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Local Property (β) in Orlicz Spaces

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In this paper, criteria of β points in Orlicz spaces are given. Furthermore, we have obtained an equivalent condition that Orlicz space has local property (β).

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1. Introduction

Let X be a Banach space and X^* be the dual space of X. Denote by $B(X)$ and $S(X)$ the closed unit ball and the unit sphere of X, respectively. Denote by N and R the set of natural and real numbers, respectively. Let (G, Σ, μ) be a measure space with a finite atomless measure μ. Denote by L^0 the set of all μ-equivalence classes of real valued measureable functions defined on G. l^0 stands for the space of all real sequences. For any subset A of X by $conv(A)$ ($\overline{conv}(A)$) we denote the convex hull (the closed convex hull) of A. Clarkson [5] introduced the concept of uniform convexity.

The norm $\|.\|$ is called *uniformly convex* (write (**UC**)), if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $x, y \in S(X)$, inequality $\|x - y\| \geq \varepsilon$ implies

$$\|\frac{1}{2}(x+y)\| < 1 - \delta. \tag{1.1}$$

For any $x \notin B(X)$, the *drop* determined by x is the set

$$D(x, B(X)) = conv(\{x\} \cup B(X)).$$

Rolewicz in [21], basing on Danes drop theorem [6], introduced the notion of drop property for Banach spaces. A Banach space X is said to have the *drop property*, if for every closed set C disjoint with $B(X)$ there exists an element $x \in C$ such that

$$D(x, B(x)) \cap C = \{x\}.$$

A Banach space X is said to have the *Kadec - Klee property* (or *property* **(H)**), if every weakly convergent sequence on the unit sphere is convergent in norm.

In [21] Rolewicz proved that if the Banach space has the drop property, then X is reflexive. V. Montesions [16] extended this result by showing that X has the drop property if and only if and if X is reflexive and the property **(H)**.

Recall that a sequence $(x_n) \subset X$ is said to be an ε- separate sequence for some $\varepsilon > 0$ if

$$sep(x_n) = \inf\{\|x_n - x_m\| : n \neq m\} \geq \varepsilon.$$

A Banach space X is said to have the *uniform Kadec - Klee property* (write **(UKK)**), if for each $\varepsilon > 0$ there is a $\delta > 0$ such that if x is a weak limit of norm one ε-separate sequence then $\|x\| < 1 - \delta$.

A Banach space X is said to be *nearly uniform convex* (write(**NUC**)) if for each $\varepsilon > 0$ there is a $\delta \in (0,1)$ such that for $(x_n) \subset B(X)$ with $sep(x_n) > \varepsilon$, we have

$$conv(\{x_n\}) \cap (1-\delta)B(X) \neq \emptyset.$$

It is easy to see that every **(NUC)** space has the **(UKK)**, and every Banach space with **(UKK)** property has property **(H)**. Huff [9] proved that a Banach space X is **(NUC)** if and only if X is reflexive and X has **(UKK)**.

For any subset C of X, the *Kuratowski measure* $\alpha(C)$ of C is the infimum of those $\varepsilon > 0$ for which there is a covering of C by a finite number of sets of diameter less than ε.

Goebel and Sekowski [8] extend the definition convexity replacing condition (1) by condition involving the *Kuratowski measure* of noncompactness.

The norm $\|.\|$ in Banach space X is Δ-*uniformly convex* (write $(\Delta$**UC**$)$) if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for each convex set E contained in the closed unit ball $B(X)$ such that $\alpha(E) > \varepsilon$, we have

$$\inf\{\|x\| : x \in E\} < 1 - \delta.$$

A Banach space X is said to have property (β) if for any $\varepsilon > 0$ there exists a $\delta \in (0,1)$ such that

$$\alpha(D(x, B(X)) \setminus B(X)) < \varepsilon$$

whenever $< \|x\| < 1 + \delta$ (see [20]).

Very helpful for our considerations is the following equivalent form of the property (β) (see [12]). A Banach space X has property (β) iff for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $x \in S(X)$ and each sequence $(x_n) \subset S(X)$ with $sep(x_n) > \varepsilon$, there is an index k satisfying $\|\frac{x_k+x}{2}\| < 1 - \delta$.

Rolewicz [21] has showed that the property (β) follows from (**UC**) and that the property (β) implies (**ΔUC**). It was proved in Kutzarova [24,25] that property (β) is isomerphically different from **UC** and **NUC**.

Sum up the above discussion we have

$$(UC) \Rightarrow (\beta) \Rightarrow (\Delta UC) \Leftrightarrow (NUC) \Leftrightarrow [(\mathrm{Rfx}) \cap (UKK)] \Rightarrow [(\mathrm{Rfx}) \cap (H)] \Leftrightarrow (D)$$

where (Rfx) denote the property of reflexivity.

A Banach space X is said to have normal structure if $r(A) < diam(A)$ for every non-singleton bounded subset A of X, where $r(A) = \inf\{\sup\{\|x-y\| : y \in A\} : x \in conv(A)\}$.

It is well known that if X has property (β) then X^* has normal structure (see [11]).

A $x \in S(X)$ is said to be a local uniform point it for any sequence $(x_n) \subset S(X)$ with $\lim_{n\to\infty} \|x_n + x\| = 2$ implies $\lim_{n\to\infty} \|x_n + x\| = 2$ (see [22]).

A Banach space X is said to be *local uniform convex* (write (**LUC**)) if every point on $S(X)$ is a local uniform point (see [18]).

To study the relationships between (**LUC**) and property (β), D. Kutzarova and P.L. Papini introduced local property (β). They have obtained some important results (see [10]). More generally, similar local properties were studied in Kutzarova and Lin [26]. For details studying the relationship between (**LUC**) and local property (β), we introduced the concept of β-point. In this paper, we have got that there exist some Banach space with local property (β) are not (**LUC**).

A $x \in S(X)$ is said to be β-point if for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon, x) > 0$ such that for any sequence $(x_n) \subset S(X)$ with $sep(x) \geq \varepsilon$, we have $\|\frac{x_k+x}{2}\| < 1-\delta$ for some index k.

It is obvious that a local uniform point is a β-point.

A Banach space X has local property (β) (for short $L(\beta)$) if and only if every point on $S(X)$ is a β-point (see [10]).

A map $\Phi : R \to [0, \infty)$ is said to be an N-function if Φ is vanishing only at 0, even, convex, $\lim_{u\to 0} \frac{\Phi(u)}{u} = 0$ and $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$. Let $p(u)$ be the right-side derivative of $\Phi(u)$ and $p_-(u)$ be the left-side derivative of $\Phi(u)$. For every *Orlicz function* Φ we define the *complementary function* $\Psi : R \to [0, \infty)$ by the formula

$$\Psi(v) = \sup_{u>0}\{u|v| - \Phi(u)\}$$

for every $v \in R$. The *complementary function* Ψ is also an N-function.

