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On Semi-Generalized Recurrent Manifold

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0. Introduction

In a previous paper U. G. De and N. Guha [1] introduced a generalized recurrent manifold. In this paper, we consider a nonflat Riemannian manifold (M^n, g) $(n \geq 2)$ whose curvature tensor R satisfies the condition

$$(D_X R)(Y,Z)W = A(X)R(Y,Z)W + B(X)g(Z,W)Y, \tag{1}$$

where A and B are two 1-forms, B is non-zero, P and Q are two vector fields such that

$$g(X,P) = A(X), \tag{2}$$

$$g(X,Q) = B(X), \tag{3}$$

for every vector field X and D denotes the operator of covariant differentiation with respect to the metric g. Such a manifold may be called a semi-generalized recurrent manifold and the 1-form B may be called its associated 1-form. An n-dimensional semi-generalized recurrent manifold shall be denoted by $(SGK)_n$. If the 1-form B in (1) becomes zero, then the manifold reduces to a recurrent manifold [2].

In this paper the nessecary and sufficinet condition for constant scalar curvature of $(SGK)_n$ is obtained. $(SGK)_n$ with Codazzi type of Ricci tensor and cyclic Ricci tensor are studied. Finally it is shown that if $(SGK)_n$ admits a parallel vector field V then V is orthogonal to Q and if it admits a concurrent vector field V then V is not orthogonal to Q.

1. Preliminaries

Let Ric and r denote the Ricci tensor of type (0,2) and scalar curvature respectively and L denote the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor, i.e.

$$g(LX,Y) = Ric(X,Y), \tag{1.1}$$

for any vector field X and Y.

From (1), we get

$$(D_X Ric)(Z,W) = A(X)Ric(Z,W) + nB(X)g(Z,W). \tag{1.2}$$

Contracting (1.2), we obtain

$$dr(X) = A(X)r + n^2\dot{B}(X). \tag{1.3}$$

2. Scalar curvature of a $(\mathrm{SGK})_n$

From (1.3), we see that if $r = 0$, then

$$B(X) = 0.$$

But $B(X)$ cannot be zero. Hence we have

Theorem 1. *The scalar curvature of a $(SGK)_n$ $(n > 2)$ cannot be zero.*

Now let us asuume that $(SGK)_n$ is of constant scalar curvature. Then from (1.3), we find

$$A(X)r + n^2B(X) = 0,$$

or

$$B(X) = -\frac{r}{n^2}A(X). \tag{2.1}$$

Again if (2.1) holds, then from (1.3), we get

$$r = \text{const}.$$

Hence, we can state the following theorem.

Theorem 2. *A $(SGK)_n$ $(n > .2)$ is of constant curvature if and only if (2.1) holds.*

3. $(SGK)_n$ with Codazzi type of Ricci-tensor

In this section we consider a $(SGK)_n$ in which the Ricci-tensor is a Codazzi tensor [3]

$$(D_X Ric)(Y,Z) = (D_Z Ric)(Y,X). \tag{3.1}$$

By virtue of Bianchi identity and (3.1),

$$(divR)(X,Y)Z = 0, \tag{3.2}$$

where *div* denotes the divergence with respect to D. In view of (1), we get on contraction

$$(divR)(Y,Z)W = A(R(Y,Z)W)) + B(g(Z,W)Y). \tag{3.3}$$

Now using (3.2) in (3.3), we obtain

$$A(R(Y,Z)W) = -B(g(Z,W)Y). \tag{3.4}$$

Putting $Z = W = e_i$ in (3.4), where $\{e_i\}$, $i = 1,2,\dots,n$ is an orthonormal basis of the tangent space at any point, we get by the sum of $1 \le i \le n$ of the relation (3.4),

$$A(LY) = -nB(Y) \tag{3.5}$$

where L is defined in (1.1).

