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Strengthened Cauchy Inequality for Bilinear Forms over Curved Domains. Cubic Case

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The strengthened Cauchy - Buniakowskii - Schwarz inequality for elliptic bilinear forms over curved domains and 10-node 2-simplex elements is considered. It is proven that the inequality holds uniformly with respect to the finite element spaces. The results have applications in multilevel method for solving elliptic boundary-value problems. Upper bound for contraction number is found.

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1. Introduction

Let U, V be two linear finite-dimensional spaces, $U \cap V = \{0\}$ and let there exists a constant $\gamma \in [0,1)$ depending only on the spaces U and V, but not dependent on the choice of the elements $u \in U$, and $v \in V$, such that

$$|(u,v)| \leq \gamma\sqrt{(u,u)}\sqrt{(v,v)}.$$

The last inequality is the so-called strengthened Cauchy-Buniakowskii-Schwarz (C.B.S.) inequality. Among authors who have used the strengthened C.B.S. inequality in two-level method we mention Bank and Dupon [3], Braess [4,5], Maitre and Musy [10], Axelsson [1], and Axelsson and Gustafsson [2]. The inequality has been used in connection with the two-grid FAC-preconditioner by McCormick [12], McCormick and Thomas [13]. The C.B.S. inequality is applied in works of Bramble et al. [6] and Mandel and McCormick [11]. The role of the C.B.S. inequality in multilevel methods is considered in detail by Eijkhout and Vassilevski [9]. Computation of constants in the strengthened C.B.S. inequality we can find in [14]. Our goal is to study the behaviour of the

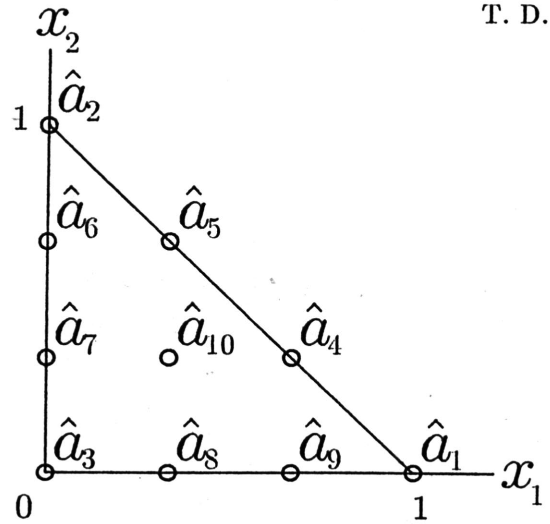

Figure 1: **Finite element of reference**

constant in the strengthened C.B.S. inequality for a class of 10-node curvilinear triangle finite elements.

Let $H^1(\Omega)$ be the usual Sobolev's space.

We consider an elliptic bilinear form $a(\cdot,\cdot)$:

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad u,v \in H^1(\Omega)$$

with $u, v = 0$ on $\Gamma_D \subset \Gamma = \partial\Omega$, $meas(\Gamma_D) \neq 0$. The set Ω is open subset of $\mathbf{R}^2$, with Lipschitz - continuous boundary. We assume that Γ is piecewise $(P_3)^2$, where P_k is the space of all polynomials of degree, not exceeding k-th.

We denote the point (x_1, x_2) by x, and the vector with the same coordinates by $\underline{x}$.

Let $(\hat{T}, \hat{P}, \hat{\Sigma})$ be the 10-node 2-simplex finite element of reference (Fig. 1) defined as follows:

$\hat{T} = \{(\hat{x}_1, \hat{x}_2) \mid \hat{x}_1 \geq 0,\ \hat{x}_2 \geq 0,\ \hat{x}_1 + \hat{x}_2 \leq 1\}$ is the unit 2-simplex;

$\hat{P} = \hat{P}_3$, where $\hat{P}_k$ is the space of all polynomials of degree, not exceeding k-th on $\hat{T}$;

$\hat{\Sigma} = \{(\hat{x}_1, \hat{x}_2) \mid \hat{x}_1 = \frac{i}{3},\ \hat{x}_2 = \frac{j}{3};\ i + j \leq 3;\ i, j \in \{0,1,2,3\}\}$ is the set of all Lagrangian interpolation nodes.

