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On Some Fixed Point Theorems for Compatible Mappings

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In this paper, first we present a general common fixed point theorem for four compatible mappings, which extend the results of Jungck and Rhoades [2] and Telci, Tas and Fisher [5]–[7]. Secondly, we extend our result for sequences of mappings.

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Let S and T be two self mappings of a metric space (X, d). Sessa [3] defines S and T to be weakly commuting if $d(STx, TSx) \leq d(Tx, Sx)$ for all x in X. Jungck [1] defines S and T to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequuence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$ for some $x \in X$. Clearly, commuting mappings are weakly commuting and weakly commuting are compatible, but neither implications is reversible. (Ex.1, [4] and Ex.2.2, [1]).

Lemma 1. *Let f and g be the self mappings of the set $X = \{x, x'\}$ with any metric d. If the range of g contains the range of f, then the following statements are equivalent:*

1) f and g commute,

2) f and g weakly commute,

3) f and g are compatible.

By Lemma 1 we suppose that X contains at least three points.

Lemma 2. ([1]) *Let f and g be compatible self mappings on a metric space (X, d). If $f(t) = g(t)$, then $fg(t) = gf(t)$.*

The following theorem is proved in [2].

Theorem 1. *Let $\{S, I\}$ and $\{T, J\}$ be two pairs of compatible self mappings of a complete metric space (X, d) such that*

$$d(Sx, Ty) \leq g(d(Ix, Ty), d(Ix, Sx), d(Jy, Ty)) \tag{1}$$

for any $x, y \in X$, where $g : R_+^3 \to R_+$ is continuous and satisfies:

(i) $g(1,1,1) = h < 1$ and

(ii) whenever $u, v \geq 0$ and either $u \leq g(u, v, v), u \leq g(v, u, v)$ or $u \leq g(v, v, u)$, then $u \leq h.v$.

If $T(X) \subset I(X)$, $S(X) \subset J(X)$ and if one of I, J, S or T is continuous, then I, J, S and T have a unique common fixed point z. Further, z is unique common fixed point of I and S and of J and T.

Let $\mathcal{H}$ be the set of real upper semi-continuous functions $g(t_1, \ldots, t_5) : R_+^5 \to R_+$ satisfying the following conditions:

H_1: g is non-decreasing in variables t_4 and t_5,

H_2: $g(u, 0, 0, u, u) < u, \forall u > 0$,

H_3: there exists $0 \leq h < 1$ such that for every $u, v \geq 0$ with

H_a: $u \leq g(v, v, u, u + v, 0)$ or

H_b: $u \leq g(v, u, v, 0, u + v)$,

we have $u \leq h.v$.

Example 1. $g(t_1, \ldots, t_5) = a \max\{t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5), b\sqrt{t_4.t_5}\}$, where $0 \leq a < 1$, $b \geq 0$ and $a.b < 1$.

H_1: Obviously.

H_2: $g(u, 0, 0, u, u) = a. \max\{u, bu\} = ab.u < u$ for $u > 0$.

H_3: g satisfies (H_3) with $h = a$.

Example 2. $g(t_1, \ldots, t_5) = [c_1. \max\{t_1^2, t_2^2, t_3^2\} + c_2. \max\{t_2 t_4, t_3 t_5\} + c_3.t_4 t_5]^{\frac{1}{2}}$, where $c_1 > 0, c_2, c_3 \geq 0, c_1 + 2c_2 < 1$ and $c_1 + c_3 < 1$.

H_1: Obviously.

H_2: $g(u, 0, 0, u, u) = \sqrt{c_1 + c_3}.u < u$ for $u > 0$.

H_3: If $u \leq g(v, v, u, u + v, 0)$ then $u^2 \leq c_1. \max\{u^2, v^2\} + c_2.v(u + v)$. If $u \geq v$ then $u^2 \leq (c_1 + 2c_2)u^2 < u^2$ for $u > 0$, a contradiction. Then $u < v$. Thus there exists $h_1 \in [0, 1)$ such that $u \leq h_1 v$. Similarly, if $u \leq g(v, u, v, 0, u + v)$ there exists $h_2 \in [0, 1)$ such that $u \leq h_2.v$. Therefore, $u \leq h.v$, where $h = \max\{h_1, h_2\} < 1$.

