# Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol. 14, 2000, Fasc. 1-2

# On Polygroups and Permutation Polygroups

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In this paper some aspects of polygroups are studied. By using the concept of generalized permutation, we define permutation polygroups and some concepts related to it. We obtain a generalization of Cayley's theorem as well as some interesting related results. We also define the concept of weak polygroups which is a generalization of polygroups and some concepts related to it. Moreover, we define a semi-direct hyperproduct of two weak polygroups in order to obtain an extension of weak polygroups by weak polygroups.

*AMS Subj. Classification*: 20N20

*Key Words*: polygroups, generalized permutations, permutation polygroups, $P$-map, $P$-isomorphism, strong homomorphism, fundamental relation, fundamental group, semi-direct hyperproduct

## 1. Introduction

The concept of a hypergroup, which is a generalization of the concept of ordinary group, first was introduced by Marty [8]. Application of hypergroups have mainly appeared in special subclasses. For example, polygroups which are certain subclasses of hypergroups, are introduced by Ioulidis in [6] and are used to study color algebra [1],[2].

The definitions and properties of the action of a group on a set, permutation group, orbit, stabilizer can be found in every text book (see [10],[15]), and we know every abstract group is isomorphic to a permutation group, hence with respect to algebraic structures, there is no difference between abstract groups and permutation groups.

In Section 4 of this paper, we define the concept of the action of a polygroup on a set and introduce permutation polygroups which is a generalization of the concept of permutation groups. We obtain a generalization of Cayley's theorem, as well as some interesting results with this respect.

$H_v$-structures first were introduced by Vougiouklis [13]. Actually, Vougiouklis has replaced some axioms concerning the hypergroup and hyperring by their corresponding weak axioms. And then some researchers followed him, for example see [4],[9].

In Section 5 of this paper we deal with a new class of hyperstructures called weak polygroups. The concept of a weak polygroup is a generalization of the concept of polygroup. We define the fundamental relation $\beta^*$ on the weak polygroups in a similar way as in the case of hyperstructures, and we obtain some results in this respect. We also define semi-direct hyperproduct of two weak polygroups.

## 2. Preliminaries

We recall the following definition from [2].

**Definition 2.1.** A *polygroup* is a multivalued system $\mathcal{M} =< P, ., e, ^{-1} >$, where $e \in P$, $^{-1} : P \longrightarrow P$, $\cdot : P \times P \longrightarrow \mathcal{P}^*(P)$, where the following axioms hold for all $x, y, z$ in $P$:

(i) $(x.y).z = x.(y.z)$,

(ii) $e.x = x.e = x$,

(iii) $x \in y.z$ implies $y \in x.z^{-1}$ and $z \in y^{-1}.x$.

In the above definition, $\mathcal{P}^*(P)$ is the set of all the non-empty subsets of P, and if $x \in P$ and $A, B \subseteq P$, then

$$A.B = \bigcup_{a\in A, b\in B} a.b, \quad x.B = \{x\}.B, \quad A.x = A.\{x\}.$$

And for simplicity of notations we sometimes write $ab$ instead of $a.b$.

If $K$ be a nonempty subset of $P$, then $K$ is called a *subpolygroup* of $P$ if $e \in K$ and $< K, ., e, ^{-1} >$ is a polygroup. The subpolygroup $N$ of $P$ is said to be *normal* in $P$ iff

$$a^{-1}.N.a \subseteq N, \quad \text{for every } a \in P.$$

If $N$ be a normal subpolygroup of $P$, the following elementary facts follows easily from the axioms:

(i) $Na = aN, \quad \forall a \in P$,

(ii) $(Na)(Nb) = Nab, \quad \forall a, b \in P$,

(iii) $Na = Nb, \quad \forall b \in Na$.

For $a$ subpolygroup $K$ of $P$ and $x \in P$ , the right coset of $K$ is defined as usual and is denoted by $Kx$ and $P/K$ is the set of all right cosets of $K$ in $P$. If $N$ be a normal subpolygroup of $P$, then $< P/N, \odot, N, ^{-I} >$ is a polygroup where $Na \odot Nb = \{Nc | c \in Nab\}$ and $(Na)^{-I} = Na^{-1}$.

**Definition 2.2.** (see [7]) Let $\mathcal{M}_1 = < P_1, ., e_1, ^{-1} >$ and $\mathcal{M}_2 = < P_2, *, e_2, ^{-1} >$ be polygroups. A mapping $f$ from $P_1$ into $P_2$ is said to be a strong homomorphism if for all $a, b \in P_1$,

(i) $f(e_1) = e_2$,

(ii) $f(ab) = f(a) * f(b)$.

A strong homomorphism $f$ is said to be an isomorphism if $f$ is one to one and onto. Two polygroups $P_1, P_2$ are said to be isomorphic if there is an isomorphism from $P_1$ onto $P_2$. In this case we write $P_1 \cong P_2$.

**Corollary 2.3.** *From the above definition, we have*

$$f(a^{-1}) = f(a)^{-1}, \ \forall a \in P_1.$$

Proof. Since $e_1 \in aa^{-1}$, then $f(e_1) \in f(a)*f(a^{-1})$ or $e_2 \in f(a)*f(a^{-1})$. Now by condition (iii) of Definition 2.1, we get $f(a^{-1}) \in f(a)^{-1} * e_2$, therefore $f(a^{-1}) = f(a)^{-1}$. ■

**Definition 2.4.** If $f$ is a strong homomorphism from $P_1$ into $P_2$, the *kernel* of $f$, is defined by

$$kerf = \{x \in P_1 | \quad f(x) = e_2\}.$$

It is easy to see that $kerf$ is a subpolygroup of $P_1$, but in general is not normal.

