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On a Two Variables Analogue of Bessel Polynomials

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The present paper deals with a study of a two variables analogue of Bessel polynomials. Certain integral representations, a Schläfli's contour integral, a fractional integral, Laplace transformations, some generating functions and double generating functions have been obtained.

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1. Introduction

In 1949 Krall and Frink [11] initiated a study of what they called Bessel polynomials. In their terminology the simple Bessel polynomial is

$$Y_n(x) = {}_2F_0[-n, 1+n; -; -\frac{x}{2}], \tag{1.1}$$

and the generalized one is

$$Y_n(a,b,x) = {}_2F_0[-n, a-1+n; -; -\frac{x}{b}]. \tag{1.2}$$

These polynomials were introduced by them in connection with the solution of the wave equation in spherical coordinates. They are the polynomial solutions of the differential equation

$$x^2 y''(x) + (ax+b)y'(x) = n(n+a-1)y(x), \tag{1.3}$$

where n is a positive integer and a and b are arbitrary parameters. These polynomials are orthogonal on the unit circle with respect to the weight function

$$\rho(x,\alpha)=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha)}{\Gamma(\alpha+n-1)}(-\frac{2}{x})^n. \tag{1.4}$$

Several authors including Agarwal [1], Al-Salam [2], Brafman [3], Burchnall [4], Carlitz [5], Chatterjea [6], Dickinson [7], Eweida [9], Grosswald [10], Rainville [14], and Toscano [18] have contributed to the study of the Bessel polynomials.

The aim of the present paper is to introduce a two variables analogue $Y_n^{(\alpha,\beta)}(x,y)$ of the Bessel polynomials $Y_n^{(\alpha)}(x)$ defined by

$$Y_n^{(\alpha)}(x)={}_2F_0[-n,\alpha+n+1;-;-\frac{x}{2}] \tag{1.5}$$

and to obtain certain results involving the two variable Bessel polynomial

$$Y_n^{(\alpha,\beta)}(x,y).$$

2. The polynomial $Y_n^{(\alpha,\beta)}(x,y)$

The Bessel polynomial of two variables $Y_n^{(\alpha,\beta)}(x,y)$ is defined as follows:

$$Y_n^{(\alpha,\beta)}(x,y)=\sum_{n=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(\alpha+n+1)_s(\beta+n+1)_r}{r!\ s!}(-\frac{x}{2})^s(-\frac{y}{2})^r. \tag{2.1}$$

For $y=0$, (2.1) reduces to the Bessel polynomial $Y_n^{(\alpha)}(x)$.

$$Y_n^{(\alpha,\beta)}(x,0)=Y_n^{(\alpha)}(x). \tag{2.2}$$

where $Y_n^{(\alpha)}(x)$ is defined by (1.5).

Similarly,

$$Y_n^{(\alpha,\beta)}(0,y)=Y_n^{(\beta)}(y). \tag{2.3}$$

Also, for $\alpha=-n-1$, we have

$$Y_n^{(-n-1,\beta)}(x,y)=Y_n^{(\beta)}(y). \tag{2.4}$$

Similarly,

$$Y_n^{(\alpha,-n-1)}(x,y)=Y_n^{(\alpha)}(x). \tag{2.5}$$

3. Integral representations

It is easy to show that the polynomial $Y_n^{(\alpha,\beta)}(x,y)$ has the following integral representations:

$$\frac{1}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}\int_0^\infty\int_0^\infty u^{\alpha+n}v^{\beta+n}(1+\frac{xu}{2}+\frac{yv}{2})^n e^{-u-v}dudv$$

$$= Y_n^{(\alpha,\beta)}(x,y). \tag{3.1}$$

For $y=0$, (3.1) reduces to

$$Y_n^{(\alpha)}(x) = \frac{1}{\Gamma(\alpha+n+1)}\int_0^\infty u^{\alpha+n}(1+\frac{xu}{2})^n e^{-u}du. \tag{3.2}$$

For $y=0$, $\alpha=a-2$ and x replaced by $\frac{2x}{b}$, (3.1) becomes

$$Y_n(a,b,x) = \frac{1}{\Gamma(a-1+n)}\int_0^\infty t^{a-2+n}(1+\frac{xt}{b})^n e^{-t}dt \tag{3.3}$$

a result due to Agarwal [1].