Example 3.

$$g(t_1, \ldots, t_5) = \frac{p. \max\{t_2.t_3, t_4.t_5\} + f(\max\{t_1, t_2, t_3, 1/2.(t_4 + t_5)\})}{1 + pt_1},$$

where $p \geq 0$ and $f : R_+^5 \to R_+$, non-decreasing and upper-continuous with $f(t) < t$ for all $t > 0$.

H_1: Obviously.

H_2: $g(u,0,0,u,u) = \frac{pu^2+f(u)}{1+pu} < \frac{pu^2+u}{1+pu} = u$ for $u > 0$.

H_3: If $u \leq g(v,v,u,u+v,0)$, then $u \leq \frac{puv+f(\max\{u,v,\frac{1}{2}(u+v)\}}{1+pv}$. If $u \geq v$, then $u \leq \frac{puv+f(u)}{1+pv} < \frac{puv+u}{1+v} = u$ a contradiction. Then $u < v$. Thus there exists $h_1 \in [0,1)$ such that $u \leq h_1 v$. Similarly if $u \leq g(v,u,v,0,u+v)$ there exists $h_2 \in [0,1)$ such that $u \leq h_2 v$. Therefore, $u \leq h.v$, where $h = \max\{h_1, h_2\} < 1$.

Remark. There exists function $g \in \mathcal{H}$ which is decreasing in variables t_2 or t_3.

Example 4.

$$g(t_1,\ldots,t_5) = [at_1^2 + \frac{bt_4t_5}{t_2^2+t_3^2+1}]^{\frac{1}{2}},$$

where $a > 0, b \geq 0$ and $a + b < 1$.

H_1: Obviously. g is decreasing in variables t_2 and t_3.

H_2: $g(u,0,0,u,u) = \sqrt{(a+b)u} < u$ for $u > 0$.

H_3: If $u \leq g(v,v,u,u+v,0)$, then $u^2 \leq a.v^2$ and $u \leq a^{\frac{1}{2}}.v = h.v$, where $h = a^{\frac{1}{2}} < 1$. If $u \leq g(v,u,v,0,u+v)$, then $u \leq h.v$, where $h = a^{\frac{1}{2}} < 1$.

The following theorems are recently proved.

Theorem 2. ([6]) *Let S and T be self mappings of a complete metric space (X,d) satisfying the inequality*

$$d(Sx,Ty) \leq a \max\Big\{d(x,y), d(x,Sx), d(y,Ty),$$

$$\frac{1}{2}(d(x,Ty)+d(y,Sx)), b\sqrt{d(x,Ty).d(y,Sx)}\Big\}$$

for all x,y in X, where $0 \leq a < 1$ and $b \geq 0$. Then S and T have a common fixed point. Further, if $a.b < 1$, then the fixed point is unique.

Theorem 3. ([7]) *Let S,T,I and J self mappings a complete metric space (X,d) satisfying the conditions:*

1^0 *$T(X) \subset I(X)$ and $S(X) \subset J(X)$,*

2^0 *One of S,T,I and J is continuous,*

3^0 *S and T weakly commute with I and J, respectively,*

4^0 *The inequality*

$$[I + p.d(Ix,Jy)]d(Sx,Ty)$$

$$\leq p.\max\{d(Ix,Sx).d(Jy,Ty),d(Ix,Ty).d(Jy,Sx)\}$$

$$+g(\max\{d(Ix,Jy),d(Ix,Sx),d(Jy,Ty),\frac{1}{2}(d(Ix,Ty)+d(Jy,Sx))\})$$

holds for all x, y in X, where $p \geq 0$ and $g : R_+ \rightarrow R_+$, nondegecreasing and upper semi-continuous with $g(t) < t$ for all $t > 0$, then S, T, I and J have a common fixed point z. Further, z is the unique common fixed point for S and I and T and J.