**Theorem 2.5.** (Fundamental theorem of homomorphism) *Let $f$ be a strong homomorphism from $P_1$ onto $P_2$ with kernel $K$ such that $gg^{-1} \subseteq K$, for all $g \in P_1$, then*

$$P_1/K \cong P_2.$$

Proof. We define $\phi : P_1/K \to P_2$ by setting

$$\phi(Kx) = f(x), \quad \forall x \in P_1.$$

It is easy to see that $K$ is a normal subplygroup of $P_1$ . We show that $\phi$ is well-defined.

If $Kx = Ky$, then $f(Kx) = f(Ky)$ which implies $f(x) = f(y)$ and so $\phi(Kx) = \phi(Ky)$, consequently $f$ is well-defined. Now for every $Kx, Ky \in P_1/K$, we have

$$\begin{aligned} \phi(Kx \odot Ky) &= \phi(\{Kz | z \in Kxy\}) \\ &= \{f(z) | z \in Kxy\} = f(Kxy) \\ &= f(K)f(x)f(y) = f(x)f(y) \\ &= \phi(Kx)\phi(Ky) \end{aligned}$$

and $\phi(Kx^{-1}) = f(x^{-1}) = f(x)^{-1} = \phi(Kx)^{-1}$. Therefore $\phi$ is a strong homomorphism.

Furthermore, if $\phi(Kx) = \phi(Ky)$, then $f(x) = f(y)$ which implies $f(xy^{-1}) = e_2$ and so $xy^{-1} \subseteq K$. Therefore $Kx = Kxe \subseteq Kxy^{-1}y \subseteq KKy = Ky$, similarly $Ky = Kx$. Thus $\phi$ is a one to one mapping.

Finally, it is easy to see that $\phi$ is onto. Hence $\phi$ is an isomorphism of $P_1/K$ onto $P_2$ and $P_1/K \cong P_2$, this proves the theorem. ■

### 3. Generalized permutations

According to [5] a generalized permutation is defined as follows (see also [11],[14]).

**Definition 3.1.** Let $\Omega$ be a non-empty set. A map $f : \Omega \to \mathcal{P}^*(\Omega)$ is called a generalized permutation on $\Omega$, if

$$\bigcup_{\omega \in \Omega} f(\omega) = f(\Omega) = \Omega,$$

where $\mathcal{P}^*(\Omega)$ is the set of all the non-empty subsets of $\Omega$. We write $f = \begin{pmatrix} x \\ f(x) \end{pmatrix}$ for the generalized permutation $f$. Denote $M_\Omega$ the set of all the generalized permutations on $\Omega$.

**Proposition 3.2.** *Let $\Theta$ be a one to one function from a set $\Omega_1$ onto a set $\Omega_2$. For a generalized permutation $f$ on $\Omega_1$, we define a function $\Theta(f)$ on $\Omega_2$ by the formula*

$$\Theta(f)(y) = \Theta(f(\Theta^{-1}(y))), \forall y \in \Omega_2.$$

*Then $\Theta(f)$ is a genralized permutation on $\Omega_2$.*

Proof. For every $y \in \Omega_2$, we have $\Theta^{-1}(y) \in \Omega_1$ which implies $f(\Theta^{-1}(y)) \subseteq \Omega_1$, and so $\Theta(f(\Theta^{-1}(y))) \subseteq \Omega_2$. Furthermore,

$$\bigcup_{y\in\Omega_2} \Theta(f)(y) = \bigcup_{y\in\Omega_2} \Theta(f(\Theta^{-1}(y))) = \Theta(\bigcup_{y\in\Omega_2} f(\Theta^{-1}(y))) = \Theta(\Omega_1) = \Omega_2.$$

■

**Definition 3.3.** ([14]) Let $f_1, f_2 \in M_\Omega$. We say that $f_1$ is a subpermutation of $f_2$, or $f_2$ contains $f_1$, and write $f_1 \subseteq f_2$, if $f_1(x) \subseteq f_2(x)$ for every $x$ in $\Omega$. The mapping $g$ for which $g(x) = \Omega$ for all $x \in \Omega$, is called the universal generalized permutation and contains all the elements of $M_\Omega$.

Every map $f : \Omega \longrightarrow \mathcal{P}^*(\Omega)$ which contains a generalized permutation is itself a generalized permutation.

The map $i : \Omega \longrightarrow \mathcal{P}^*(\Omega)$, where $i(x) = \{x\} =: x, \forall x \in \Omega$, is a generalized permutation.

We can define operation $o$, the usual composition on $M_\Omega$, i.e. if $f, g \in M_\Omega$,

$$fog(x) = \bigcup_{y \in g(x)} f(y), \quad \text{for all} \quad x \in \Omega.$$

Now we define a hyperoperation $*$ on $M_\Omega$ as follows.

**Definition 3.4.** Let $\begin{pmatrix} x \\ f(x) \end{pmatrix}$ and $\begin{pmatrix} x \\ g(x) \end{pmatrix}$ be two elements of $M_\Omega$, we define $* : M_\Omega \times M_\Omega \to \mathcal{P}^*(M_\Omega)$ by setting

$$\begin{pmatrix} x \\ g(x) \end{pmatrix} * \begin{pmatrix} x \\ f(x) \end{pmatrix} = \{\begin{pmatrix} x \\ h(x) \end{pmatrix} | h \subseteq fog, \bigcup_{x \in \Omega} h(x) = \Omega\}.$$

The generalized permutation $\begin{pmatrix} x \\ i(x) \end{pmatrix}$ serves as a scalar element, since

$$\begin{pmatrix} x \\ f(x) \end{pmatrix} \in \begin{pmatrix} x \\ f(x) \end{pmatrix} * \begin{pmatrix} x \\ i(x) \end{pmatrix} = \begin{pmatrix} x \\ i(x) \end{pmatrix} * \begin{pmatrix} x \\ f(x) \end{pmatrix}.$$

For $f \in M_\Omega$ the inverse of $f$ which is denoted by $\overline{f}$, is the generalized permutation defined as follows: $\overline{f}(y) = \{x \in \Omega | y \in f(x)\}$. It is clear that

$$\begin{pmatrix} x \\ i(x) \end{pmatrix} \in \begin{pmatrix} x \\ f(x) \end{pmatrix} * \begin{pmatrix} x \\ \overline{f}(x) \end{pmatrix} \bigcap \begin{pmatrix} x \\ \overline{f}(x) \end{pmatrix} * \begin{pmatrix} x \\ f(x) \end{pmatrix}.$$

**Proposition 3.5.** *The hyperoperation $*$ is associative.*

P r o o f. Since $o$ is associative, it follows that $*$ is associative. ■

**Definition 3.6.** $< P, ., e, ^{-1} >$ is called *poly-monoid* if the following conditions hold:

(i) $(x.y).z = x.(y.z), \quad \forall x, y, z \in P$,
(ii) $x \in e.x = x.e, \quad \forall x \in P$,
(iii) $e \in x.x^{-1} \bigcap x^{-1}.x, \quad \forall x \in P$.