By the *Orlicz function space* L_Φ we mean

$$L_\Phi = \{x \in L^0 : I_\Phi(cx) = \int_G \Phi(cx(t))d\mu < \infty, \quad \text{for some} \quad c > 0\}$$

and the subspace of the *Orlicz function space*

$$E_\Phi = \{x \in L^0 : I_\Phi(cx) = \int_G \Phi(cx(t))d\mu < \infty, \quad \text{for any} \quad c > 0\}.$$

Analogously, we define the *Orlicz sequence space* and the subspaces of *Orlicz sequence space* by the following formula:

$$l_\Phi = \{x \in l^0 : I_\Phi(cx) = \sum_{i=1}^{\infty} \Phi(cx(i)) < \infty \quad \text{for some} \quad c > 0\}$$

$$\text{and} \quad h_\Phi = \{x \in l^0 : I_\Phi(cx) = \sum_{i=1}^{\infty} \Phi(cx(i)) < \infty \quad \text{for any} \quad c > 0\}.$$

$L_\Phi(l_\Phi)$ are equipped with so called the *Luxemburg norm:*

$$\|x\| = \inf\{k > 0 : I_\Phi(\frac{x}{k}) \le 1\}$$

or equipped with one equivalent norm

$$\|x\|^0 = \inf_{k>0}\{\frac{1}{k}(1 + I_\Phi(kx))\} \tag{1.2}$$

called the *Orlicz norm.* To simplify notations, we put $L_\Phi = (L_\Phi, \|\ \|)$, $l_\Phi = (l_\Phi, \|\ \|)$, $L_\Phi^0 = (L_\Phi, \|\ \|^0)$, $l_\Phi^0 = (l_\Phi, \|\ \|^0)$.

For $x \in L_\Phi^0$ (or l_Φ^0) and $x \neq 0$, we denote $k_x^* = \inf\{k > 0 : I_\Psi(p(k|x|)) \ge 1\}$, $k_x^{**} = \sup\{k > 0 : I_\Psi(p(k|x|)) \le 1\}$ and $K(x) = [k_x^*, k_x^{**}]$. It is well known $\|x\|^0 = \frac{1}{k}(1 + I_\Phi(kx))$ iff $k \in K(x)$ (see [3]).

We say that the *Orlicz function* Φ satisfies Δ_2-condition ($\Phi \in \Delta_2$, for short), if there exist constants $k > 2$ and $u_0 > 0$ such that

$\Phi(2u) \le k\Phi(u)$ for $|u| \ge u_0$ (or $|u| \le u_0$ in the case of sequence space).

We say that the *Orlicz function* Φ satisfies ∇_2-condition ($\Phi \in \nabla_2$, for short), if its *complementary function* Ψ satisfies Δ_2-condition.

We say that a interval $[a, b]$ is an *affine interval* of Φ, if

$$\Phi(\frac{a+b}{2}) = \frac{1}{2}(\Phi(a) + \Phi(b)), \Phi(\frac{a-\varepsilon+b}{2}) < \frac{1}{2}(\Phi(a-\varepsilon) + \Phi(b))$$

$$\text{and} \quad \Phi(\frac{a+b+\varepsilon}{2}) < \frac{1}{2}(\Phi(a) + \Phi(b+\varepsilon)), \quad \text{for any} \quad \varepsilon > 0.$$

We say that a point x is *strictly convex point* of Φ, if $x = \frac{1}{2}(u+v)$ and $u \neq v$ imply $\Phi(\frac{u+v}{2}) < \frac{1}{2}(\Phi(u) + \Phi(v))$.

For more details of Orlicz spaces, we refer to [3], [17] and [19].

2. Results

Theorem 1. *A $x \in S(l_\Phi^0)$ is a β point if and only if:*
(1) $\Phi \in \Delta_2$,
(2) $\Phi \in \nabla_2$.

Proof. **Necessity.**
(1) For convenience, we introduce the following notion

$$[x]_n^m = (0, \ldots, 0, x(n+1), \ldots, x(m), 0, 0, \ldots)$$

for any $x \in S(l_\Phi^0)$ when $m > n$.

Suppose that $\Phi \notin \Delta_2$. We will consider the following two cases:

Case I: $x \notin S(h_\Phi^0)$.

In this case we have $\lim_{i\to\infty} \|[x]_i^\infty\|^0 = d(x, h_\Phi^0) = d > 0$.

Denote $i_0 = 0$.

Since $\|[x]_{i_0}^\infty\|^0 > d$, there is $i_1 > i_0$ such that $\|[x]_{i_0}^{i_1}\|^0 > d$.

Since $\|[x]_{i_1}^\infty\|^0 > d$, there is $i_2 > i_1$ such that $\|[x]_{i_1}^{i_2}\|^0 > d$; etc.

In such a way, we get a sequence $i_0 < i_1 < i_2 < \ldots$ such that $\|[x]_{i_{n-1}}^{i_n}\|^0 > d \quad (n = 1, 2, \ldots)$.

Now, let us consider the sequence $x]_0^{i_n} \quad (n = 1, 2, \ldots)$. It is obviously $\|[x]_0^{i_n}\|^0 \to \|x\|^0 = 1 \quad (n \to \infty)$ and $\|[x]_0^{i_n} - [x]_0^{i_m}\|^0 > \|[x]_{i_n}^{i_n+1}\|^0 > d$ for any $m > n$. But

$$\|x + [x]_0^{i_n}\|^0 \geq 2\|[x]_0^{i_n}\|^0 \to 2.$$

This contradiction shows that $\Phi \in \Delta_2$.

Case II: $x \in S(h_\Phi^0)$.

Take $k > 1$ such that

$$1 = \|x\|^0 = \frac{1}{k}(1 + I_\Phi(kx)).$$

Take $z \in S(l_\Phi^0 \setminus h_\Phi^0)$ such that $I_\Phi(z) < \infty$ and $d(z, h_\Phi^0) = d > 0$. By the same arguments as above we have $i_0 < i_1 < i_2 < \ldots$ such that $\|[x]_{i_{n-1}}^{i_n}\|^0 > d, \quad (n = 1, 2, \ldots)$. Put

$$x_k = (x(1), \ldots, x(i_{n-1}), \frac{z(i_{n-1}+1)}{k}, \frac{z(i_{n-1}+2)}{k}, \ldots, \frac{z(i_n)}{k}, x(i_n{+}1), x(i_n{+}2), \ldots)$$

By $\|x_n\|^0 \geq \|[x]_0^{i_{n-1}}\|^0 \to \|x\|^0 = 1$ and

$$\|x_n\|^0 \leq \frac{1}{k}(1 + I_\Phi(kx) + \sum_{i=i_{n-1}+1}^{i_n} \Phi(z(i))) = \|x\|^0 + \frac{1}{k} \sum_{i=i_{n-1}+1}^{i_n} \Phi(z(i)) \to 1,$$

we get $\lim_{n\to\infty}\|x_n\|^0 = 1$. Take n_0 such that $\|[x]_{i_{n_0}}^{\infty}\|^0 < \frac{d}{2k}$ thanks to $x \in S(h_\Phi^0)$. Then,

$$\|x_m - x_n\|^0 \geq \frac{1}{k}\|[z]_{i_n}^{i_n+1}\|^0 - \|[x]_{i_n}^{i_n+1}\|^0 \geq \frac{d}{k} - \frac{d}{2k} = \frac{d}{2k} \quad \text{whenever} \quad m > n \geq n_0.$$

But $\|x_n+x\|^0 \geq 2\|[x]_0^{i_{n-1}}\| \to 2\|x\|^0 = 2$. This contradiction shows that $\Phi \in \Delta_2$.

(2) Otherwise, we assume that $\Phi \notin \nabla_2$, i.e. $\Psi \notin \Delta_2$. It is easy to find a sequence $y_n = [y_n]_{i_{n-1}}^{i_n} \in h_\Psi$ such that $I_\Psi(y_n) < \frac{1}{n}$ and $\frac{n}{n+1} < \|y_n\|_\Psi \leq (n = 1,2,\ldots)$. Since $y_n \in h_\Psi$, there are a sequence $x_n \in S(h_\Phi^0)$ such that $x_n = [x]_{i_{n-1}}^{i_n}$ and $\langle x_n, y_n\rangle = \|y_n\|_\Psi$ for $n = 1,2,\ldots$. Obviously, $\|x_n - x_m\|^0 \geq \|x_n\|^0 =$ when $m \neq n$.