Theorem 4. ([5]) *Let A, B, S and T be self mappings of a complete metric space (X, d) such that*

a) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

b) One of A, B, S and T is continuous,

c) A and B are compatible with S and T, respectively,

d) The inequality

$$[d(Ax,By)]^2 \leq c_1.\max\{d^2(Sx,Ty),d^2(Sx,Ax),d^2(Ty,By)\}$$

$$+c_2.\max\{d(Sx,Ax)d(Sx,By),d(Ax,Ty)d(By,Ty)\}+c_3d(Sx,By)d(Ty,Ax)$$

holds for all x, y in X, where $c_1 > 0, c_2, c_3 \geq 0, c_1 + 2c_2 < 1$ and $c_1 + c_3 < 1$, then A, B, S and T have common fixed point z. Further, z is the unique common fixed point of A and S and of B and T.

The purpose of this paper is to prove some theorems which generalize Theorems 1–4 for compatible mappings.

Theorem 5. *Let S, T, I and J be mappings from a complete metric space (X, d) into itself satisfying the conditions:*

a) $S(X) \subset J(X)$ and $T(X) \subset I(X)$,

b) One of S, T, I and J is continuous,

c) S and I as well T and J are compatible,

d) The inequality

$$d(Sx,Ty) \leq g(d(Ix,Jy),d(Ix,Sx),d(Jy,Ty),d(Ix,Ty),d(Jy,Sx)) \quad (2)$$

holds for all x, y in X, where $g \in \mathcal{H}$, then S, T, I and J have a common fixed point z. Further, z is unique common fixed point of S and I and of T and J.

Proof. Suppose x_0 an arbitrary point in X. Then since (a) holds, we can define a sequence

$$\{Sx_0, Tx_1, Sx_2, Tx_3, \ldots, Sx_{2n}, Tx_{2n+1}, \ldots\} \quad (3)$$

inductively, such that $Sx_{2n} = Jx_{2n+1}$, $Tx_{2n+1} = Ix_{2n+2}$ for $n = 0, 1, 2, \ldots$. Using inequality (2), we have

$$d(Sx_{2n}, Tx_{2n+1}) \leq g(d(Ix_{2n}, Jx_{2n+1}),$$

$$d(Ix_{2n}, Sx_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), d(Ix_{2n}, Tx_{2n+1}), d(Jx_{2n+1}, Sx_{2n}))$$

$$\leq g(d(Tx_{2n-1}, Sx_{2n}), d(Tx_{2n-1}, SX_{2n}), d(Sx_{2n}, Tx_{2n+1}),$$

$$d(Tx_{2n-1}, Sx_{2n}) + d(Sx_{2n}, Tx_{2n+1}), 0).$$

By (H_a) we have

$$d(Sx_{2n}, Tx_{2n+1}) \leq h.d(Tx_{2n+1}, Sx_{2n}).$$

Similarly, by (H_b) we have

$$d(Sx_{2n}, Tx_{2n-1}) \leq h.d(S_{2n-2}, Sx_{2n})$$

and so

$$d(Sx_{2n}, Tx_{2n+1}) \leq (h)^{2n}.d(Sx_0, Tx_1) \text{ for } n = 0, 1, 2, \ldots.$$

By a routine calculation it follows that the sequence (3) is a Cauchy sequence. Since X is complete, the sequence (3) converge to a point z in X. Hence z is also the limit of the sequence $\{Sx_{2n}\} = \{Jx_{2n+1}\}$ and $\{Tx_{2n-1}\} = \{Ix_{2n}\}$ of (3).

Let us now suppose that I is continuous, so that the sequence $\{IS_{2n}\}$ converge to Iz. We have

$$d(SIx_{2n}, Iz) \leq d(SIx_{2n}, ISx_{2n}) + d(ISx_{2n}, Iz).$$

Since I is continuous and S and I are compatible, letting n tend to infinity it follows that the sequence $\{SIx_{2n}\}$ also converge to Iz. Using (2) we have

$$d(SIx_{2n}, Tx_{2n+1}) \leq g(d(I^2x_{2n}, Jx_{2n+1}),$$

$$d(I^2x_{2n}, SIx_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), d(I^2x_{2n}, Tx_{2n+1}), d(Jx_{2n+1}, SI(x_{2n})).$$

Letting n tend to infinity and since g is upper semi-continuous, we have

$$d(Iz, z) \leq g(d(Iz, z), 0, 0, d(Iz, z), d(Iz, z)).$$

By (H_2) we have $d(Iz, z) = 0$ and so $Iz = z$.

