Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol. 14, 2000, Fasc. 1-2

Study of $L(r,s)$-colourings for Caterpillars and for Trees

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In this paper, we study the chromatic numbers $\chi_{s,s}$ and $\chi_{r,s}$ for caterpillars and for trees.

AMS Subj. Classification: 05C05, 05C15, 05C20

Key Words: graph theory, directed graphs, colouring, trees, caterpillars

1. Introduction

Let $G = (V, S)$ be a non directed graph. Given a set H of colours and two positive integers s, r with $s \geq r$, we say that $K : V \to H$ is an $L(r,s)$-colouring of G, if in every simple chain of G of length $s-1$ there are at least vertices $x_1, x_2, \ldots, x_r$ such that $|K(x_1, x_2, \ldots, x_r)| = r$.

The minimum number of colours so that G has an $L(r,s)$-colouring is said to be the (r,s)-chromatic number of G and it is indicated by $\chi_{r,s}$, [3].

A colouring $L(r,r)$ is the classical vertex-colouring of a graph. A caterpillar is a tree T_1 such that $T' = (x_1, x_2, \ldots, x_{|T'|})$, obtained deleting the verteces of T' of degree one, is a path. T' is called the spine of T' and its verteces are the spinal modes of T_1.

The neighborhood $\Gamma(x)$ of $x \in V(T_1)$ is the set of all vertices adjacent to x. The end-neighborhood $\Gamma_1(x)$ consists of all the leaves in $\Gamma(x)$. Let $d_1(x) = |\Gamma_1(x)|$, $\Gamma_1(G) = T_1 - T_1'$.

Let T be the following tree: $T = T_1 \cup T_2 \cup \ldots \cup T_n$, where T_i, $1 \leq i \leq n$ is a caterpillar, $T_i' = \{x_{i0}, x_{i1}, x_{i2}, \ldots, x_{i|T_i'|}\}$ is a spine of the T_i-caterpillar and $d(x_{ij}) \geq 2$ for $j = 1, \ldots, |T_i'|$, $h = |T_i'|$, $h \geq 1$. T will be called an h-caterpillar with spine $T' = \{x_{10}, x_{21}, x_{30}, \ldots, x_{i|T'|_0}\}$.

At last we indicate by y_{ijk} the vertices of T_i such that $d_1(x_{ij}) = |\Gamma_1(x_{ij})| = k$, for $i = 1, \ldots, n$, $j = 1, \ldots, |T_i'| - 1$, and $k \geq 0$.

2. Study of $\chi_{s,s}$

Theorem 2.1. *Let T be an h-caterpillar. If $s > 3, h+2 \leq s \leq 2(h+1)$ and $|T'| \geq 3s-8$, then:*

i) $\chi_{s,s} = \frac{s(s+2)}{4}$ for s even,

ii) $\chi_{s,s} = \frac{(s+1)^2}{4}$ for s odd.

Proof. Suppose $s > 3, h+2 \leq s \leq 2(h+1)$ and $|T'| \geq 3s-8$. Let K be the $L(s,s)$-colouring of T, defined as follows:

$$K(x_{qs+i}) = \omega_i \quad i = 1, \ldots .s, \quad q \geq 0.$$

Consider the vertices x_{p1}, $p = 1, \ldots, |T'|$. We define $K(x_{p1}) = \omega_{5+p}$ $p = 1, \ldots, s-2$ and

$$K(x_{p1}) = K(y_{p+k}) \quad k \geq 0.$$

Now, since it results $s \geq h+2$, it is immediate to colour the remaining vertices x_{p1} and y_{p1k}, $s-1 \leq p \leq |T'|$ $K \geq 0$, repeating cyclically the new colours $\omega_{s+1}, \omega_{s+2}, \ldots, \omega_{s+(s-2)}$.

Consider the vertices x_{p2}, $p = 1, \ldots, |T'|$.

We define

$$K(x_{p2}) = \omega_{2(s-1)+p} \quad p = 1, \ldots, s-4 \text{ and } K(y) = K(x_{j-1})$$

for each

$$y \in \Gamma(x_j) \cup \Gamma_1(G), \quad j \notin (s-2, s-1, \ldots, |T'| - (s-3))$$

and $\omega_0 = \omega_s$.

It follows $x_{s,s} = s$.

$\Rightarrow$ Suppose that there exists a vertex $x_i, s-2 \leq i \leq |T'| - (s-3)$ such that $d_1(x_i) > 0$. It is immediate to see that in the graph there is a path of lenght $s-1$ not colourable by s distinct colours.