Using of $\Phi \in \Delta_2$, there exists a $y \in S(l_\Psi)$ such that y is a supporting functional of x. Put

$$z_n = \frac{n}{n+1}(y(1),\ldots,y(i_{n-1}),y_n(i_{n-1}+1),\ldots,y_n(i_n+1),y(i_n+1),y(i_n+2),\ldots).$$

Then

$$I_\Psi(z_n) < \frac{n}{n+1}(I_\Psi(y) + I_\Psi(y_n)) < \frac{n}{n+1}(1+\frac{1}{n}) = 1.$$

Hence

$$\|x_n + x\|^0 \geq \langle x + x_n, z_n\rangle$$

$$\geq \frac{n}{1+n}(\langle x_n, y_n\rangle + \langle x, y\rangle - \sum_{i=i_{n-1}+1}^{i_n} |x(i)|(|y(i)| + |y_n(i)|)$$

$$> \frac{n}{n+1}(\frac{n}{n+1} + 1 - \|[x]_{i_{n-1}}^{i_n}\|^0(\|y\|_\Psi + \|y_n\|_\Psi)) \to 2.$$

This means that x is not a β-point. So, we get $\Phi \in \nabla_2$.

Sufficiency. For any $x_0 \in S(l_\Phi^0)$, we are going to prove that it is a β-point.

Suppose that a sequence $(x_n) \subset S(l_\Phi^0)$ with $sep(x_n) \geq \varepsilon$ for any given $\varepsilon > 0$.

Take a sequence $k_n > 1$ such that

$$1 = \|x_n\|^0 = \frac{1}{k_n}(1 + I_\Phi(k_n x_n)) \quad (n = 1,2,\ldots.) \tag{2.1}$$

Since $\Phi \in \nabla_2$, we know $\bar{k} = \sup\{k_n : n = 1,2,\ldots\} < \infty$.

By $\Phi \in \nabla_2$ again, there exists $\theta \in (0,1)$ such that

$$\Phi(\frac{k_0 k_n}{k_0 + \frac{k_n}{2}} x_n(i)) \leq (1-\theta)\frac{k_0}{k_0+\frac{k_n}{2}}\Phi(k_n x_n(i)) \quad (n = 1, 2, \ldots), \tag{2.2}$$

whenever $|x_n(i)| \leq \Phi^{-1}(1)$.

Using $\Phi \in \Delta_2$, there is $\delta > 0$ such that

$$\|x\|^0 \geq \frac{\varepsilon}{3} \quad \text{implying} \quad I_\Phi(x) \geq \delta. \tag{2.3}$$

Using $\Phi \in \Delta_2$ again, there exists $i_0 \in N$ such that

$$\sum_{i>i_0} \Phi(2k_0 x_0(i)) < \frac{k_0\theta\delta}{k} \tag{2.4}$$

Passing a subsequence, if necessary, we may assume that $\lim_{n\to\infty} x_n(i) = a_i$ for $i = 1, 2, \ldots$. Hence there is $n_0 \in N$ such that

$$\|[x_n]_0^{i_0} - [x_m]_0^{i_0}\|^0 < \frac{\varepsilon}{3} \quad \text{when} \quad n, m > n_0. \tag{2.5}$$

So,

$$\begin{aligned}\varepsilon \leq \|x_n - x_m\|^0 &\leq \|[x_n]_0^{i_0} - [x_m]_0^{i_0}\|^0 + \|[x_n]_{i_0}^{\infty}\|^0 + \|[x_m]_{i_0}^{\infty}\|^0 \\ &< \frac{\varepsilon}{3} + \|[x_n]_{i_0}^{\infty}\|^0 + \|[x_m]_{i_0}^{\infty}\|^0\end{aligned}$$

for any $n, m > n_0$.

Therefore, there exists at least one element in $\{x_n\}$, without loss of generality, we may assume that it is x_n such that $\|[x_n]_{i_0}^{\infty}\|^0 \geq \frac{\varepsilon}{3}$. It follows from (2.3) that

$$I_\Phi([x_n]_{i_0}^{\infty}) = \sum_{i=i_0+1}^{\infty} \Phi(x_n(i)). \geq \delta \tag{2.6}$$

Using the convexity of $\Phi(u)$ and (2.2), (2.1), (2.6), (2.4) we have

$$\begin{aligned}\|x_n + x_0\|^0 \leq \frac{k_n + k_0}{k_n k_0}\{1 &+ \sum_{i=1}^{i_0} \Phi(\frac{k_n k_0}{k_n + k_0}(x_n(i) + x_0(i))) \\ &+ \sum_{i=i_0+1}^{\infty} \Phi(\frac{\frac{k_n}{2} + k_0}{k_n + k_0}\frac{k_0}{\frac{k_n}{2} + k_0} k_n x_n(i) + \frac{\frac{k_n}{2}}{k_n + k_0} 2k_0 x_0(i))\} \\ \leq \frac{k_n + k_0}{k_n k_0}\{1 &+ \sum_{i=1}^{i_0} \Phi(\frac{k_n k_0}{k_n + k_0}(x_n(i) + x_0(i)))\end{aligned}$$

$$+\sum_{i=i_0+1}^{\infty}\frac{\frac{k_n}{2}+k_0}{k_n+k_0}\Phi(\frac{k_0}{\frac{k_n}{2}+k_0}k_nx_n(i)+\frac{2}{k_n+k_0}\Phi(2k_0x_0(i))\}$$

$$\leq\frac{k_n+k_0}{k_nk_0}\{1+\sum_{i=1}^{i_0}(\frac{k_0}{k_n+k_0}\Phi(k_nx_n(i))+\frac{k_n}{k_n+k_0}\Phi(k_0x_0(i))$$

$$+\sum_{i=i_0+1}^{\infty}(\frac{k_n+2k_0}{2k_n+2k_0}(1-\theta)(\frac{2k_0}{k_n+2k_0}\Phi(k_nx_n(i))$$

$$+\frac{k_n}{2k_n+2k_0}\Phi(2k_0x_0(i)))\}$$

$$\leq\frac{1}{k_n}(1+\sum_{i=1}^{\infty}\Phi(k_nx_n(i)))+\frac{1}{k_0}(1+\sum_{i=1}^{\infty}\Phi(k_0x_0(i)))$$

$$-\frac{\theta}{k_n}\sum_{i=i_0+1}^{\infty}\Phi(k_nx_n(i))+\frac{1}{2k_0}\sum_{i=i_0+1}^{\infty}\Phi(2k_0x_0(i))$$

$$\leq\|x_n\|^0+\|x_0\|^0-\frac{\theta\delta}{\bar{k}}+\frac{1}{2}\frac{\theta\delta}{\bar{k}}=2-\frac{\theta\delta}{2\bar{k}}.$$

Because $\bar{k}$, δ and θ are independent on choosing $\{x_n\}$, we get that x_0 is a β-point. ■

Corollary 1. *Orlicz sequence space l_Φ^0 has $L(\beta)$ if and only if $\Phi\in\Delta_2\cap\nabla_2$.*

Theorem 2. *A $x_0\in S(L_\Phi^0)$ is β-point if and only if:*
(1) $\Phi\in\Delta_2\cap\nabla_2$
(2) For any affine interval $a,b]$ of Φ and $k\in K(x)$, we have

$$\mu\{t\in G:k|x(t)|\in(a,b)\}=0;$$

(3) (i) if a left endpoint of affine interval $a,b]$ of Φ with $p_-(a)=p(a)$, then

$$\mu\{t\in G:k|x(t)|=a\}=0;$$

(ii) if b is right endpoint of affine interval $a,b]$ of Φ with $p_-(b)=p(b)$, then

$$\mu\{t\in G:k|x(t)|=b\}=0;$$

(4) (i) if a is left endpoint of affine interval $a,b]$ of Φ with $p_-(a)<p(a)$ and $\mu\{t\in G:k|x(t)|=a\}>0$, then $I_\Psi(p_-(kx))=1$.