**Corollary 3.7.** $< M_\Omega, *, i, ^{-} >$ *is a poly-monoid.*

**Definition 3.8.** Let $\mathcal{M} =< P, ., e, ^{-1} >$ be a polygroup and $M_\Omega$ be the set of all generalized permutations on the non-empty set $\Omega$. A mapping $\psi : P \rightarrow M_\Omega$ with the properties $\psi(x.y) = \psi(x) * \psi(y)$ and $\psi(x^{-1}) = \overline{\psi}(x), \forall x, y \in P$, is called a representation of $P$ by generalized permutations.

## 4. Permutation polygroups

**Definition 4.1.** Let $\mathcal{M} =< P, ., e, ^{-1} >$ be a polygroup and $\Omega$ be a non-empty set. A map $f : \Omega \times P \rightarrow \mathcal{P}^*(\Omega)$ is called an *action* $P$ on $\Omega$, if the following axioms hold:

(i) $f(\omega, e) = \{\omega\} = \omega, \quad \forall \omega \in \Omega,$
(ii) $f(f(\omega, g), h) = \bigcup\limits_{\alpha \in g.h} f(\omega, \alpha), \quad \forall g, h \in P, \forall \omega \in \Omega,$
(iii) $\bigcup\limits_{\omega \in \Omega} f(\omega, g) = \Omega, \quad \forall g \in P,$
(iv) $\forall g \in P, \alpha \in f(\beta, g) \Rightarrow \beta \in f(\alpha, g^{-1}).$

From the second condition, we get

$$\bigcup_{\omega_0 \in f(\omega, g)} f(\omega_0, h) = \bigcup_{\alpha \in g.h} f(\omega, \alpha).$$

For $\omega \in \Omega$, we write $\omega^g := f(\omega, g)$, therefore we have:

(i) $\omega^e = \omega,$
(ii) $(\omega^g)^h = \omega^{g.h}$, where

$$A^g = \bigcup_{a \in A} a^g \text{ and } \omega^B = \bigcup_{\alpha \in B} \omega^\alpha, \qquad \forall A \subseteq \Omega, B \subseteq P,$$

(iii) $\bigcup\limits_{\omega \in \Omega} \omega^g = \Omega,$
(iv) $\forall g \in P, \quad \alpha \in \beta^g \Rightarrow \beta \in \alpha^{g^{-1}}.$

In this case, we say that $P$ is a permutation polygroup on a set $\Omega$ and it is said that $P$ acts on $\Omega$.

It is easy to see that if $P$ is a permutation polygroup on two sets $\Omega_1$ and $\Omega_2$, then $P$ is a permutation polygroup on the set $\Omega_1 \times \Omega_2$ with the action defined by $(\omega_1, \omega_2)^g = \{(a, b) | a \in \omega_1^g, b \in \omega_2^g\}, \forall (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2, \quad \forall g \in P.$

The polygroup $P$ acts on itself as a permutation polygroup, if we define $x^g = x.g$ or $x^g = g^{-1}.x, \quad \forall g \in P, \quad \forall x \in P.$

**Proposition 4.2.** *Let $N$ be a normal subpolygroup of a polygroup $P$. Let $\Omega$ denote the set of all the right cosets $Nx$ $(x \in P)$, and define*

$$(Nx)^g = \{Nz | z \in Nxg\}, \quad \forall g \in P,$$

*then $P$ is a permutation polygroup on $\Omega$.*

Proof. It is easy to see that $(Nx)^e = Nx$. Now, let $g, h \in P$, then:

$$\begin{aligned} ((Nx)^g)^h &= (\{Nz|z \in Nxg\})^h \\ &= \bigcup_{z \in Nxg} \{Nt|t \in Nzh\} \\ &= \{Nt|t \in Nxgh\} \\ &= \bigcup_{\alpha \in g.h} \{Nt|t \in Nx\alpha\} \\ &= \bigcup_{\alpha \in g.h} (Nx)^\alpha = (Nx)^{g.h}. \end{aligned}$$

Therefore the second condition of Definition 4.1 is satisfied.

Now, we prove that $\bigcup_{Nx \in \Omega} (Nx)^g = \Omega$. Suppose that $Ny \in \Omega$, where $y \in P$. Since $P$ is a hypergroup, there exists $a \in P$ such that $y \in ag$ which implies $y \in Nag$, and so $Ny \in (Na)^g$, therefore $Ny \in \bigcup_{Nx \in \Omega} (Nx)^g$.