Hence $\chi_{s,s} = s+1$.

ii) $\Leftarrow$ Let K be the $L(s,s)$-colouring of T so defined:

$$K(x_i) = \omega_i \quad \text{for each} \quad i = 1, 2, \ldots, s$$

$$K(x_{ps+i}) = K(x_i) \quad \text{for each} \quad i = 1, 2 \ldots, s-1, \quad p \geq 0$$

$$K(y) = K(x_{j-1}) \quad \text{for each} \quad y \in \Gamma(x_j) \cap \Gamma_1(G),$$

$$j \notin (s-2, s-1, \ldots, |T'| - (s-3)) \quad \text{and} \quad \omega_0 = \omega_s$$

$K(y) = \omega_{s+1}$ for each $y \in \Gamma(x_j) \cap \Gamma_1(G)$, $y \in (s-2, s-1, \dots, |T'| - (s-3))$.

It follows $\chi_{s,s} \leq s+1$.

Now suppose $\chi_{s,s} < s+1$.

Necessarily from 1) and 2), in the graph there is a path of lenght $s-1$ not colourable by s distinct colours. Hence we have $\chi_{s,s} = s+1$.

$\Rightarrow$ Suppose that for each x_i, $s-2 \leq i \leq |T'| - (s-3)$, $d_1(x_i) = 0$.

It follows $\chi_{s,s} = s$.

Moreover if x_k, x_j, for $s-2 \leq k < j \leq |T'| - (s-3)$, are two vertices such that $d_1(x_k) > 0$, $d_1(x_j) > 0$ and $d(x_k, x_j) = s-3$, then there is in

$$r_3(W) + r_0(W) = \frac{v(v-1)}{6} - \frac{p(v-1)}{2} + \frac{p(p-1)}{2}.$$

If R_i is the family of blocks of Σ such that $b \in R_i$ if and only if $|b \cap W| = i$, for $i = 0, 1, 2, 3$, and $D = R_3 \cup R_0$, $D = \cup_{b \in B-D} b$, then the set $W \cap S$ is a blocking set for the partial system $\Sigma' = (D, B-D)$.

If $p = \frac{v+h}{2}$, where h is an odd number, it follows:

$$r_3(W) + r_0(W) = \frac{v^2 - 4v + 3h^2}{24}$$

hence:

$$\beta(\Sigma) = \frac{v^2 - 4v + 3}{24},$$

where we consider a set $W \subset S$, $|W| = \frac{v+1}{2}$. ■

Theorem 2.2. *For every* $S(4,5,v)$, $v > 5$, *it is:*

$$\beta = \frac{v^4 - 16v^3 + 86v^2 - 176v + 105}{1.920}$$

Proof. Let $\Sigma = (S, B)$ be an $S(4,5,v)$ system, $v > 5$.
If $W \subseteq S$ and $|W| = p > 0$, $m = |B|$ we have

$$m = \sum_{i-0}^{5} r_i(W), \quad r_4(W) + 5r_5(W) = \binom{p}{4},$$

$$\sum_{i=1}^{5} i r_1(W) = p \frac{(v-1)(v-2)(v-3)}{24}.$$

■

3. Study of $\chi_{r,s}$

Theorem 3.1. *Let T be an h-caterpillar. If $|T'| \geq 3s-8$, $s>3$, and $4 \leq s \leq 2(h+1)$, then:*
i) $\chi_{r,s} \leq r + (r-\frac{s}{2})(r-\frac{s}{2}-1)$ for s even,
ii) $\chi_{r,s} \leq r + (r-\frac{s+1}{2})^2$ for s odd.

Proof. Suppose $4 \leq s \leq 2(h+1)$ and $|T'| \geq 3s-8$. Let K be the $L(r,s)$-colouring of T, defined as follows:

$$K(x_{p0}) = \omega_i, \quad i = 1,2,\dots,r, \quad p = qr+1, \quad q \geq 0$$

$$K(x_{pj}) = K(y_{pjk}) = K(x_{p-j,0}) \quad 1 \leq j \leq s-r \leq r, \quad k \geq 0$$

$$p = qr+i, \quad i = 1,\dots,r, \quad q \geq 0 \text{ and } \omega_0 = \omega_r.$$

If $s-r = qr+t$ $q \geq 0, t < r$, then we repeate the colours ω_i, $i = 1,\dots,r$.
Consider the vertices $x_{p,s-r+1}$, $p = 1,2,\dots,|T'|$.
We define

$$K(x_{p,s-r+1}) = \omega_{r+p} \quad p = 1,\dots,s-2(s-r)-2 \text{ and}$$

$$K(x_{p,s-r+1}) = K(y_{p,s-r+1,k}), \quad k \geq 0.$$

Now it is immediate to colour the remaining vertices $x_{p,s-r+1}$, $p = s-2(s-r)-1,\dots,|T'|$ and $y_{p,s-r+1,k}$, $k \geq 0$, repeating cyclically the new colours $\omega_{r+1}, \omega_{r+2}, \omega_{r+3}, \dots, \omega_{r+s-2(s-r)-2}$.