(ii) If b is right endpoint of affine interval $a,b]$ of Φ with $p_-(b) < p(b)$ and $\mu\{t \in G : k|x(t)| = b\} > 0$, then $I_\Psi(p(kx)) = 1$.

Proof. **Sufficiency**. In this condition, x is a local uniform rotund point thanks to Theorem 2 of [22]. Hence, x is β-point.

Necessity. For convenience, without loss of generality, we may assume that $x(t) \geq 0$, i.e. $t \in G$.

(A) Using the same argument as the proof of Theorem 1, it is enough to prove that the condition (1) holds.

(B) If the condition (2) does not hold, there exists an affine interval $a, b]$ of Φ and $k \in K(x)$ such that $\mu\{t \in G : kx(t) \in (a,b)\} > 0$.

Take $\varepsilon > 0$ small enough such that $\mu E = \mu\{t \in G : kx(t) \in [a+\varepsilon, b-\varepsilon]\} > 0$.

Divide E into two substs E_1^1 and E_2^1 such that $\mu E_1^1 = \mu E_2^1$, $E_1^1 \cap E_2^1 = \emptyset$, $E_1^1 \cup E_2^1 = E$;

Divide E_1^1 into two substs E_1^2 and E_2^2 such that $\mu E_1^2 = \mu E_2^2$, $E_1^2 \cap E_2^2 = \emptyset$, $E_1^2 \cup E_2^2 = E_1^1$;

Divide E_2^1 into two substs E_3^2 and E_4^2 such that $\mu E_3^2 = \mu E_4^2$, $E_3^2 \cap E_4^2 = \emptyset$, $E_3^2 \cup E_4^2 = E_2^1$; etc.

Divide E_k^{n-1} into two substs E_{2k-1}^n and E_{2k}^n such that $\mu E_{2k-1}^n = \mu E_{2k}^n$, $E_{2k-1}^n \cap E_{2k}^n = \emptyset$, $E_{2k-1}^n \cup E_{2k}^n = E_k^{n-1}$; etc. $(n = 2,3,\ldots; k = 1,2,\ldots,2^{n-1})$.

Put

$$x_n(t) = \begin{cases} x(t) & t \in G \setminus E \\ x(t) - \frac{\varepsilon}{k} & t \in \cup_{i=1}^{2^{n-1}} E_{2i-1}^n \\ x(t) + \frac{\varepsilon}{k} & t \in \cup_{i=1}^{2^{n-1}} E_{2i}^n \end{cases} \quad \text{for } i = 1,2,\ldots,2^{n-1}, (n = 2,3,\ldots)$$

Obviously, $\|x_n - x_m\|^0 = \frac{2\varepsilon}{k}\frac{\mu E}{2}\Psi^{-1}(\frac{\mu E}{2})$ for $m \neq n$.

For any $0 < \eta < \frac{\varepsilon}{b}$,

$$I_\Psi(p(1+\eta)kx_n)) \geq I_\Psi(p((1+\eta)kx)) \geq 1;$$

$$I_\Psi(p(1-\eta)kx_n)) \leq I_\Psi(p((1-\eta)kx)) \leq 1.$$

So we have $k \in K(x_n)$ $(n = 1,2,\ldots)$. Then

$$\|x^n\|^0 = \frac{1}{k}(1 + I_\Phi(kx))$$

$$= \frac{1}{k}\{1 + I_\Phi(kx\chi_{G\setminus E}) + \sum_{i=1}^{2^{n-1}}(\int_{E_{2i-1}^n} \Phi(kx(t) - \varepsilon)dt + \int_{E_{2i}^n} \Phi(kx(t) + \varepsilon)dt)\}$$

$$= \frac{1}{k}\{1 + I_\Phi(kx\chi_{G\setminus E}) + \sum_{i=1}^{2^{n-1}}(\int_{E_{2i-1}^n} \Phi(kx(t))dt - \varepsilon p(a)\mu E_{2i-1}^n$$

$$+ \int_{E_{2i}^n} \Phi(kx(t))dt + \varepsilon p(a)\mu E_{2i}^n)\}$$

$$= \frac{1}{k}(1 + I_\Phi(kx) = \|x\|^0 = \quad \text{for } n = 1,2,\ldots$$

Using the same argument, we get $k \in K(\frac{x_n+x}{2})$ for $n = 1,2,\ldots$. Then

$$\|\frac{x_n + x}{2}\|^0 = \frac{1}{k}(1 + I_\Phi(k\frac{x_n + x}{2}))$$

$$= \frac{1}{k}\{1 + I_\Phi(kx\chi_{G\setminus E}) + \int_E \Phi(\frac{kx(t) + kx_n(t)}{2})dt\}$$

$$= \frac{1}{k}\{1 + I_\Phi(kx\chi_{G\setminus E}) + \int_E \frac{\Phi(kx(t)) + \Phi(kx_n(t))}{2}dt\}$$

$$= \frac{1}{2k}(1 + I_\Phi(kx)) + \frac{1}{2k}(1 + I_\Phi(kx_n)) = \frac{\|x\|^0 + \|x_n\|^0}{2} = 1,$$

which leads to a contradiction.

(C) Now, we are going to prove that (i) in the condition (3) holds. If there exists a, it is left endpoint of affine interval $a, b]$ of Φ with $p_-(a) = p(a)$ such that $\mu E = \mu\{t \in G : kx(t) = a\} > 0$.

Let $\{E_i^n\}_{i=1}^{2^n}$ be a partition of the E as (B) in the above proof.

Put

$$x_n(t) = \begin{cases} x(t) & t \in G \setminus E \\ \frac{a}{k} & t \in \cup_{i=1}^{2^{n-1}} E_{2i-1}^n \quad \text{for } i = 1,2,\ldots,2^{n-1} \quad (n = 2,3,\ldots) \\ \frac{b}{k} & t \in \cup_{i=1}^{2^{n-1}} E_{2i}^n \end{cases}$$

For any $\eta > 0$, we have

$$I_\Psi(p(1-\eta)kx_n)) \leq I_\Psi(p((1-\eta)kx\chi_{G\setminus E})) + \Psi(p((1-\eta)b))\mu E$$

$$\leq I_\Psi(p_-(kx\chi_{G\setminus E})) + \Psi(p(a))\mu E$$

$$= I_\Psi(p_-(kx\chi_{G\setminus E})) + \Psi(p_-(a))\mu E = I_\Psi(p_-(kx) \leq 1.$$

So, we get $k \in K(x_n)$ $(n = 1,2,\ldots)$.

Clearly, $\|x_n\|^0 = \|x_m\|^0$ for any $n, m = 1,2,\ldots$. So $k' = k\|x_n\|^0 \in K(\frac{x_n}{\|x_n\|^0})$ for $n = 1,2,\ldots$.