Now, we show that

$$Nx \in (Ny)^g \Rightarrow Ny \in (Nx)^{g^{-1}}.$$

To prove this, we observe that since $Nx \in \{Nz|z \in Nyg\}$, there exists $z_0 \in Nyg$ such that $Nx = Nz_0$. From $z_0 \in Nyg$, we get $g \in y^{-1}Nz_0$, hence $y^{-1} \in gz_0^{-1}N$ and so $y \in Nz_0g^{-1}$. Therefore $y \in Nxg^{-1}$ which implies $Ny \in (Nx)^{g^{-1}}$. ∎

**Theorem 4.3.** (Generalization of Cayley's Theorem) *Let $P$ be a polygroup acting on a non-empty finite set $\Omega$. Then there is a subset of $M_\Omega$ which is a polygroup under the induced action of $P$ and is isomorphism to $P$.*

Proof. We define the subset $S_\Omega$ of $M_\Omega$ as follows:

$$S_\Omega = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g \end{pmatrix} | g \in P \right\}.$$

The hyperoperation $o$ on $S_\Omega$ is defined as follows:

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g \end{pmatrix} o \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^h & \alpha_2^h & \cdots & \alpha_{|\Omega|}^h \end{pmatrix}$$
$$= \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^f & \alpha_2^f & \cdots & \alpha_{|\Omega|}^f \end{pmatrix} | f \in g.h \right\},$$

then $< S_\Omega, o, i, ^{-I} >$ is a polygroup, where

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g \end{pmatrix}^{-I} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^{g^{-1}} & \alpha_2^{g^{-1}} & \cdots & \alpha_{|\Omega|}^{g^{-1}} \end{pmatrix}.$$

Now, we define $\phi : P \longrightarrow S_\Omega$ by

$$\phi(g) = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g \end{pmatrix}.$$

It is easy to see that $\phi$ is well defined, one to one and onto. Moreover $\phi$ is a strong homomorphism because, for every $g, h \in P$, we have

$$\phi(g.h) = \{\phi(f) | f \in g.h\}$$

$$= \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^f & \alpha_2^f & \cdots & \alpha_{|\Omega|}^f \end{pmatrix} | f \in g.h \right\}$$
$$= \phi(g) o \phi(h).$$

Therefore $P \cong S_\Omega$. ■

The above theorem is true when $\Omega$ is infinite.

**Definition 4.4.** Let $P$ be a polygroup acting on non-empty sets $\Omega_1$ and $\Omega_2$. A map $\Theta : \Omega_1 \to \Omega_2$ is called a *P-map*, if $\Theta(x^g) = \Theta(x)^g$, for all $g \in P$ and all $x \in \Omega_1$.

If $\Theta$ is also a one to one correspondence, then $\Theta$ is called a *P-isomorphism* and $\Omega_1, \Omega_2$ are called *isomorphic.*

**Proposition 4.5.** *If $P$ is a polygroup and $\lambda, \rho$ are the left and the right regular representation of $P$, i.e. $\lambda_g : x \to g^{-1}.x$, $\rho_g : x \to x.g$, then $(P, \lambda)$ and $(P, \rho)$ are isomorphic.*

Proof. We define $\Theta : P \to P$ by $\Theta(x) = x^{-1}$, then

$$\begin{aligned}\Theta(\rho_g x) &= \Theta(x.g) = \{\Theta(y) | y \in x.g\} = \{y^{-1} | y \in x.g\} \\ &= (x.g)^{-1} = g^{-1}.x^{-1} = g^{-1}.\Theta(x) = \lambda_g \Theta(x).\end{aligned}$$

Since $\Theta$ is one to one therefore, it is a $P$-isomorphism. ■

**Definition 4.6.** Let $P$ be a permutation polygroup on a non-empty set $\Omega$. For $\alpha, \beta \in \Omega$ define $\sim$ by

$$\alpha \sim \beta \text{ iff } \alpha \in \beta^g \text{ for some } g \in P.$$

**Lemma 4.7.** *The relation defined above is an equivalence relation on* $\Omega$.

Now let $\Omega = \bigcup_{\alpha \in I} \Delta(\alpha)$ be the partition of $\Omega$ with respect to this relation. Then the sets $\Delta(\alpha), \alpha \in I$, are called *orbits* of $P$ on $\Omega$.

**Definition 4.8.** Let $P$ be a permutation polygroup on a non-empty set $\Omega$. If $P$ has only one orbit; that is, if $\alpha \sim \beta$ for every $\alpha, \beta \in \Omega$, we say that the polygroup is *transitive* on $\Omega$. If $|I| > 1$, we say that $P$ is *intransitive*.

**Theorem 4.9.** *Let $P$ be a permutation polygroup on a non-empty set $\Omega$, then the polygroup $P$ is transitive on every orbit.*

Proof. Let $\Delta(\alpha)$ be the orbit containing $\alpha \in \Omega$. Clearly, for the set $\Delta(\alpha)$, conditions (i), (ii) and (iv) of Definition 4.1 hold. Therefore we prove the third condition of Definition 4.1, i.e.

$$\bigcup_{\omega \in \Delta(\alpha)} \omega^g = \Delta(\alpha), \quad \forall g \in P.$$

To prove this suppose $\beta \in \bigcup_{\omega \in \Delta(\alpha)} \omega^g$, then $\beta \in \omega_0^g$ for some $\omega_0 \in \Delta(\alpha)$. So $\beta \sim \omega_0$ and $\omega_0 \sim \alpha$ which imply $\beta \sim \alpha$ or $\beta \in \Delta(\alpha)$, therefore $\bigcup_{\omega \in \Delta(\alpha)} \omega^g \subseteq \Delta(\alpha)$.

If $\Delta(\alpha_i), i \in I$, are all the disjoint orbits, then

$$\bigcup_{\omega \in \Delta(\alpha_i)} \omega^g \subseteq \Delta(\alpha_i), \quad \forall i \in I.$$

But, $\bigcup_{i \in I} \Delta(\alpha_i) = \Omega$ and $\Delta(\alpha_i) \cap \Delta(\alpha_j) = \Phi, \quad \forall i \neq j$. So $\bigcup_{\omega \in \Delta(\alpha_i)} \omega^g = \Delta(\alpha_i), \quad \forall_{i \in I}$.
Therefore $P$ acts on $\Delta(\alpha)$.