$$\int_0^t\int_0^s x^\alpha(s-x)^{n-1}y^\beta(t-y)^{n-1}Y_n^{(\alpha,\beta)}(x,y)dxdy$$

$$= \frac{s^{\alpha+n}t^{\beta+n}\{\Gamma(n)\}^2}{(\alpha+1)_n(\beta+1)_n}Y_n^{(\alpha-n,\beta-n)}(x,t), \tag{3.4}$$

$$\int_0^1\int_0^1 u^{\gamma-1}(1-u)^{\delta-1}v^{\lambda-1}(1-v)^{\mu-1}Y_n^{(\alpha,\beta)}(xu,yv)dudv =$$

(3.5)
$$\frac{\Gamma(\gamma)\Gamma(\delta)\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\gamma+\delta)\Gamma(\lambda+\mu)}F^{1:2;2}_{-:1;1}\left[\begin{matrix}-n; & \alpha+n+1, & \gamma; & \beta+n+1, & \lambda; \\ -: & \gamma+\delta; & & \lambda+\mu; & \end{matrix}\; -\frac{x}{2},-\frac{y}{2}\right],$$

$$\int_0^1\int_0^1 u^{\gamma-1}(1-u)^{\delta-1}v^{\lambda-1}(1-v)^{\mu-1}Y_n^{(\alpha,\beta)}(x(1-u),yv)dudv =$$

(3.6)
$$\frac{\Gamma(\gamma)\Gamma(\delta)\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\gamma+\delta)\Gamma(\lambda+\mu)}F^{1:2;2}_{-:1;1}\left[\begin{matrix}-n; & \alpha+n+1, & \delta; & \beta+n+1, & \lambda; \\ -: & \gamma+\delta; & & \lambda+\mu; & \end{matrix}\; -\frac{x}{2},-\frac{y}{2}\right],$$

$$\int_0^1\int_0^1 u^{\gamma-1}(1-u)^{\delta-1}v^{\lambda-1}(1-v)^{\mu-1}Y_n^{(\alpha,\beta)}(xu,y(1-v))dudv =$$

(3.7)
$$\frac{\Gamma(\gamma)\Gamma(\delta)\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\gamma+\delta)\Gamma(\lambda+\mu)}F^{1:2;2}_{-:1;1}\left[\begin{matrix}-n; & \alpha+n+1, & \gamma; & \beta+n+1, & \mu; \\ -: & & \gamma+\delta; & & \lambda+\mu;\end{matrix}\; -\frac{x}{2},-\frac{y}{2}\right],$$

$$\int_0^1\int_0^1 u^{\gamma-1}(1-u)^{\delta-1}v^{\lambda-1}(1-v)^{\mu-1}Y_n^{(\alpha,\beta)}(x(1-u),y(1-v))dudv =$$

(3.8)
$$\frac{\Gamma(\gamma)\Gamma(\delta)\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\gamma+\delta)\Gamma(\lambda+\mu)}F^{1:2;2}_{-:1;1}\left[\begin{matrix}-n; & \alpha+n+1, & \delta; & \beta+n+1, & \lambda; \\ -: & & \gamma+\delta & & \lambda+\mu;\end{matrix}\; -\frac{x}{2},-\frac{y}{2}\right],$$

Particular Cases:

Some interesting particular cases of the above results are as follows:
(i) Taking $\gamma=\alpha+1$, $\lambda=\beta+1$, $\delta=\mu=n$ in (3.5), we obtain

$$\int_0^1\int_0^1 u^{\alpha}(1-u)^{n-1}v^{\beta}(1-v)^{n-1}Y_n^{(\alpha,\beta)}(xu,yv)dudv$$

$$= \frac{\{\Gamma(n)\}^2}{(\alpha+1)_n(\beta+1)_n}Y_n^{(\alpha-n,\beta-n)}(x,y) \tag{3.9}$$

which is equivalent to (3.4).