From (1.2) and (3.1), we get

$$A(X)Ric(Y,Z) - A(Z)Ric(Y,X) + n[B(X)g(Y,Z) \\ -B(Z)g(Y,X)] = 0. \tag{3.6}$$

On contraction of (3.6), we find

$$A(X)r = A(L(X)) - n(n-1)B(X). \tag{3.7}$$

Using (3.5) and (3.7) in (1.3), we have

$$dr(X) = 0. \tag{3.8}$$

Again it is known [4] that in Riemannian manifold (M^n, g) $(n > 3)$,

$$(divC)(X,Y)Z = \frac{n-3}{n-2}[(D_X Ric)(Y,Z) - (D_Z Ric)(Y,X)] \tag{3.9}$$

$$+\frac{1}{2(n-1)}[g(X,Y)dr(Z)-g(Y,Z)dr(X)],$$

where C denotes the conformal curvature.

As a consequence of (3.1) and (3.8), (3.9) reduces to

$$(divC)(X,Y)Z=0 \tag{3.10}$$

which shows that the tensor C is conservative [5].

Hence we can state the following theorem.

Theorem 3. *If in a $(SGK)_n$ $(n>3)$ the Ricci-tensor is a Codazzi tensor then its conformal curvature tensor is conservative.*

4. $(SGK)_n$ with cyclic Ricci-tensor

In this section we consider a $(SGK)_n$ in which the Ricci-tensor is a cyclic tensor, i.e.

$$(D_X Ric)(Y,Z)+(D_Y Ric)(Z,X)+(D_Z Ric)(X,Y)=0, \tag{4.1}$$

which implies that

$$dr(X)=0. \tag{4.2}$$

By the definition of $(SGK)_n$, we have

$$dr(X)=A(X)r+n^2B(X). \tag{4.3}$$

Therefore from (4.2) and (4.3), we get

$$A(X)r+n^2B(X)=0. \tag{4.4}$$

From (4.1), we have

$$A(X)Ric(Y,Z)+A(Y)Ric(Z,X)+A(Z)Ric(X,Y)$$
$$+n[B(X)g(Y,Z)+B(Y)g(Z,X)+B(Z)g(X,Y)]=0,$$

which yields on contraction

$$A(X)r+2A(L(X))+n[nB(X)+2B(X)]=0. \tag{4.5}$$

Now in view of (4.4) and (4.5), we find

$$A(L(X))=\frac{r}{n}A(X)$$

or

$$Ric(X,P) = \frac{r}{n}g(X,P).$$

Hence, we have the following theorem -

Theorem 4. *If a $(SGK)_n$ has cyclic Ricci-tensor, then r/n is an eigen value of the Ricci tensor Ric and P is an eigen vector corresponding to the eigen value.*

5. $(SGK)_n$ with concurrent and parallel vector fields

In this section first we supppose that the $(SGK)_n$ admits a concurrent vector field V, [6]. Then

(5.1) $$D_X V = \rho X, \quad \text{where } \rho \text{ is a non-zero constant.}$$

By Ricci-identity, we obtain

(5.2) $$R(X,Y)V = 0.$$

Taking covariant derivative of (5.2), we get

(5.3) $$(D_W R)(X,Y)V = -\rho R(X,Y)W.$$

Also by definition of $(SGK)_n$, we find

(5.4) $$(D_W R)(X,Z)Y = A(W)R(X,Z)Y + B(W)g(Z,Y)X.$$

In view of (5.2), (5.3) and (5.4), we have

$$-\rho R(X,Y)W = B(W)g(Z,V)X.$$

On contraction, we find

$$-\rho Ric(Z,W) = nB(W)g(Z,V).$$

Again on contraction, we get

(5.5) $$-\rho r = ng(Q,V),$$

since $\rho \neq 0$ and $r \neq 0$, then $g(Q,V) \neq 0$.

Hence, we have

Theorem 5. *If a $(SGK)_n$ admits a concurrent vector field V, then V is not orthogonal to Q, where Q is the associated vector field to the 1-form B.*

If in particular $\rho = 0$, then the vector field V becomes parallel [6], i.e.

$$D_X V = 0.$$

Then (5.5) yields

$$g(Q, V) = 0.$$

Thus, we get the following theorem.

Theorem 6. *If a $(SGK)_n$ $(n > 2)$ admits a parallel vector field V, then V is orthogonal to Q.*

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