+1)}[\frac{c^2(1-\alpha)}{2(n+1)}$$

$$+\frac{4(1-c)}{(n+2)}]r^2+\frac{(1-c)(1-\alpha)^2}{(n+1)^2(n+2)}[\frac{4(1-c)}{(n+2)}+\frac{c^2(1-\alpha)}{2(n+1)}]r^4\}^{\frac{1}{2}} \tag{4.17}$$

with equality for

$$\theta=\cos^{-1}(\frac{2c(1-c)(1-\alpha)r^2-c(n+1)(n+2)}{8(1-c)(n+1)r}), \tag{4.18}$$

where

$$c_1=\frac{1}{4(1-\alpha)}\{-[(n+1)(n+2)+8(n+1)-2(1-\alpha)]$$

$$+\sqrt{[(n+1)(n+2)+8(n+1)-2(1-\alpha)]^2+64(1-\alpha)(n+1)}\} \tag{4.19}$$

and

$$r_1=\frac{1}{2c(1-c)(1-\alpha)}\{-4(1-c)(n+1)$$

$$+\sqrt{16(1-c)^2(n+1)^2+2c^2(1-c)(1-\alpha)(n+1)(n+2)}\}. \tag{4.20}$$

The proof of Lemma 4 is given in the same way as Lemma 2.

Theorem 6. *Let the function $f(z)$ defined by (1.12) be in the class $Q_{n,c}(\alpha)$. Then, for $0 \le r < 1$,*

$$|f'(re^{i\theta})| \ge 1-\frac{c(1-\alpha)}{(n+1)}r-\frac{2(1-c)(1-\alpha)}{(n+1)(n+2)}r^2 \tag{4.21}$$

with equality for $f_3'(z)$ at $z=r$, and

$$|f'(re^{i\theta})| \le \max\{\max_{\theta}|f_3'(re^{i\theta})|, f_4'(-r)\}, \tag{4.22}$$

where $\max_\theta |f_3'(re^{i\theta})|$ is given by Lemma 4.

Remark . Putting $c=1$ in Theorem 6 we obtain the following result.

Corollary 4. *Let the function $f(z)$ defined by (1.8) be in the class $Q_n(\alpha)$. Then for $|z|=r<1$, we have*

$$1-\frac{(1-\alpha)}{(n+1)}r \le |f'(z)| \le 1+\frac{(1-\alpha)}{(n+1)}r. \tag{4.23}$$

The result is sharp.

5. Radii of starlikeness and convexity

Theorem 7. *Let the function $f(z)$ defined by (1.12) be in the class $Q_{n,c}(\alpha)$. Then $f(z)$ is starlike of order ρ $(0 \leq \rho < 1)$ in the disc $|z| < r_1(n,c,\alpha,\rho)$ where $r_1(n,c,\alpha,\rho)$ is the largest value for which*

$$\frac{c(1-\alpha)(2-\rho)}{2(n+1)}r + \frac{(1-c)(1-\alpha)(k-\rho)}{k\delta(n,k)}r^{k-1} \leq 1-\rho, \tag{5.1}$$

for $k \geq 3$. The result is sharp with the extremal function

$$f_k(z) = z - \frac{c(1-\alpha)}{2(n+1)}z^2 - \frac{(1-c)(1-\alpha)}{k\delta(n,k)}z^k \tag{5.2}$$

for some k.

Proof. It suffices to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\rho \quad (0 \leq \rho < 1)$$

for $|z| < r_1(n,c,\alpha,\rho)$. Note that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\frac{c(1-\alpha)}{2(n+1)}r + \sum_{k=3}^{\infty}(k-1)a_k r^{k-1}}{1 - \frac{c(1-\alpha)}{2(n+1)}r - \sum_{k=3}^{\infty} a_k r^{k-1}} \leq 1-\rho \tag{5.3}$$

for $|z| \leq r$ if and only if

$$\frac{c(1-\alpha)(2-\rho)}{2(n+1)}r + \sum_{k=3}^{\infty}(k-\rho)a_k r^{k-1} \leq 1-\rho. \tag{5.4}$$