Denote

$$z_n = \frac{x_n}{\|x_n\|^0} \quad (n = 1, 2, \ldots).$$

Then $z_n \in S(L_\Phi^0)$ and for any $m, n \in N$ with $m \neq n$

$$\|z_n - z_m\|^0 = \frac{b-a}{k\|x_n\|^0} \frac{\mu E}{2} \Psi^{-1}(\frac{2}{\mu E}) = \frac{b-a}{2k'} \mu E \Psi^{-1}(\frac{2}{\mu E}).$$

Notice that

$$\frac{kk'}{k+k'}(x(t) + z_n(t)) = \frac{k'}{k+k'} kx(t) + \frac{k}{k+k'} k' x_n(t)$$

$$= \begin{cases} kx(t) & t \in G \setminus E \\ a & t \in \cup_{i=1}^{2^{n-1}} E_{2i-1}^n \\ \frac{k'}{k+k'} a + \frac{k}{k+k'} b & t \in \cup_{i=1}^{2^{n-1}} E_{2i}^n \end{cases} \quad \text{for } n = 1, 2, \ldots$$

It is easy to prove that $\frac{kk'}{k+k'} \in K(x + z_n)$ for $n = 1, 2, \ldots$. Hence

$$\|x + z_n\|^0 = \frac{k+k'}{kk'}(1 + I_\Phi(\frac{kk'}{k+k'}(x + z_n)))$$

$$= \frac{k+k'}{kk'}\{1 + I_\Phi(kx\chi_{G\setminus E}) + \int_E \Phi(\frac{k'}{k+k'} kx(t) + \frac{k}{k+k'} k' x_n(t)) dt\}$$

$$= \frac{k+k'}{kk'}\{1 + I_\Phi(kx\chi_{G\setminus E}) + \int_E (\frac{k'}{k+k'} \Phi(kx(t)) + \frac{k}{k+k'} \Phi(k' x_n(t))) dt\}$$

$$= \frac{1}{k}(1 + I_\Phi(kx)) + \frac{1}{k'}(1 + I_\Phi(k' z_n)) = 2.$$

This contradicts with that x is a β-point. Using the same method we can prove that (ii) of the condition (3) also holds.

Next, we will prove that (i) of the condition (4) holds.

(D) Otherwise, there exists a, it is left endpoint of affine interval $a, b]$ of Φ with $p_-(a) < p(a)$ such that $\mu \tilde{E} = \mu\{t \in G : kx(t) = a\} > 0$ and $I_\Psi(p_-(kx)) < 1$.

Notice that

$$I_\Psi(p_-(kx\chi_{G\setminus \tilde{E}})) + \Psi(p_-(a))\mu \tilde{E} < 1.$$

If $I_\Psi(p_-(kx\chi_{G\setminus \tilde{E}})) + \Psi(p(a))\mu \tilde{E} \leq 1$, then we let $E = \tilde{E}$;

If $I_\Psi(p_-(kx\chi_{G\setminus \tilde{E}})) + \Psi(p(a))\mu \tilde{E} > 1$, then we take $E \subset \tilde{E}$ such that $\mu E > 0$ and $I_\Psi(p_-(kx\chi_{G\setminus E})) + \Psi(p(a))\mu E = 1$.

Let $\{E_i^n\}_{i=1}^{2^n}$ be a partition of the E as (B) in this proof of Theorem 2. For any $\eta > 0$, we have

$$I_\Psi(p(1+\eta)kx_n) \geq I_\Psi(p(1+\eta)kx)) > 1;$$

$$I_\Psi(p(1-\eta)kx_n) \leq I_\Psi(p_-(kx\chi_{G\setminus E})) + \Psi(p(a))\mu E \leq,$$

i.e. $k \in K(x_n)$ for $n = 1, 2, \ldots$.

By the argument as above, we can finish the proof of (4). Here omit the later procedure of the proof. ■

Corollary 2. *Orlicz function space L_Φ^0 has $L(\beta)$ if and only if:*
(1) $\Phi \in \Delta_2 \cap \nabla_2$,
(2) Φ is strictly convex on the whole line.

Denote by $\{a_i\}_{i=1}^m$ and $\{b_i\}_{i=1}^m$ (m is finite or infinite) the set of left endpoint and right endpoint of all affine intervals of Φ, respectively.

Theorem 3. *A $x \in S(L_\Phi)$ is β-point if and only if:*
(1) $\Phi \in \Delta_2$
(2) For any affine interval $a, b]$ of Φ, we have

$$\mu\{t \in G : |x(t)| \in (a, b)\} = 0$$

(3) $\mu\{t \in G : |x(t)| \in \{a_i\}\} = 0$ and $\Phi \in \Delta_2$ or $\mu\{t \in G : |x(t)| \in \{b_i\}\} = 0$.

Proof. **Sufficiency.** In this conditions, x is an **UR**-point thanks to Theorem 2 of [23]. Of course, x is a β-point.

Necessity.

(A) If the condition (1) does not hold, the $L_\Phi \neq E_\Phi$. We consider two cases:

Case I: $x \notin E_\Phi$. Then we have $\lim_{n\to\infty} \|x\chi_{G\setminus G_n}\| = d(x, E_\Phi) = d > 0$, where $G_n = \{t \in G : |x(t)| \leq n\}$.

Since $\|x\| > \frac{d}{2}$, there exists $G_{n_1} \subset G$ such that $\|x\chi_{G_{n_1}}\| > \frac{d}{2}$;

Since $\|x\chi_{G\setminus G_{n_1}}\| > \frac{d}{2}$, there exists $G_{n_2} \supset G_{n_1}$ such that $\|x\chi_{G_{n_2}\setminus G_{n_1}}\| > \frac{d}{2}$; etc.

In such a way, we can get a sequence $\{G_{n_i}\}_{i=1}^\infty$ of subsets of G such that

$$G_{n_i} \subset G_{n_{i+1}} \text{ and } \|x\chi_{G_{n_{i+1}}\setminus G_{n_i}}\| > \frac{d}{2} \text{ for } i = 1, 2, \ldots.$$

Obviously, $\lim_{n\to\infty} \|x\chi_{G_{n_i}}\| = \|x\| = 1$ and

$$\lim_{n\to\infty} \|x + x\chi_{G_{n_i}}\| \geq 2 \lim_{n\to\infty} \|x\chi_{G_{n_i}}\| = 2.$$

But

$$\|x\chi_{G_{n_i}} - x\chi_{G_{n_j}}\| = \|x\chi_{G_{n_i}\setminus G_{n_j}}\| > \|x\chi_{G_{n_{j+1}}\setminus G_{n_j}}\| > \frac{d}{2} \text{ for any } i > j.$$

<u>Case II:</u> $z \in E_\Phi$. Take $x \notin E_\Phi$ such that $I_\Phi(x) < \infty$ and $d(x, E_\Phi) = d > 0$.