(ii) Taking $\gamma=n+1$, $\delta=\alpha$, $\lambda=n+1$, $\mu=\beta$ in (3.5), we get

$$\int_0^1\int_0^1 u^{n}(1-u)^{\alpha-1}v^{n}(1-v)^{\beta-1}Y_n^{(\alpha,\beta)}(xu,yv)dudv$$

$$= \frac{(n!)^2}{(\alpha)_{n+1}(\beta)_{n+1}}Y_n^{(0,0)}(x,y). \tag{3.10}$$

(iii) Replacing γ by $\alpha-\gamma+n+1$, λ by $\beta-\lambda+n+1$ and putting $\delta=\gamma$ and $\mu=\lambda$ in (3.5), we get

$$\int_0^1\int_0^1 u^{\alpha-\gamma+n}(1-u)^{\gamma-1}v^{\beta-\lambda+n}(1-v)^{\lambda-1}Y_n^{(\alpha,\beta)}(xu,yv)dudv$$

$$= \frac{\Gamma(\alpha-\gamma+n+1)\Gamma(\gamma)\Gamma(\beta-\lambda+n+1)\Gamma(\lambda)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}Y_n^{(\alpha-\gamma,\beta-\lambda)}(x,y). \tag{3.11}$$

Similar particular cases hold for (3.6), (3.7) and (3.8) also.
Also, using the integral (see Erdélyi et al. [8], vol I, p.14),

$$2i\sin\pi \quad z\Gamma(z) = -\int_{\infty}^{(0+)} (-t)^{z-1} e^{-t} dt \tag{3.12}$$

and the fact that

$$(1-x-y)^n = \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s} y^r x^s}{r!\ s!}, \tag{3.13}$$

we can easily derive the following integral representations for $Y_n^{(\alpha,\beta)}(x,y)$:

$$-\int_{\infty}^{(0+)}\int_{\infty}^{(0+)} (-u)^{\alpha+n}(-v)^{\beta+n} e^{-u-v}(1+\frac{xu}{2}+\frac{yv}{2})^n dudv$$

$$= 4\sin\pi\alpha \quad \sin\pi\beta \quad \Gamma(\alpha+n+1)\Gamma(\beta+n+1)Y_n^{(\alpha,\beta)}(x,y), \tag{3.14}$$

$$\frac{\sin\pi\alpha \quad \sin\pi\beta \quad \Gamma(1+\alpha+n)\Gamma(1+\beta+n)}{\pi^2} \int_0^\infty \int_0^\infty u^{-\alpha-n-1} v^{-\beta-n-1} e^{-u-v}$$

$$\times Y_n^{(\alpha,\beta)}(\frac{2x}{u},\frac{2y}{v}) dudv = (1-x-y)^n. \tag{3.15}$$

4. Schlafli's contour integral

It is easy to show that

$$\int_{-\infty}^{(0+)}\int_{-\infty}^{(0+)} u^{\alpha+n} v^{\beta+n} e^{u+v}(1-\frac{xu}{2}-\frac{yv}{2})^n dudv$$

$$= -4\sin\pi\alpha \quad \sin\pi\beta \quad \Gamma(1+\alpha+n)\Gamma(1+\beta+n)Y_n^{(\alpha,\beta)}(x,y). \tag{4.1}$$

Proof of (4.1): We have

$$\frac{1}{(2\pi i)^2}\int_{-\infty}^{(0+)}\int_{-\infty}^{(0+)} u^{\alpha+n} v^{\beta+n} e^{u+v}(1-\frac{xu}{2}-\frac{yv}{2})^n dudv$$

$$= \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s}(\frac{x}{2})^s(\frac{y}{2})^r}{r!\ s!} \frac{1}{(2\pi i)^2}\int_{-\infty}^{(0+)}\int_{-\infty}^{(0+)} u^{\alpha+n+s} v^{\beta+n+r} e^{u+v} dudv$$

$$= \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} (\frac{x}{2})^s (\frac{y}{2})^2}{r! \;\; s! \Gamma(-\alpha - n - s) \Gamma(-\beta - n - r)},$$

using Hankel's formula (see A. Erdélyi et al. [8], 1.6 (2))