Now suppose that $\beta_1, \beta_2 \in \Delta(\alpha)$, then $\beta_1 \in \alpha^g, \beta_2 \in \alpha^h$ for some $g, h \in P$. By the forth condition of Definition 4.1 , we have $\alpha \in \beta_2^{h^{-1}}$ and so

$$\beta_1 \in (\beta_2^{h^{-1}})^g = \beta_2^{h^{-1}g}.$$

Therefore there exists $x \in h^{-1}g$ such that $\beta_1 \in \beta_2^x$. ■

**Corollary 4.10.** *Let $P$ be a permutation polygroup on a finite set $\Omega$. If $\triangle$ is an orbit of $P$ such that $|\Omega| = |\triangle|$, then $P$ is transitive on $\Omega$.*

**Definition 4.11.** Let $P$ be a permutation polygroup on a non-empty set $\Omega$ and let $\omega \in \Omega$. The set

$$P_\omega = \{g \in P | \omega^g = \omega^{g^{-1}} = \{\omega\}\} \subseteq P$$

is called the *stabilizer* of $\omega$.

**Corollary 4.12.** *The stabilizer $P_\omega$ is a subpolygroup of $P$ for each $\omega \in \Omega$.*

**Definition 4.13.** Let $P$ be a permutation polygroup on a non-empty set $\Omega$ and $\omega_1, \cdots, \omega_k \in \Omega$, then the stabilizer $P_{\omega_1,\cdots,\omega_k}$ is the subpolygroup

$$P_{\omega_1,\cdots,\omega_k} = \{g \in P | \omega_i^g = \omega_i^{g^{-1}} = \{\omega_i\}, \quad \text{for all } i = 1, \cdots, k\}.$$

This may be expressed also by

$$P_{\omega_1,\cdots,\omega_k} = \bigcap_{i=1}^{k} P_{\omega_i}.$$

It follows that if $P$ acts on $\Omega$ and $\omega_1, \omega_2 \in \Omega$, then $P_{\omega_1}$ and $P_{\omega_2}$ act on $\Omega$ and

$$(P_{\omega_1})_{\omega_2} = P_{\omega_1,\omega_2} = (P_{\omega_2})_{\omega_1}.$$

**Definition 4.14.** Let $P$ be a polygroup and $P$ acts on $\Omega$. The *kernel* of action is defined as follows:

$$H = \{g \in P | \omega^g = \{\omega\}, \quad \forall \omega \in \Omega\}.$$

**Proposition 4.15.** *Let $N$ be a normal subpolygroup of a polygroup $P$. If $\Omega$ is the set of all the right cosets of $N$ in $P$, $P$ acts on $\Omega$ and the kernel of this action is*

$$H = \bigcap_{x \in P} x^{-1}Nx.$$

Proof. Suppose that $g \in H$, then

$$(Nx)^g = \{Nz | z \in Nxg\} = \{Nx\}.$$

From $z \in Nxg$, we get $g \in x^{-1}Nz = x^{-1}Nx$, therefore

$$g \in \bigcap_{x \in P} x^{-1}Nx.$$

Now let $g \in \bigcap_{x \in P} x^{-1}Nx$, then $g \in x^{-1}Nx$, for every $x \in P$.

We have

$$\begin{aligned}(Nx)^g &= \{Nz \mid z \in Nxg\} \\ &\subseteq \{Nz \mid z \in Nxx^{-1}Nx\} \\ &\subseteq \{Nz \mid z \in NxN\}, \quad \text{since } N \lhd P \\ &= \{Nz \mid z \in NNx\} \\ &= \{Nz \mid z \in Nx\} = \{Nx\}.\end{aligned}$$

Therefore $g \in H$. ■

## 5. Weak polygroups

**Definition 5.1.** A multivalued system $\mathcal{M} =< P, ., e, ^{-1} >$, where $e \in P$, $^{-1} : P \longrightarrow P$, $\cdot : P \times P \longrightarrow \mathcal{P}^*(P)$ is called a *weak polygroup*, if the following axioms hold for all $x, y, z \in P$ :

(i) $(x.y).z \cap x.(y.z) \neq \emptyset$ (*weak associative* ),
(ii) $x.e = x = e.x$,
(iii) $x \in y.z$ implies $y \in x.z^{-1}$ and $z \in y^{-1}.x$.

The following elementary facts about weak polygroups follow easily from the axioms:

$$e \in x.x^{-1} \cap x^{-1}.x, \quad e^{-1} = e, \quad (x^{-1})^{-1} = x.$$

**Proposition 5.2.** *Let $(G, .)$ be a group and $\theta$ be an equivalence relation on $G$ such that:*

*(i) $x\theta e$ implies $x = e$,*
*(ii) $x\theta y$ implies $x^{-1}\theta y^{-1}$.*

*Let $\theta(x)$ be the equivalence class of the element $x \in G$. Suppose that $G/\theta = \{\theta(x) | x \in G\}$. Then $< G/\theta, \odot, \theta(e), ^{-I} >$ is a weak polygroup, when the hyperoperation $\odot$ is defined as follows:*

$$\odot : G/\theta \times G/\theta \to \mathcal{P}^*(G/\theta)$$
$$\theta(x) \odot \theta(y) = \{\theta(z) | z \in \theta(x).\theta(y)\},$$

*and* $\theta(x)^{-I} = \theta(x^{-1})$.

Proof. For all $x, y, z \in G$, we have

$$x.(y.z) \in \theta(x) \odot (\theta(y) \odot \theta(z)),$$
$$(x.y).z \in (\theta(x) \odot \theta(y)) \odot \theta(z),$$

therefore $\odot$ is weak associative. It is easy to see that $\theta(e)$ is the identity element in $G/\theta$ and $\theta(x^{-1})$ is the inverse of $\theta(x)$ in $G/\theta$. Now, we show that:

$\theta(z) \in \theta(x) \odot \theta(y)$ implies $\theta(x) \in \theta(z) \odot \theta(y^{-1})$ and $\theta(y) \in \theta(x^{-1}) \odot \theta(z)$.