Repeat the procedure of the above method, we can get a sequence $\{G_{n_i}\}_{i=1}^{\infty}$ of subsets of G such that

$$G_{n_i} \subset G_{n_{i+1}} \quad \text{and} \quad \|x\chi_{G_{n_{i+1}}\setminus G_{n_i}}\| > \frac{d}{2} \quad \text{for} \quad i = 1, 2, \ldots.$$

Put

$$z_i(t) = \begin{cases} z(t) & t \in (G \setminus G_{n_{i+1}}) \cup G_{n_i} \\ x(t) & t \in G_{n_{i+1}} \setminus G_{n_i} \end{cases} \qquad i = 1, 2, \ldots.$$

Then $\|z_i\| \geq \|z\chi_{G_{n_i}}\| \to \|z\| =$. So, we have $\lim_{i\to\infty} \inf \|z_i\| \geq 1$. Since $I_\Phi(z_i) \leq I_\Phi(z) + I_\Phi(x\chi_{G\setminus G_{n_i}}) \to I_\Phi(z) = 1$, we get $\lim_{i\to\infty} \sup \|z_i\| \leq 1$. Hence, $\lim_{i\to\infty} \|z_i\| =$. Because of $z \in E_\Phi$, we can take $i_0 \in N$ such that $\|z\chi_{G_{n_i}}\| < \frac{d}{4}$ when $i \geq i_0$.

So

$$\|z_i - z_j\| \geq \|(x - z)\chi_{G_{n_{i+1}}\setminus G_{n_i}}\| \geq \|x\chi_{G_{n_{i+1}}\setminus G_{n_i}}\| - \|z\chi_{G_{n_{i+1}}\setminus G_{n_i}}\| > \frac{d}{2} - \frac{d}{4} = \frac{d}{4},$$

when $j > i \geq i_0$.

It is obviously that

$$\|z + z_i\| \geq 2\|z\chi_{G_{n_i}}\| \to \|z\| = 2.$$

This contradiction shows that the condition (1) holds.

(B) Suppose that there exist an affine interval $a, b]$ of Φ such that $\mu\{t \in G : x(t) \in (a, b)\} > 0$.

Take $\varepsilon > 0$ small enough such that $\mu E = \mu\{t \in G : x(t) \in [a+\varepsilon, b-\varepsilon]\} > 0$.

Divide E into two subsets E_1^1 and E_2^1 such that $\mu E_1^1 = \mu E_2^1$, $E_1^1 \cap E_2^1 = \emptyset$, $E_1^1 \cup E_2^1 = E$;

Divide E_1^1 into two subsets E_1^2 and E_2^2 such that $\mu E_1^2 = \mu E_2^2$, $E_1^2 \cap E_2^2 = \emptyset$, $E_1^2 \cup E_2^2 = E_1^1$;

Divide E_2^1 into two subsets E_2^3 and E_4^2 such that $\mu E_3^2 = \mu E_4^2$, $E_3^2 \cap E_4^2 = \emptyset$, $E_3^2 \cup E_4^2 = E_2^1$; etc.

Divide E_k^{n-1} into two subsets E_{2k-1}^n and E_n^{2k} such that $\mu E_{2k-1}^n = \mu E_{2k}^n$, $E_{2k-1}^n \cap E_{2k}^n = \emptyset$, $E_{2k}^n \cup E_{2k-1}^n = E_k^{n-1}$; etc. for $(n = 2, 3, \ldots; k = 1, 2, \ldots 2^{n-1})$.

Put

$$x_n(t) = \begin{cases} x(t) & t \in G\setminus E \\ x(t)-\varepsilon & t \in \bigcup_{i=1}^{2^{n-1}} E_{2i-1}^n \\ x(t)+\varepsilon & t \in \bigcup_{i=1}^{2^{n-1}} E_{2i}^n \end{cases} \quad n = 2,3,\dots$$

Then

$$I_\Phi(x_n) = I_\Phi(x\chi_{G\setminus E}) + \sum_{i=1}^{2^{n-1}} \Big(\int_{E_{2i-1}^n} \Phi(x(t)-\varepsilon)dt + \int_{E_{2i}^n} \Phi(x(t)+\varepsilon)dt \Big)$$

$$= I_\Phi(x\chi_{G\setminus E}) + \sum_{i=1}^{2^{n-1}} \Big(\int_{E_{2i-1}^n} \Phi(x(t))dt - \varepsilon p(a)\mu E_{2i-1}^n + \int_{E_{2i}^n} \Phi(x(t))dt + \varepsilon p(a)\mu E_{2i}^n \Big)$$

$$= I_\Phi(x) = \quad \text{for} \quad n = 1,2,\dots.$$

So, we get $\|x_n\| = 1$ for all $n \in N$.

Since

$$I_\Phi\Big(\frac{x_n + x}{2}\Big) = I_\Phi(x\chi_{G\setminus E}) + \int_E \Phi\Big(\frac{x(t) + x_n(t)}{2}\Big)dt$$

$$= I_\Phi(x\chi_{G\setminus E}) + \int_E \frac{\Phi(x(t)) + \Phi(x_n(t))}{2}dt = \frac{I_\Phi(x) + I_\Phi(x_n)}{2} = 1$$

So $\|x_n + x\| = 2$. But,

$$\|x_m - x_n\| = 2\varepsilon\Phi^{-1}\Big(\frac{2}{\mu E}\Big) \quad \text{for any} \quad m \neq n.$$

This shows that x is not β-point, which leads to a contradiction.

(C) If the condition (3) does not hold, there are the following two cases.
Case I: There exist $a \in \{a_i\}_{i=1}^m$ and $b \in \{b_i\}_{i=1}^m$ such that

$$\mu E = \mu\{t \in G : x(t) = a\} > 0;$$

and $\mu F = \mu\{t \in G : x(t) = b'\} \geq 0$.

Take $b > a$ and $a' < b'$ for which $[a,b]$ and $[a',b']$ are affine intervals of Φ with

$$\Phi(b) - \Phi(a) = \Phi(b') - \Phi(a').$$

If $\mu E < \mu F$, we take subset of F, we still denote it as F, such that $\mu E = \mu F$. Let $\{E_i^n\}_{i=1}^{2^n}$ and $\{F_i^n\}_{i=1}^{2^n}$ $(n = 1,2,\dots)$ be a partition of the E and F as in the proof of Theorem 2, respectively.

Put

$$x_n(t) = \begin{cases} x(t) & t \in G \setminus (E \cup F) \\ a & t \in \cup_{i=1}^{2^{n-1}} E_{2i-1}^n \\ b & t \in \cup_{i=1}^{2^{n-1}} E_{2i}^n \\ a' & t \in \cup_{i=1}^{2^{n-1}} F_{2i-1}^n \\ b' & t \in \cup_{i=1}^{2^{n-1}} F_{2i}^n \end{cases} \qquad (n = 1, 2, \ldots)$$

Then $I_\Phi(x_n) = I_\Phi(x\chi_{G\setminus(E\cup F)}) + (\Phi(a) + \Phi(b))\frac{\mu E}{2} + (\Phi(a') + \Phi(b'))\frac{\mu F}{2}$

$$= I_\Phi(x\chi_{G\setminus(E\cup F)}) + \Phi(a)\mu E + \Phi(b'))\mu F = 1$$

So, $\|x_n\| = 1 \ \ (n = 1, 2, \ldots)$. Similarly,

$$I_\Phi(\frac{x_n + x}{2})$$

$$= I_\Phi(x\chi_{G\setminus(E\cup F)}) + \Phi(a)\frac{\mu E}{2} + \Phi(\frac{a+b}{2})\frac{\mu E}{2} + \Phi(b')\frac{\mu F}{2} + \Phi(\frac{a'+b'}{2})\frac{\mu F}{2}$$

$$= I_\Phi(x\chi_{G\setminus(E\cup F)}) + (\frac{3}{2}\Phi(a) + \frac{1}{2}\Phi(b) + \frac{3}{2}\Phi(b') + \frac{1}{2}\Phi(a'))\frac{\mu E}{2}$$

$$= I_\Phi(x\chi_{G\setminus(E\cup F)}) + (\frac{3}{2}(\Phi(a) + \Phi(b')) + \frac{1}{2}(\Phi(b) + \Phi(a')))\frac{\mu E}{2} = I_\Phi(x) = 1$$

i.e., $\|x_n - x_m\| = 2$. But,

$$\|x_n + x\| > (b - a)\Phi^{-1}(\frac{2}{\mu E}) \text{ for any } n \neq m.$$

So we prove that Case I does not exist.