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^t t^{-z} dt. \tag{4.2}$$

Finally, (4.1) follows from (2.1) after using the result

$$\Gamma(z)\Gamma(1-z) = \pi \;\; \text{cosec} \pi z. \tag{4.3}$$

5. Fractional integrals

Let L denote the linear space of (equivalent classes of) complex-valued functions $f(x)$ which are Lebegue-integrabe on $[0, \alpha]$, $\alpha < \infty$. For $f(x) \in L$ and complex number μ with $\Re\mu > 0$, the Riemann-Liouville fractional integral of order μ is defined as (see Prabhakar [12], p.72)

$$I^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t) dt \quad \text{for almost all } x \in [0, \alpha] \tag{5.1}$$

Using the operator I^μ, Prabhakar [13] obtained the following result for $\Re\mu > 0$ and $\Re\alpha > -1$:

$$I^\mu [x^\alpha Z_n^\alpha (x; k)] = \frac{\Gamma(kn + \alpha + 1)}{\Gamma(kn + \alpha + \mu + 1)} x^{\alpha+\mu} Z_n^{\alpha+\mu}(x; k), \tag{5.2}$$

where $Z_n^\alpha(x; k)$ is Konhauser's biorthogonal polynomial.

In an attempt to obtain a result analogous to (5.2) for the polynomial $Y_n^{(\alpha,\beta)}(x, y)$ we first seek a two variable analogue of (5.1).

A two variable analogue of I^μ may be defined as

$$I^{\lambda,\mu}[f(x,y)] = \frac{1}{\Gamma(\lambda)\Gamma(\mu)} \int_0^x \int_0^y (x-u)^{\lambda-1} (y-v)^{\mu-1} f(u,v) du dv. \tag{5.3}$$

Putting $f(x, y) = x^{\alpha+n-\lambda} y^{\beta+n-\mu} Y_n^{(\alpha,\beta)}(x, y)$ in (5.3), we obtain

$$I^{\lambda,\mu}[x^{\alpha+n-\lambda} y^{\beta+n-\mu} Y_n^{(\alpha,\beta)}(x, y)]$$

$$= \frac{1}{\Gamma(\lambda)\Gamma(\mu)} \int_0^x \int_0^y (x-u)^{\lambda-1} (y-v)^{\mu-1} u^{\alpha+n-\lambda} v^{\beta+n-\mu} Y_n^{(\alpha,\beta)}(u, v) du dv$$

$$= \frac{x^{\alpha+n}y^{\beta+n}}{\Gamma(\lambda)\Gamma(\mu)} \int_0^1 \int_0^1 (1-t)^{\lambda-1}(1-\omega)^{\mu-1} t^{\alpha+n-\lambda}\omega^{\beta+n-\mu} Y_n^{(\alpha,\beta)}(xt, y\omega)dtd\omega$$

(by putting $u = xt$ and $v = y\omega$)

$$= x^{\alpha+n}y^{\beta+n} \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s}\Gamma(\alpha-\lambda+n+1+s)\Gamma(\beta-\mu+n+1+r)}{r!\ \ s!\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}$$

$$\times (-\frac{x}{2})^s(-\frac{y}{2})^r.$$

We thus arrive at

$$I^{\lambda,\mu}[x^{\alpha+n-\lambda}y^{\beta+n-\mu}Y_n^{(\alpha,\beta)}(x,y)]$$

$$= \frac{x^{\alpha+n}y^{\beta+n}\Gamma(\alpha-\lambda+n+1)\Gamma(\beta-\mu+n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)} Y_n^{(\alpha-\lambda,\beta-\mu)}(x,y). \tag{5.4}$$

6. Laplace transform

In the usual notation the Laplace transform is given by

$$L\{f(t):s\} = \int_0^\infty e^{-st}f(t)dt, \quad \Re(s-a) > 0, \tag{6.1}$$

where $f \in L(0,R)$ for every $R > 0$ and $f(t) = O(e^{st}),\ t \to \infty$.

Using (6.1) Srivastava [16] proved

$$L\{t^\beta Z_n^\alpha(xt;k):s\}$$

$$= \frac{(\alpha+1)_{kn}\Gamma(\beta+1)}{s^{\beta+1}n!}\, {}_{k+1}F_k \left[\begin{matrix} -n, & \Delta(k,\beta+1); \\ & \Delta(k,\alpha+1); \end{matrix}\ (\frac{x}{s})^s\right], \tag{6.2}$$

provided that $\Re(s) > 0$ and $\Re(\beta > -1$.