We have $\theta(z) \in \theta(x) \odot \theta(y) = \{\theta(a) | a \in \theta(x).\theta(y)\}$, hence $\theta(z) = \theta(a)$ for some $a \in \theta(x).\theta(y)$, therefore there exist $b \in \theta(x)$ and $c \in \theta(y)$ such that $a = b.c$, so $b = a.c^{-1}$ which implies $\theta(b) = \theta(a.c^{-1}) \in \theta(a) \odot \theta(c^{-1})$. Therefore

$$\theta(x) \in \theta(z) \odot \theta(y^{-1})$$

In a similar way, we get

$$\theta(y) \in \theta(x^{-1}) \odot \theta(z).$$

Therefore $< G/\theta, \odot, \theta(e), ^{-I} >$ is a weak polygroup. ■

An extension of polygroups by polygroups has been introduced in [1]. In the following we define an extention of a weak polygroup by another weak polygroup.

**Definition 5.3.** *(An extension construction)* Suppose that $\mathcal{A} =< A, ., e, ^{-1} >$ and $\mathcal{B} =< B, ., e, ^{-1} >$ are weak polygroups whose elements have been renamed so that $A \cap B = \{e\}$ where $e$ is the identity of both $\mathcal{A}$ and $\mathcal{B}$. A new system $\mathcal{A}[\mathcal{B}] =< M, *, e, ^{-I} >$, which is called the extension of $\mathcal{A}$ by $\mathcal{B}$, is defined in the following way:

Set $M = A \cup B$ and let $e^{-I} = e$, $x^{-I} = x^{-1}$, $e * x = x * e = x$ for all $x \in M$, and for all $x, y \in M - \{e\}$,

$$x * y = \begin{cases} x.y & \text{if } x, y \in A, \\ x & \text{if } x \in B,\ y \in A, \\ y & \text{if } x \in A,\ y \in B, \\ x.y & \text{if } x, y \in B,\ y \neq x^{-1}, \\ x.y \cup A & \text{if } x, y \in B,\ y = x^{-1}. \end{cases}$$

Now, we show that the extension construction will always yield a weak polygroup.

**Theorem 5.4.** *$\mathcal{A}[\mathcal{B}]$ is a weak polygroup.*

Proof. The verification of the third condition of Definition 5.1 is similar to the proof of Theorem 1 in [1]. Therefore we show that weak associativity is valid. For all $x, y, z$ in $M$, we consider the following cases:

(1) If $x, y, z \in A$, then $(x.y).z = (x * y) * z$ and $x.(y.z) = x * (y * z)$,
(2) If $x, y, z \in B$, then $(x.y).z \subseteq (x * y) * z$ and $x.(y.z) \subseteq x * (y * z)$,
(3) If $x \in A, y, z \in B$, then $(y.z) \subseteq (x * y) * z$ and $y.z \subseteq x * (y * z)$,
(4) If $x \in A, y \in B, z \in A$, then $y \in (x * y) * z$ and $y \in x * (y * z)$,
(5) If $x \in A, y \in A, z \in B$, then $z \in (x * y) * z$ and $z \in x * (y * z)$,
(6) If $x \in B, y, z \in A$, then $x \in (x * y) * z$ and $x \in x * (y * z)$,
(7) If $x \in B, y \in B, z \in A$, then $x.y \subseteq (x * y) * z$ and $x.y \subseteq x * (y * z)$,
(8) If $x \in B, y \in A, z \in B$, then $x.z \subseteq (x * y) * z$ and $x.z \subseteq x * (y * z)$.

Thus, $*$ is weak associative. ■

The following definition, first was defined in [2] for polygroups.

**Definition 5.5.** The equivalence relation $\theta$ on a weak polygroup $\mathcal{M}$ is called a *full conjugation* on $\mathcal{M}$, if:

(i) $x\theta y$ implies $x^{-1}\theta y^{-1}$,

(ii) $z \in x.y$ and $z_1\theta z$ imply $z_1 \in x_1.y_1$ for some $x_1$ and $y_1$ where $\theta(x_1) = \theta(x)$ and $\theta(y_1) = \theta(y)$.

The collection of all $\theta$-classes, with the induced operation from $\mathcal{M}$, forms a weak polygroup.

**Proposition 5.6.** *Let $\mathcal{M}$ be a weak polygroup, then $\theta$ is full conjugation on $\mathcal{M}$ if and only if*

*(i) $(\theta(x))^{-1} = \theta(x^{-1})$ and*
*(ii) $\theta(\theta(x)y) = \theta(x)\theta(y)$.*

Proof. The proof is similar to the proof of Lemma 3 of [2]. ■

**Definition 5.7.** (see [7],[14]). Let $\mathcal{A} =< A, ., e_1, ^{-1} >, \mathcal{B} =< B, *, e_2, ^{-1} >$ be weak polygroups, and let $f$ be a mapping from $A$ into $B$, such that $f(e_1) = e_2$. Then $f$ is called:

(i) a *weak homomorphism*, if

$$f(x.y) \cap f(x) * f(y) \neq \emptyset, \quad \forall x, y \in A,$$

(ii) an *inclusion homomorphism*, if

$$f(x.y) \subseteq f(x) * f(y), \quad \forall x, y \in A,$$

(iii) a *strong homomorphism*, if

$$f(x.y) = f(x) * f(y), \quad \forall x, y \in A.$$

If $f$ is one to one, onto and strong homomorphism, then it is called an *isomorphism*. Moreover, if $f$ is defined on the same weak polygroup, then it is called an *automorphism*. The set of all automorphism of $A$, written $AutA$, is a group.