Consider the vertices $x_{p,s-r+2}$ and $y_{p,s-r+2,k}$, $p = 1,\dots,|T'|$ and $k \geq 0$. We colour the vertices $x_{1,s-r+2}, \dots, x_{s-2(s-2)-4,s-r+2}$ by $s-2(s-r)-4$ new colours and we repeate cyclically these colours for the remaining vertices, where $K(x_{p,s-r+2}) = K(y_{p,s-r+2,k})$.

In general, we consider the vertices $x_{p,s-2+j}$, $p = 1,2,\dots,|T'|$ and $j < \frac{s}{2} - (s-r)$ for s even ($j < \frac{s+1}{2} - (s-r)$ for s odd). We colour the vertices $x_{1,s-r+j}, x_{2,x-r+j}, \dots, x_{s-2(s-r)-2j,s-r+j}$ by $s-2(s-r)-2j$ new colours and repeate these colours cyclically, where it results $K(x_{p,s-r+j}) = K(y_{p,s-r+j,k})$, $k \geq 0$.

Now, given the vertices $x_{p,\frac{s}{2}}$ for s even and $x_{p,\frac{s+1}{2}}$ for s odd, it is possible to colour them as follows:

$$K(x_{p,\frac{s}{2}}) = K(x_{p+1,\frac{s}{2}-1}) \text{ and } K(x_{p+1,\frac{s}{2}}) = K(x_{p,\frac{s}{2}-1})$$

$$p = 1,2,3,\dots,|T'|-1 \text{ and } K(x_{p,\frac{s}{2}}) = K(y_{p,\frac{s}{2},k})$$

$$K(x_{p+1,\frac{s}{2}}) = K(y_{p,\frac{s}{2},k}), \quad \text{for } s \text{ even},$$

$$K(x_{p,\frac{s+1}{2}}) = K(y_{p,\frac{s+1}{2},k}) = K(x_{p,\frac{s-1}{2}}), \quad \text{for } s \text{ odd.}$$

At last, for each vertex $x_{p,\frac{s}{2}+j}$ it is immediate to see that it exists a colour $\omega_i \in (\omega_1, \omega_2, \ldots, \omega_s)$ such that

$$K(x_{p,\frac{s}{2}+j}) = K(y_{p,\frac{s}{2}+j,k}) = \omega_j, \quad k \geq 0.$$

It follows:

$$\chi_{r,s} \leq r + [s - 2(s-r) - 2] + [s - 2(s-r) - 4] + \ldots + 2$$
$$= r + (r - \frac{s}{2})(r - \frac{s}{2} - 1), \quad \text{for } s \text{ even.}$$
$$\chi_{r,s} \leq r + [s - 2(s-r) - 2] + [s - 2(s-r) - 4] + \ldots + 1$$
$$= r + (r - \frac{s+1}{2})^2, \quad \text{for } s \text{ odd.}$$

■

Corollary 3.1. *Let T^* be a sub-tree of T, $s > 3$ and $h+2 \leq s \leq 2(h+1)$, then:*

i) $s \leq \chi_{s,s} \leq \frac{s(s+2)}{4}$ *for s even;*
ii) $s \leq \chi_{s,s} \leq \frac{(s+1)^2}{4}$ *for s odd.*

Proof. It follows from Theorems 2.1, 2.2, 2.3 [5], from Theorem 2.1 [6] and from Theorem 2.1. ■

Corollary 3.2. *Let T^* be a sub-tree of T, $s > 3$ and $4 \leq s \leq 2(h+1)$, then:*

i) $r \leq \chi_{r,s} \leq r + (r - \frac{s}{2})(r - \frac{s}{2} - 1)$, *for s even;*
ii) $r \leq \chi_{r,s} \leq r + (r - \frac{s+1}{2})^2$, *for s odd.*

Proof. It follows from Theorem 2.2, [6] and Theorem 3.1. ■

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Received: 20.09.1997