In an attempt to obtain a result analogous to (6.2) for $Y_n^{(\alpha,\beta)}(x,y)$ we take a two variable analogue of (6.1) as follows:

$$L\{f(u,v):s_1,s_2\} = \int_0^\infty \int_0^\infty e^{-s_1u-s_2v}f(u,v)dudv. \tag{6.3}$$

Now, we have

$$L\{u^{-\alpha-n-1}v^{-\beta-n-1}Y_n^{(\alpha,\beta)}(\frac{2x}{us_1},\frac{2y}{vs_2}):s_1,s_2\}$$

$$(6.4) \qquad = \frac{\pi^2 s_1^{\alpha+n} s_2^{\beta+n}}{\sin\pi\alpha \quad \sin\pi\beta \quad \Gamma(\alpha+n+1)\Gamma(\beta+n+1)}(1-x-y)^n.$$

Similarly, we obtain

$$L\{u^{\alpha+n}v^{\beta+n}(1+\frac{xus_1}{2}+\frac{yvs_2}{2})^n : s_1, s_2\}$$

$$(6.5) \qquad = \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{s_1^{\alpha+n+1} s_2^{\beta+n+1}} Y_n^{(\alpha,\beta)}(x,y).$$

7. Generating functions

It is easy to derive the following generating functions for $Y_n^{(\alpha,\beta)}(x,y)$:

$$(7.1) \qquad \sum_0^\infty \frac{t^n}{n!} Y_n^{(\alpha-n,\beta-n)}(x,y) = e^t(1-\frac{xt}{2})^{-\alpha-1}(1-\frac{yt}{2})^{-\beta-1},$$

$$\sum_0^\infty \frac{t^n}{n!} Y_n^{(\alpha,\beta-n)}(x,y)$$

$$(7.2) \qquad = (1-2xt)^{-\frac{1}{2}}[\frac{2}{1+\sqrt{1-2xt}}]^\alpha[1-\frac{yt}{1+\sqrt{1-2xt}}]^{-\beta-1} e^{\frac{2t}{1+\sqrt{1-2xt}}},$$

$$\sum_0^\infty \frac{t^n}{n!} Y_n^{(\alpha-n,\beta)}(x,y)$$

$$(7.3) \qquad = (1-2yt)^{-\frac{1}{2}}[1-\frac{xt}{1+\sqrt{1-2yt}}]^{-\alpha-1}[\frac{2}{1+\sqrt{1-2yt}}]^\beta e^{\frac{2t}{1+\sqrt{1-2yt}}},$$

$$(7.4) \qquad \sum_0^\infty \frac{t^n}{n!} Y_n^{(\alpha-2n,\beta-n)}(x,y) = (1+\frac{xt}{2})^\alpha[1-\frac{yt}{2+xt}]^{-\beta-1} e^{\frac{2t}{2+xt}},$$

$$(7.5) \qquad \sum_0^\infty \frac{t^n}{n!} Y_n^{(\alpha-n,\beta-2n)}(x,y) = [1-\frac{xt}{2+yt}]^{-\alpha-1}(1+\frac{yt}{2})^\beta e^{\frac{2t}{2+yt}},$$

$$\sum_{0}^{\infty}(-\lambda)^k Y_n^{(\alpha,k-n)}(x,y) = \frac{1}{1+\lambda} Y_n^{(\alpha,-n)}(x, \frac{y}{1+\lambda}), \tag{7.6}$$

$$\sum_{0}^{\infty}(-\lambda)^k Y_n^{(k-n,\beta)}(x,y) = \frac{1}{1+\lambda} Y_n^{(-n,\beta)}(\frac{x}{1+\lambda}, y). \tag{7.7}$$