Further, by (2) we have

$$d(Sz, Tx_{2n+1}) \le g(d(Iz, Jx_{2n+1}),$$

$$d(Iz, Sz), d(Jz_{2n+1}, Tx_{2n+1}), d(Iz, Tx_{2n+1}), d(Jx_{2n+1}, Sz))$$

and letting n tend to infinity we have

$$d(Sz, z) \le g(0, d(z, Sz), 0, 0, d(z, Sz))$$

which implies by (H_b) that $z = Sz$. This means that z is in the range of S and since $S(X) \subset J(X)$, there exists a point u in X such that $Ju = z$. Thus $d(z, Tu) = d(Sz, Tu) \le g(d(Iz, Ju), d(Iz, Sz), d(Ju, Tu), d(Iz, Tu), d(Ju, Sz))$ $= g(0, 0, d(z, Tu), d(z, Tu), 0)$ which implies by (H_a) that $z = Tu$. Since $Ju = Tu = z$ by Lemma 2 it follows that $TJu = JTu$ and so $Tz = TJu = JTu = Jz$. Thus from (2) we have

$$d(z, Tz) = d(Sz, Tz)$$

$$\le g(d(Iz, Jz), d(Iz, Sz), d(Jz, Tz), d(Iz, Tz), d(Jz, Sz))$$

$$= g(d(z, Tz), 0, 0, d(z, Tz), d(z, Tz))$$

and by (H_2) $z = Tz = Jz$. We have therefore proved that z is a common fixed point of S, T, I and J. The same result holds if we assume that J is continuous instead of I.

Now suppose that S is continuous. Then the sequence $\{SIx_{2n}\}$ converges to Sz. We have

$$d(ISx_{2n}, Sz) \le d(ISx_{2n}, SIx_{2n}) + d(SIx_{2n}, Sz).$$

Since S is continuous and S and T are compatible, letting n tend to infinity, it follows that $\{ISx_{2n}\}$ converge to Sz. Using the inequality (2) we have

$$d(S^2x_{2n}, Tx_{2n+1}) \le g(d(ISx_{2n}, Jx_{2n+1}),$$

$$d(ISx_{2n}, S^2x_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), d(ISx_{2n}, Tx_{2n+1}), d(Jx_{2n+1}, S^2x_{2n})).$$

Letting n tend to infinity and since g is upper semi-continuous, we have

$$d(Sz, z) \le g(d(Sz, z), 0, 0, d(Sz, z), d(Sz, z))$$

and by (H_2) we have $d(Sz, z) = 0$ and so $Sz = z$. This means that z is in the range of S and since $S(X) \subset J(X)$, there exists a point u in X such that $Ju = z$. Thus

$$d(S^2x_{2n}, Tu)$$

$$\leq g(d(ISx_{2n}, Ju), d(ISx_{2n}, S^2x_{2n}), d(Ju, Tu), d(ISx_{2n}, Tu), d(Ju, S^2x_{2n})).$$

Letting n tend to infinity, it follows that

$$d(z, Tu) \leq g(0, 0, d(z, Tu), d(z, Tu), 0)$$

and by (H_a) it follows that $z = Tu$. Since $Ju = Tu = z$ by Lemma 2 it follows that $Tz = TJu = JTu = Jz$.

Thus from (2) we have

$$d(Sx_{2n}, Tz)$$

$$\leq g(d(Ix_{2n}, Jz), d(Ix_{2n}, Sx_{2n}), d(Jz, Tz), d(Ix_{2n}, Tz), d(Jz, Sx_{2n})).$$

Letting n tend to infinity, it follows that

$$d(z, Tz) \leq g(d(z, Tz), 0, 0, d(z, Tz), d(z, Tz))$$

and by (H_2) it follows that $z = Tz = Jz$. This means that z is in the range of T and since $T(X) \subset I(x)$, there exists $u' \in X$ such that $Iu' = z$. Thus from (2) we have

$$d(Su', z) = d(Su', Tz)$$

$$\leq g(d(Iu', Jz), d(Iu', Su'), d(Jz, Tz), d(Iu', Tz), d(Jz, Su')).$$

Thus

$$d(Su', z) \leq g(0, d(z, Su'), 0, 0, d(z, Su'))$$

and by (H_b) we have $z = Su' = Iu'$. Since $Su' = Iu' = z$ by Lemma 2 it follows that $z = Sz = SIu' = ISu' = Iz$, and thus $z = Iz$. We have therefore proved that z is a common fixed point of S, T, I and J. The same result holds if we assume that T is continuous instead of S.