Case II: There is a $b \in \{b_i\}_{i=1}^m$ such that $\mu E > 0$ and $\Phi \notin \nabla_2$, where $E = \{t : x(t) = b\}$.

Since $\Phi \notin \nabla_2$, there exists a sequence $u_n \uparrow \infty$ such that

$$\Phi(\frac{u_n}{2}) > (1 - \frac{1}{n})\frac{\Phi(u_n)}{2} \quad (n = 1, 2, \ldots).$$

Take $\varepsilon > 0$ satisfying $b - \varepsilon \in (a, b)$ and a sequence $\{F_n\}$ of subsets of E such that $F_n \cap F_m = \emptyset \ \ (n = 1, 2, \ldots; m \neq n)$.

Take a subsequence of $\{u_n\}$, we still denote it as $\{u_n\}$, for which $\Phi(u_n - b)\mu F_n > \varepsilon p_-(b)\mu E$.

Choose a subset E_n of F_n such that $\Phi(u_n - b)\mu E = \varepsilon p_-(b)\mu(E_n) \ \ (n = 1, 2, \ldots)$.

Obviously, $\lim_{n\to\infty} \mu E_n = 0$.

Put

$$x_n(t) = \begin{cases} x(t) & t \in G \setminus E \\ b - \varepsilon & t \in E \setminus E_n \\ u_n - b & t \in E_n \end{cases} \qquad (n = 1, 2, \dots)$$

Then $I_\Phi(x_n) = I_\Phi(x\chi_{G\setminus E}) + \Phi(b-\varepsilon)\mu(E \setminus E_n) + \Phi(u_n - b)\mu(E_n)$

$$= I_\Phi(I_\Phi x\chi_{G\setminus E}) + (\Phi(b) - \varepsilon p_-(b))\mu(E \setminus E_n) + \varepsilon p_-(b)\mu E$$

$$\to I_\Phi(x\chi_{G\setminus E}) + \Phi(b)\mu(E) = I_\Phi(x) = 1 \quad (n \to \infty).$$

So, $\|x_n\| \to$ as $n \to \infty$. Similarly,

$$I_\Phi(\frac{x_n + x}{2}) = I_\Phi(x\chi_{G\setminus E}) + \Phi(b - \frac{\varepsilon}{2})\mu(E \setminus E_n) + \Phi(\frac{u_n}{2})\mu(E_n)$$

$$> I_\Phi(x\chi_{G\setminus E}) + (\Phi(b) - \frac{\varepsilon}{2}p_-(b))\mu(E \setminus E_n) + (1 - \frac{1}{n})\frac{\Phi(u_n)}{2}\mu(E_n)$$

$$> I_\Phi(x\chi_{G\setminus E}) + (\Phi(b) - \frac{\varepsilon}{2}p_-(b))\mu(E \setminus E_n) + (1 - \frac{1}{n})\frac{\Phi(u_n - b)\mu(E_n)}{2}$$

$$\to I_\Phi(x\chi_{G\setminus E}) + \Phi(b)\mu(E) = I_\Phi(x) = 1.$$

So, $\|\frac{x_n+x}{2}\| \to 1$ as $n \to \infty$.

Since $\Phi \in \Delta_2$, there exists $\delta > 0$ such that

$$|I_\Phi(u) - I_\Phi(u - v)| < \frac{\varepsilon}{2}p_-(b)\mu E$$

whenever $I_\Phi(u) = \varepsilon p_-(b)\mu E$ and $I_\Phi(v) < \delta$.

Using $\lim_{n\to\infty} \Phi(b - \varepsilon)\mu E_n = 0$ there is $n_0 > 0$ such that

$$\Phi(b - \varepsilon)\mu E_n < \delta \quad \text{when} \quad n > n_0.$$

Hence,

$$I_\Phi(x_m - x_n) > \Phi(u_n - b) - (b - \varepsilon))\mu E_n$$

$$> \Phi(u_n - b)\mu E_n - \frac{\varepsilon}{2}p_-(b)\mu E$$

$$= \frac{\varepsilon}{2}p_-(b)\mu E \quad \text{for any} \quad m > n > n_0.$$

Using $\Phi \in \Delta_2$ again, there exists $\delta_1 > 0$ such that

$$\|x_m - x_n\| > \delta_1 \quad \text{for any} \quad m > n > n_0.$$

The proof is finished. ■

Corollary 3. *L_Φ has local property β if and only if $\Phi \in \Delta_2$ and Φ is strictly convex in the whole line.*

Theorem 4. *A $x \in S(l_\Phi)$ is a β-point if and only if:*
(1) $\Phi \in \Delta_2$
(2) $\{i \in N : |x(i)| \in (a, b)\} = \emptyset$ for any interval $[a, b]$ of Φ or $\Phi \in \nabla_2$.

Proof. **Necessity.** The proof is similar to Theorem 3 and so we omit it.

Sufficiency. Without loss of generality, we may assume $x(i) \geq 0$, $(i = 1, 2, \ldots)$.

For any $x_0 \in S(l_\Phi)$, we are going to prove that it is a β-point.

Suppose that a sequence $(x_n) \subset S(l_\Phi)$ with $sep(x_n) \geq \varepsilon$ for any given $\varepsilon > 0$.

Now, let us consider two cases.

Case I: $\Phi \in \Delta_2 \cap \nabla_2$.

Since $\Phi \in \Delta_2$, there is $\tau \in (0, 1)$, such that

$$\|u\| \geq \frac{\varepsilon}{3} \Rightarrow I_\Phi(u) \geq \tau. \tag{3.1}$$

Since $\Phi \in \nabla_2$, there is $0 < \theta < 1$ such that

$$\Phi(\frac{2}{3}u) \leq (1 - \theta)\frac{2}{3}\Phi(u), \tag{3.2}$$

whenever $|u| < \Phi^{-1}(1)$.

Using $\Phi \in \Delta_2$ again, there exists $0 < \xi < 1$ such that

$$I_\Phi(u) < 1 - \frac{\tau\theta}{4} \Rightarrow \|u\| \leq 1 - \xi. \tag{3.3}$$

Notice that $I_\Phi(2x) < \infty$ thanks to $\Phi \in \Delta_2$, there exists i_0 such that

$$\sum_{i > i_0} \Phi(2x(i)) < \theta\tau \tag{3.4}$$

Passing a subsequence, if necessary, we may assume that $\lim_{n\to\infty} x_n(i) = a_i$ $(i = 1, 2, \ldots)$.

Then there is $n_0 \in N$ such that

$$\|[x_n]_0^{i_0} - [x_m]_0^{i_0}\| < \frac{\varepsilon}{3} \quad \text{when} \quad m > n \geq n_0.$$

So

$$\varepsilon < \|x_n - x_m\| \leq \|[x_n]_0^{i_0} - [x_m]_0^{i_0}\| + \|[x_n]_0^{\infty}\| + \|[x_m]_0^{\infty}\| \leq \frac{\varepsilon}{3} + \|[x_n]_{i_0}^{\infty}\| + \|[x_m]_0^{\infty}\|$$

From the above inequality, we obtain that there at least exists one element in $\{x_n\}$, without loss of generality, we may assume that it is x_n such that $\|[x_n]_{i_0}^{\infty}\| \geq \frac{\varepsilon}{3}$.