Using (3.1) we can also derive the following results:

$$\sum_{0}^{\infty} \frac{(-\lambda)^k}{k!} Y_n^{(\alpha,k-n)}(x,y)$$

$$= \frac{1}{\Gamma(\alpha+n+1)} \int_0^\infty \int_0^\infty u^{\alpha+n}(1+\frac{xu}{2}+\frac{yv}{2})^n e^{-u-v} J_0(2\sqrt{\lambda v})dudv, \tag{7.8}$$

$$\sum_{0}^{\infty} \frac{(-\lambda)^k}{k!} Y_n^{(k-n,\beta)}(x,y)$$

$$= \frac{1}{\Gamma(\beta+n+1)} \int_0^\infty \int_0^\infty v^{\beta+n}(1+\frac{xu}{2}+\frac{yv}{2})^n e^{-u-v} J_0(2\sqrt{\lambda u})dudv, \tag{7.9}$$

$$\sum_{0}^{\infty}(-1)^k \lambda^{2k} Y_n^{(\alpha,2k-n)}(x,y)$$

$$= \frac{1}{\Gamma(\alpha+n+1)} \int_0^\infty \int_0^\infty u^{\alpha+n}(1+\frac{xu}{2}+\frac{yv}{2})^n e^{-u-v} \cos \lambda v dudv, \tag{7.10}$$

$$\sum_{0}^{\infty}(-1)^k \lambda^{2k} Y_n^{(2k-n,\beta)}(x,y)$$

$$= \frac{1}{\Gamma(\beta+n+1)} \int_0^\infty \int_0^\infty v^{\beta+n}(1+\frac{xu}{2}+\frac{yv}{2})^n e^{-u-v} \cos \lambda u dudv, \tag{7.11}$$

$$\sum_{0}^{\infty}(-1)^k \lambda^{2k+1} Y_n^{(\alpha,2k+1-n)}(x,y)$$

$$= \frac{1}{\Gamma(\alpha+n+1)} \int_0^\infty \int_0^\infty u^{\alpha+n}(1+\frac{xu}{2}+\frac{yv}{2})^n e^{-u-v} \sin \lambda v dudv, \tag{7.12}$$

$$\sum_{0}^{\infty}(-1)^k\lambda^{2k+1}Y_n^{(2k+1-n,\beta)}(x,y)$$

$$= \frac{1}{\Gamma(\beta+n+1)}\int_0^\infty\int_0^\infty v^{\beta+n}(1+\frac{xu}{2}+\frac{yv}{2})^n e^{-u-v}\sin\lambda u\,du\,dv. \tag{7.13}$$

8. Double generating functions

The following double generating functions for $Y_n^{(\alpha,\beta)}(x,y)$ can easily be derived by using (3.1):

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-\lambda)^m(-\mu)^k Y_n^{(m-n,k-n)}(x,y)$$

$$= \frac{1}{(1+\lambda)(1+\mu)}Y_n^{(-n,-n)}(\frac{x}{1+\lambda},\frac{y}{1+\mu}), \tag{8.1}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\lambda)^m(-\mu)^k}{m!\ k!}Y_n^{(m-n,k-n)}(x,y)$$

$$= \int_0^\infty\int_0^\infty(1+\frac{xu}{2}+\frac{yv}{2})^n e^{-u-v}J_0(2\sqrt{\lambda u})J_0(2\sqrt{\mu v})\,du\,dv, \tag{8.2}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{m+k}\lambda^{2m}\mu^{2k}Y_n^{(2m-n,2k-n)}(x,y)$$

$$= \int_0^\infty\int_0^\infty(1+\frac{xu}{2}+\frac{yv}{2})^n e^{-u-v}\cos\lambda u\cos\mu v\,du\,dv, \tag{8.3}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{m+k}\lambda^{2m+1}\mu^{2k}Y_n^{(2m+1-n,2k-n)}(x,y)$$

$$= \int_0^\infty\int_0^\infty(1+\frac{xu}{2}+\frac{yv}{2})^n e^{-u-v}\sin\lambda u\cos\mu v\,du\,dv, \tag{8.4}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{m+k}\lambda^{2m}\mu^{2k+1}Y_n^{(2m-n,2k+1-n)}(x,y)$$

$$= \int_0^\infty \int_0^\infty (1 + \frac{xu}{2} + \frac{yv}{2})^n e^{-u-v} \cos \lambda u \sin \mu v du dv, \tag{8.5}$$

$$\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{m+k} \lambda^{2m+1} \mu^{2k+1} Y_n^{(2m+1-n,2k+1-n)}(x,y)$$

$$= \int_0^\infty \int_0^\infty (1 + \frac{xu}{2} + \frac{yv}{2})^n e^{-u-v} \sin \lambda u \sin \mu v du dv. \tag{8.6}$$

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