Let $\mathcal{M} =< P, ., e, ^{-1} >$ be a weak polygroup. We define the relation $\beta^*$ as the smallest equivalence relation on $P$ such that quoient $P/\beta^*$ is a group. In this case $\beta^*$ is called the fundamenatl equivalence relation on $P$ and $P/\beta^*$ is called the fundamentl group. This relation is studied by Corsini [3] concerning hypergroups, see also [9],[13],[14]. Suppose $\beta^*(a)$ is the equivalnce class containing $a \in P$, the product $\odot$ on $P/\beta^*$ is as follow:

$$\beta^*(a) \odot \beta^*(b) = \beta^*(c), \quad \forall c \in \beta^*(a).\beta^*(b).$$

Let $\mathcal{U}_P$ be the set of all finite products of elements of $P$ and define the relation $\beta$ on $P$ as follows:

$$x\beta y \quad \text{iff} \quad \{x, y\} \subseteq u, \quad \text{for some } u \in \mathcal{U}_P.$$

Similar to the proof of Theorem 1.2.2 of [14], one can prove that the fundamental relation $\beta^*$ is the transitive closure of the relation $\beta$.

The kernel of the canonical map $\Phi : P \to P/\beta^*$ is called the core of $P$ and is denoted by $\omega_P$. Here we also denote by $\omega_P$ the unit element of $P/\beta^*$. It is easy to prove the following statements:

(i) $\omega_P = \beta^*(e)$,
(ii) $\beta^*(x)^{-1} = \beta^*(x^{-1}), \quad \forall x \in P$.

**Lemma 5.8.** *$\beta^*$ is a full conjugation.*

Proof. We verify the necessary and sufficient conditions of Proposition 5.6 to prove that $\beta^*$ is a full conjugation. The first condition of Proposition 5.6 is verified by using the above relation. To verify the second condition, we have $\beta^*(x) \subseteq \beta^*(x)$ and $\{y\} \subseteq \beta^*(y)$ which imply $\beta^*(\beta^*(x)y) = \beta^*(x.y)$. And so $\beta^*(\beta^*(x)y) = \beta^*(x) \odot \beta^*(y)$. ∎

An element $x \in P$ will be called single, if its equivalence class with respect to $\beta^*$ is singleton, i.e. $\beta^*(x) = \{x\}$. We denote the set of all the single elements of $P$ by $S_P$. It is straightforward to prove that for $a \in P$ and $x \in S_P$, if $x \in a.y$ for some $y \in P$, then $\beta^*(a) = \{z \in P | zy = x\}$.

Let $\mathcal{M}_1 =< P_1, ., e_1, ^{-1} >$ and $\mathcal{M}_2 =< P_2, *, e_2, ^{-I} >$ be two weak polygroups, then on $P_1 \times P_2$ we can define a hyperproduct as follows:

$$(x_1, y_1) o (x_2, y_2) = \{(a, b) | a \in x_1.x_2, \quad b \in y_1 * y_2\}.$$

We call this the direct product of $P_1$ and $P_2$. It is easy to see that $P_1 \times P_2$ equipped with the usual direct product operation becomes a weak polygroup.

**Theorem 5.9.** *Let $\beta_1^*, \beta_2^*$ and $\beta^*$ be fundamental equivalence relations on $P_1, P_2$ and $P_1 \times P_2$ respectively, then*

$$(P_1 \times P_2)/\beta^* \cong P_1/\beta_1^* \times P_2/\beta_2^*.$$

Proof. First we define the relation $\widetilde{\beta}$ on $P_1 \times P_2$ as follows:

$$(x_1, y_1)\widetilde{\beta}(x_2, y_2) \Longleftrightarrow x_1\beta_1^* x_2, y_1 \beta_2^* y_2.$$

$\widetilde{\beta}$ is an equivalence relation. We define $\odot$ on $(P_1 \times P_2)/\widetilde{\beta}$ as follows:

$$\widetilde{\beta}(x_1, y_1) \odot \widetilde{\beta}(x_2, y_2) = \widetilde{\beta}(a, b),$$

for all $a \in \beta_1^*(x_1).\beta_1^*(x_2)$ and $b \in \beta_2^*(y_1).\beta_2^*(y_2)$.

Since $P_1$ and $P_2$ are weak associatives, we see that is $\odot$ associative and consequently $(P_1 \times P_2)/\widetilde{\beta}$ is a group.

Now let $\theta$ be an equivalence relation on $P_1 \times P_2$ such that $(P_1 \times P_2)/\theta$ is a group. Let $\theta(x_1, y_1)$ be the class of $(x_1, y_1)$. Then $\theta(x_1, y_1) \odot \theta(x_2, y_2)$ is singleton, i.e. $\theta(x_1, y_1) \odot \theta(x_2, y_2) = \theta(a, b), \forall (a, b) \in \theta(x_1, y_1).\theta(x_2, y_2)$. But also for every $(x_1, y_1), (x_2, y_2) \in P_1 \times P_2$ and $A \subseteq \theta(x_1, y_1), B \subseteq \theta(x_2, y_2)$ we have $\theta(x_1, y_1) \odot \theta(x_2, y_2) = \theta((x_1, y_1).\theta(x_2, y_2)) = \theta(AB)$, so this relation is valid for all finite products which means that the equality $\theta(x, y) = \theta(u, v)$ for every $(u, v) \in \mathcal{U}_{P_1 \times P_2}$ and $x \in u, y \in v$ holds.

Now, if $(x, y) \in \widetilde{\beta}(a, b)$, then $x\beta_1^* a$ and $y \beta_2^* b$. We have

$x\beta_1^* a$ iff $\exists x_1, \cdots, x_{m+1}$ with $x_1 = x, x_{m+1} = a$ and $u_1, \cdots, u_m \in \mathcal{U}_{P_1}$ such that

$$\{x_i, x_{i+1}\} \subseteq u_i, \quad i = 1, \cdots, m, \quad \text{and}$$

$y\beta_2^* b$ iff $\exists y_1, \cdots, y_{m+1}$ with $y_1 = y, y_{n+1} = b$ and $v_1, \cdots, v_n \in \mathcal{U}_{P_2}$ such that