Since $f(z)$ is in $Q_{n,c}(\alpha)$, from (2.1) we may take

$$a_k = \frac{(1-c)(1-\alpha)\lambda_k}{k\delta(n,k)} \quad (k \geq 3), \tag{5.5}$$

where $\lambda_k \geq 0 \quad (k \geq 3)$ and

$$\sum_{k=3}^{\infty} \lambda_k \leq 1. \tag{5.6}$$

For each fixed r, we choose the positive integer $k_0 = k_0(r)$ for which $\frac{(k-\rho)r^{k-1}}{k\delta(n,k)}$ is maximal. Then it follows that

$$\sum_{k=3}^{\infty}(k-\rho)a_k r^{k-1} \leq \frac{(1-c)(1-\alpha)(k_0-\rho)}{k_0\delta(n,k_0)}r^{k_0-1}. \tag{5.7}$$

Hence $f(z)$ is starlike of order ρ in $|z| < r_1(n,c,\alpha,\rho)$ provided that

$$\frac{c(1-\alpha)(2-\rho)}{2(n+1)}r + \frac{(1-c)(1-\alpha)(k_0-\rho)}{k_0\delta(n,k_0)}r^{k_0-1} \leq 1-\rho. \tag{5.8}$$

We find the value $r_0 = r_0(n,c,\alpha,\rho)$ and the corresponding integer $k_0(r_0)$ so that

$$\frac{c(1-\alpha)(2-\rho)}{2(n+1)}r^0 + \frac{(1-c)(1-\alpha)(k_0-\rho)}{k_0\delta(n,k_0)}r_0^{k_0-1} = 1-\rho. \tag{5.9}$$

Then this value r_0 is the radius of starlikeness of order ρ for function $f(z)$ belonging to the class $Q_{n,c}(\alpha)$. ■

In a similar manner, we can prove the following theorem concerning the radius of convexity of order ρ for functions in the class $Q_{n,c}(\alpha)$.

Theorem 8. *Let the function $f(z)$ defined by (1.12) be in the class $Q_{n,c}(\alpha)$. Then $f(z)$ is convex of order ρ $(0 \le \rho < 1)$ in the disc $|z| < r_2(n,c,\alpha,\rho)$, where $r_2(n,c,\alpha,\rho)$ is the largest value for which*

$$\frac{c(1-\alpha)(2-\rho)}{(n+1)}r + \frac{(1-c)(1-\alpha)(k-\rho)}{\delta(n,k)}r^{k-1} \le 1-\rho \tag{5.10}$$

for $k \ge 3$. The result is sharp for the function $f(z)$ given by (5.2).

6. The class $Q_{n,c_{k,N}}(\alpha)$

Instead of fixing just the second coefficient, we can fix finitely many coefficients. Let $Q_{n,c_{k,N}}(\alpha)$ denote the class of functions in $Q_{n,c}(\alpha)$ of the form

$$f(z) = z - \sum_{k=2}^{N} \frac{c_k(1-\alpha)}{k\delta(n,k)}z^k - \sum_{k=N+1}^{\infty} a_k z_k, \tag{6.1}$$

where $0 \le \sum_{k=2}^{n} c_k = c \le 1$. Note that $Q_{n,c_{k,2}}(\alpha) = Q_{n,c}(\alpha)$.

Theorem 9. *The extreme points of $Q_{n,c_{k,N}}(\alpha)$ are the functions*

$$z - \sum_{k=2}^{N} \frac{c_k(1-\alpha)}{k\delta(n,k)}z^k$$

and

$$z - \sum_{k=2}^{N} \frac{c_k(1-\alpha)}{k\delta(n,k)}z^k - \frac{(1-c)(1-\alpha)}{k\delta(n,k)}z^k \quad \textit{for } k = n+1, N+2, \ldots$$

The details of the proof are omitted.

The characterization of the extreme points enables us to solve the standard extremal problems in the same manner as was done for $Q_{n,c}(\alpha)$. We omit the details.

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