Now let w be a second common fixed point of S and I. Using inequality (2) we have

$$d(w, z) = d(Sw, Tz) \leq g(d(Iw, Jz), d(Iw, Sw), d(Jz, Tz), d(Iw, Tz), d(Jz, Sw))$$

and thus

$$d(w, z) \leq g(d(w, z), 0, 0, d(w, z), d(w, z))$$

and by (H_2) it follows that $w = z$. Then z is the unique common fixed point of S and I. Similarly, it is proved that z is the unique common fixed point of T and J. ■

For $f : (X.d) \to (X, d)$ we denote $F_f = \{x \in X : x = f(x)\}$.

Theorem 6. *Let I, J, S, T be mappings from a metric space (X, d) into itself. If the inequality (2) holds for all x, y in X, then*

$$(F_I \cap F_J) \cap F_S = (F_I \cap F_J) \cap F_T.$$

P r o o f. Let $x \in (F_I \cap F_J) \cap F_S$. Then

$$d(x, Tx) = d(Sx, Tx)$$

$$\leq g(d(Ix, Jx), d(Ix, Sx), d(Jx, Tx), d(Ix, Tx), d(Jx, Sx)$$

$$= g(0, 0, d(x, Tx), d(x, Tx), 0)$$

which implies by (H_a) that $x = Tx$. Thus $(F_I \cap F_J) \cap F_S \subset (F_I \cap F_J) \cap F_T$. Similarly, we have by (H_b) that $(F_I \cap F_J) \cap F_T \subset (F_I \cap F_J) \cap F_S$. ∎

Theorem 7. *Let I, J and $\{T_i\}_{i \in N^*}$ be mappings from a complete metric space into itself such that*

a) $T_2(X) \subset I(X)$ and $T_1(X) \subset J(X)$,

b) One of I, J, T_1 and T_2 is continuous,

c) The pairs (T_1, I) and (T_2, J) are compatible,

d) The inequality

$$d(T_i x, T_{i+1} y)$$

$$\leq g(d(Ix, Jy), d(Ix, T_i x), d(Jy, T_{i+1} y), d(Ix, T_{i+1} y), d(Jy, T_i x))$$

holds for each x, y in X, $\forall i \in N^*$ and $g \in \mathcal{H}$.

Then I, J and $\{T_i\}_{i \in N^*}$ have a unique common fixed point.

P r o o f. Follows from Theorems 5 and 6. ∎

References

[1] G. J u n g c k, Compatible mappings and common fixed points, *Internat. J. Math. and Math. Sci.*, **9** (1986), 771-779.

[2] G. J u n g c k and B. E. R h o a d e s, Some fixed point theorems for compatible maps, *Internat. J. Math. and Math. Sci.*, **16** (1993), 417-428.

[3] S. S e s s a, On a weak commutativity condition in a fixed point considerations, *Publ. Inst. Math.*, **32(46)** (1982), 149-153.

[4] S. S e s s a and B. F i s h e r, Common fixed point of weakly commutating mappings, *Bull. Polish Acad. Sci. Math.*, **36** (1987), 341-349.

[5] K. T a s, M. T e l c i and B. F i s h e r, Common fixed point theorems for compatible mappings, *Internat. J. Math. Sci.*, **19** (1996), 451-456.

[6] M. T e l c i, K. T a s and B. F i s h e r, A generalization of the fixed point theorem of Bhola and Sharma, *Math. Balkanica*, **9** (1995), 113-116.

[7] M. T e l c i, K. T a s and B. F i s h e r, Common fixed point theorems for weakly commuting mappings (Submitted).

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