$$\{y_j, y_{j+1}\} \subseteq v_j, \quad j = 1, \cdots, n.$$

Therefore, $(x_i, y_j) \in (u_i, v_j), (x_{i+1}, y_{j+1}) \in (u_i, v_j), \quad i = 1, \cdots, m, \quad j = 1, \cdots, n$. And so $\theta(x_i, y_j) = \theta(u_i, v_j), \theta(x_{i+1}, y_{j+1}) = \theta(u_i, v_j), \quad i = 1, \cdots, m, \quad j = 1, \cdots, n$
which implies $\theta(x_i, y_j) = \theta(x_{i+1}, y_{j+1}), \quad i = 1, \cdots, m, \quad j = 1, \cdots, n$. Therefore $\theta(x, y) = \theta(a, b)$ or $(x, y) \in \theta(a, b)$. So we get

$$(x, y)\widetilde{\beta}(a, b) \Longrightarrow (x, y) \in \theta(a, b).$$

Thus, the relation $\widetilde{\beta}$ is the smallest equivalence relation on $P_1 \times P_2$ such that $(P_1 \times P_2)/\widetilde{\beta}$ is a group, i.e. $\widetilde{\beta} = \beta^*$.
Now, we consider the map

$$f : P_1/\beta_1^* \times P_2/\beta_2^* \longrightarrow (P_1 \times P_2)/\beta^*$$

with $f(\beta_1^*(x), \beta_2^*(y)) = \beta^*(x, y)$. It is easy to see that $f$ is an isomorphism. ■

Using the fundamental equivalence relation we define semi-direct hyperproduct of weak polygroups.

**Definition 5.10.** Let $\mathcal{A} =< A, ., e_1, ^{-1} >$ and $\mathcal{B} =< B, *, e_2, ^{-1} >$ be weak polygroups. We consider the group $AutA$ and the fundamental group $B/\beta_B^*$, let

$$\widehat{\ } : B/\beta_B^* \to AutA$$
$$\beta^*(b) \to \widehat{\beta^*(b)} = \widehat{b}$$

be a homomorphism of groups. Then on $\mathcal{A} \times \mathcal{B}$ we define a hyperproduct as follows:

$$(a_1, b_1)o(a_2, b_2) = \{(x, y) | x \in a_1.\widehat{b_1}(a_2), y \in b_1 * b_2\}$$

and we call this the *semi-direct hyperproduct* of weak polygroups $\mathcal{A}$ and $\mathcal{B}$.

The above definition, first was introduced by Vougiouklis [12] concerning hypergroups.

**Theorem 5.11.** *$\mathcal{A} \times \mathcal{B}$ equipped with the semi-direct hyperproduct is a weak polygroup.*

Proof. Similarly to Theorem 2.4.1 of [10], weak associativity is valid. Since $a = a.\widehat{b}(e_1), a = e_1.\widehat{e_2}(a)$ and $b = b * e_2 = e_2 * b$, we have $(a,b)o(e_1,e_2) = (a,b) = (e_1,e_2)o(a,b)$, i.e. $(e_1,e_2)$ is the identity element in $\mathcal{A} \times \mathcal{B}$, and we can check that $(\widehat{b^{-1}}(a^{-1}), b^{-1})$ is the inverse of $(a,b)$ in $\mathcal{A} \times \mathcal{B}$.

Now, we show that $(z_1,z_2) \in (x_1,x_2)o(y_1,y_2) \Rightarrow (x_1,x_2) \in (z_1,z_2)o(y_1,y_2)^{-I}$ and $(y_1,y_2) \in (x_1,x_2)^{-I}o(z_1,z_2)$. We have $(z_1,z_2) \in (x_1,x_2)o(y_1,y_2) = \{(a,b) | a \in x_1.\widehat{x_2}(y_1), b \in x_2 * y_2\}$ which implies $z_1 \in x_1.\widehat{x_2}(y_1)$ and $z_2 \in x_2 * y_2$. Since $z_1 \in x_1.\widehat{x_2}(y_1)$ we get $x_1 \in z_1.\widehat{x_2}(y_1)^{-1}$ or $x_1 \in z_1.\widehat{x_2}(y_1^{-1})$. Since $z_2 \in x_2 * y_2$, then $x_2 \in z_2 * y_2^{-1}$. Therefore $\beta^*(x_2) = \beta^*(z_2) \odot \beta^*(y_2^{-1})$ and so $\widehat{\beta^*(x_2)} = \widehat{\beta^*(z_2) \odot \beta^*(y_2^{-1})} = \widehat{\beta^*(z_2)}.\widehat{\beta^*(y_2^{-1})}$ or $\widehat{x_2} = \widehat{z_2}\widehat{y_2^{-1}}$. Therefore we get $x_1 \in z_1.\widehat{z_2}\widehat{y_2^{-1}}(y_1^{-1})$. Now , we have $(x_1,x_2) \in \{(a,b) | a \in z_1.\widehat{z_2}\widehat{y_2^{-1}}(y_1^{-1}), b \in z_2 * y_2^{-1}\}$ or $(x_1,x_2) \in (z_1,z_2)o(y_1,y_2)^{-I}$.

On the other hand, we have

$$\begin{aligned}(x_1,x_2)^{-I}o(z_1,z_2) &= (\widehat{x_2^{-1}}(x_1^{-1}), x_2^{-1})o(z_1,z_2) \\ &= \{(a,b) | a \in \widehat{x_2^{-1}}(x_1^{-1}).\widehat{x_2^{-1}}(z_1), b \in x_2^{-1} * z_2\} \\ &= \{(a,b) | a \in \widehat{x_2^{-1}}(x_1^{-1}.z_1), b \in x_2^{-1} * z_2\}.\end{aligned}$$

Since $z_1 \in x_1.\widehat{x_2}(y_1)$ implies $\widehat{x_2}(y_1) \in x_1^{-1}.z_1$ hence $y_1 \in \widehat{x_2^{-1}}(x_1^{-1}.z_1)$. Therefore $(y_1,y_2) \in (x_1,x_2)^{-I}o(z_1,z_2)$. ∎

**Acknowledgments**

The author greatefully acknowledges the support from Institute for Studies in Theoretical Physics and Mathematics (IPM). The author would like to thank Professor M.R. Darafsheh for his guidance and helpful discussions throughout this work.

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