It follows from (1) that we get

$$\sum_{i=i_0+1}^{\infty} \Phi(x_n(i)) = I_\Phi([x_n]_{i_0}^{\infty}) \geq \tau. \tag{3.5}$$

Combining (3.2), (3.4), with (3.5), we get

$$I_\Phi(\frac{x_n + x}{2}) = \sum_{i=1}^{i_0} \Phi(\frac{x_n(i) + x(i)}{2}) + \sum_{i=i_0+1}^{\infty} \Phi(\frac{3}{4}\frac{2}{3}x_n(i) + \frac{1}{4}2x(i))$$

$$\leq \sum_{i=1}^{i_0} \frac{\Phi(x_n(i)) + \Phi(x(i))}{2} + \sum_{i=i_0+1}^{\infty} (\frac{3}{4}\Phi(\frac{2}{3}x_n(i)) + \frac{1}{4}\Phi(2x(i)))$$

$$\leq \sum_{i=1}^{i_0} \frac{\Phi(x_n(i)) + \Phi(x(i))}{2} - \sum_{i=i_0+1}^{\infty} (\frac{1}{2}(1-\theta)\Phi(x_n(i)) + \frac{1}{4}\Phi(2x(i)))$$

$$\leq \frac{I_\Phi(x_n) + I_\Phi(x)}{2} + \frac{\theta}{2} \sum_{i=i_0+1}^{\infty} \Phi(x(i)) + \frac{1}{4} \sum_{i=i_0+1}^{\infty} \Phi(2x(i))$$

$$\leq 1 - \frac{\theta\tau}{2} + \frac{\theta\tau}{4} = 1 - \frac{\theta\tau}{4}.$$

Using (3.3), we obtain $\|x_n + x\| \leq 2(1-\xi)$.

Case II: $\Phi \in \Delta_2$ and $\{i \in N : |x(i)| \in (a,b)\} = \emptyset$ for any interval $[a,b]$ of Φ.

Since $\Phi \in \Delta_2$, there is $\tau > 0$ such that

$$\|u\| \geq \frac{\varepsilon}{2} \Rightarrow I_\Phi(u) \geq 2\tau. \tag{3.6}$$

Since $I_\Phi(u) = 1$, there is i_0 such that

$$\sum_{i>i_0} \Phi(x(i)) < \frac{\tau}{2}. \tag{3.7}$$

For $x(i)$ with $x(i) > \frac{\tau}{2i_0 p(\Phi^{-1}(1))}$, noticing that $x(i) \notin \cup(a,b]$, we have

$$\Phi(\frac{x(i)+u}{2}) < \frac{\Phi(x(i))+\Phi(u)}{2} \quad \text{when} \quad u < x(i).$$

So, there exists a $\delta_i \in (0,1)$ such that

$$\Phi(\frac{x(i)+u}{2}) < (1-\delta_i)\frac{\Phi(x(i))+\Phi(u)}{2}$$

when

$$|u| \le x(i) - \frac{\tau}{2i_0 p(\Phi^{-1}(1))} \quad (i = 1,2,\ldots).$$

Take $\delta = \min_{1\le i\le i_0} \delta_i$, Then

$$\Phi(\frac{x(i)+u}{2}) < (1-\delta)\frac{\Phi(x(i))+\Phi(u)}{2} \tag{3.8}$$

holds for $|u| \le x(i) - \frac{\tau}{2i_0 p(\Phi^{-1}(1))}$, where $i \in \{i \le i_0 : x(i) > \frac{\tau}{2i_0 p(\Phi^{-1}(1))}\}$.

Using $\Phi \in \Delta_2$ again, there exists $0 < \xi < 1$ such that

$$I_\Phi < 1 - \frac{\delta}{2}\Phi(\frac{\tau}{2i_0 p(\Phi^{-1}(1))}) \Rightarrow \|u\| \le 1 - \xi. \tag{3.9}$$

Since $sep(x_n) > \varepsilon$, there always exist x_n for which $\|x_n - x\| \ge \frac{\varepsilon}{2}$. Take use of (3.6), we get $I_\Phi(x_n - x) \ge 2\tau$.

Denote $N^+ = \{i \in N : x(i) > |x_n(i)|\}$.

Notice that $\Phi(b-a) \le |\Phi(b) - \Phi(a)|$ and $I_\Phi(x_n) = I_\Phi(x) = 1$,

$$2\tau \le I_\Phi(x_n - x) \le \sum_{i=1}^{\infty} |\Phi(x_n(i)) - \Phi(x(i))|$$

$$= \sum_{i\in N^+} (\Phi(x(i)) - \Phi(x_n(i))) + \sum_{i\in N\setminus N^+} (\Phi(x_n(i)) - \Phi(x(i)))$$

$$= 2\sum_{i\in N^+} (\Phi(x(i)) - \Phi(x_n(i))).$$

From (3.7), we get $\sum_{i\in N^+, i>i_0}(\Phi(x(i) - \Phi(x_n(i)) \le \sum_{i>i_0} \Phi(x(i)) < \frac{\tau}{2}$. So,

$$\frac{\tau}{2} \le \sum_{i\in N^+, i\le i_0} (\Phi(x(i)) - \Phi(x_n(i)))$$

$$\le \sum_{i\in N^+, i\le i_0} (x(i) - |x_n(i)|)p(x(i))$$

$$\leq p(\Phi^{-1}(1)) \sum_{i\in N^+, i\leq i_0} (x(i) - |x_n(i)|)$$

Hence, there at least exists one $i' \leq i_0$ such that $x(i') - |x(i')| \geq \frac{\tau}{2i_0 p(\Phi^{-1}(1))}$.

Using (3.8), we obtain

$$\Phi(\frac{x(i') + x_n(i')}{2}) \leq (1-\delta)\frac{\Phi(x(i')) + \Phi(x_n(i'))}{2}.$$

Therefore

$$1 - I_\Phi(\frac{x_n + x}{2}) = \frac{I_\Phi(x_n) + I_\phi(x)}{2} - I_\Phi(\frac{x_n + x}{2})$$

$$= \sum_{i=1}^{\infty}(\frac{\Phi(x_n(i)) + \Phi(x(i))}{2} - \Phi(\frac{x_n(i) + x(i)}{2}))$$

$$\geq \frac{\Phi(x_n(i')) + \Phi(x(i'))}{2} - \Phi(\frac{x_n(i') + x(i')}{2})$$

$$\geq \frac{\delta}{2}(\Phi(x_n(i')) + \Phi(x(i')) \geq \frac{\delta}{2}\Phi(x(i')) \geq \frac{\delta}{2}\Phi(\frac{\tau}{2i_0 p(\Phi^{-1}(1))})$$

i.e., $I_\Phi(\frac{x_n+x}{2}) \leq 1 - \frac{\delta}{2}\Phi(\frac{\tau}{2i_0(\Phi^{-1}(1))})$. We get $\|x_n + x\| \leq 2(1-\xi)$ that follows from (3.9).

Thus, the proof is finished. ■

Corollary 4. *Orlicz sequence space l_Φ has $L(\beta)$ if and only if:*
(1) $\Phi \in \Delta_2$,
(2) $\Phi \in SC[0, \Phi^{-1}(1)]$ *or* $\Phi \in \nabla_2$.

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