Let $a_{iT}(a_{1iT}, a_{2iT}),\ \ i = 1, 2, ..., 10$ be the nodes of the element T, $\mathcal{A}_{F_T}$ be the matrix

$$\mathcal{A}_{F_T} = \begin{pmatrix} a_{11T} & a_{12T} & a_{13T} & a_{14T} & a_{15T} & a_{16T} & a_{17T} & a_{18T} & a_{19T} & a_{1,10T} \\ a_{21T} & a_{22T} & a_{23T} & a_{24T} & a_{25T} & a_{26T} & a_{27T} & a_{28T} & a_{29T} & a_{2,10T} \end{pmatrix},$$

and

$$\underline{\Phi}(\hat{x}) = \Big(\frac{1}{2}\hat{x}_1(3\hat{x}_1-1)(3\hat{x}_1-2), \frac{1}{2}\hat{x}_2(3\hat{x}_2-1)(3\hat{x}_2-2),$$

$$\frac{1}{2}\hat{x}_3(3\hat{x}_3-1)(3\hat{x}_3-2), \frac{9}{2}\hat{x}_1\hat{x}_2(3\hat{x}_1-1), \frac{9}{2}\hat{x}_1\hat{x}_2(3\hat{x}_2-1), \frac{9}{2}\hat{x}_2\hat{x}_3(3\hat{x}_2-1),$$

$$\frac{9}{2}\hat{x}_2\hat{x}_3(3\hat{x}_3-1), \frac{9}{2}\hat{x}_1\hat{x}_3(3\hat{x}_3-1), \frac{9}{2}\hat{x}_1\hat{x}_3(3\hat{x}_1-1), 27\hat{x}_1\hat{x}_2\hat{x}_3\Big)^t,$$

$\hat{x}_3 = 1 - \hat{x}_1 - \hat{x}_2$ be the vector whose coordinates are the nodal basis functions of the element $\hat{T}$, then we can write the cubic transformation:

$$F_T = \mathcal{A}_{F_T}\underline{\Phi}(\hat{x}).$$

An arbitrary 10-node 2-simplex element (T, P_T, Σ_T) is defined by $T = F_T(\hat{T})$, where F_T is invertible transformation.

Let τ_h be an initial triangulation of the set Ω by 10-node 2-simplex elements. Since the boundary is piecewise $(P_3)^2$ we can write $\overline{\Omega} = \bigcup_{T\in\tau_h} T$. We consider a family of finite-element spaces $(\mathbf{V}_h)$:

$$\mathbf{V}_h = \{v_h \in H_0^1(\overline{\Omega}) \mid v_{h|T} = p(x) : p = \hat{p}\circ F_T^{-1}, \hat{p} \in \hat{P}, T \in \tau_h\},$$

where it is understood, that the parameter h is the defining parameter of the family and has limit zero.

We make hierarchical refinement of τ_h, dividing each element to four finite elements of the same class as shown in Fig. 2. Thus we obtain triangulation τ_{h_1} of the domain Ω. The space $\mathbf{V}_{h_1}$ is finite element space associated with τ_{h_1}. We denote the set of the nodes of the triangulations τ_h, τ_{h_1} accordingly by N_h, N_{h_1}. Let $\{\varphi_i^{(1)}\}$ be the nodal basis in $\mathbf{V}_{h_1}$ associated with the set N_{h_1}, excluding Dirichlet boundary points. We define the hierarchical space

$$\widetilde{\mathbf{V}}_{h_1} = Span\{\varphi_i^{(1)}\}_{i\,:\,a_i\in N_{h_1}\setminus N_h}$$

in addition to $\mathbf{V}_h \subset \mathbf{V}_{h_1}$. As it is well-known [1, 9, 10] for polygonal domains holds the restricted strengthened C.B.S. inequality

$$|a_T(v,w)| \le \gamma_T\sqrt{a_T(v,v)}\sqrt{a_T(w,w)}, \quad \forall v \in \mathbf{V}_h,\ \forall w \in \widetilde{\mathbf{V}}_{h_1},$$

where

$$a_T(u,v) = \int_T \nabla u \cdot \nabla v dx,$$

is the restricted bilinear form.

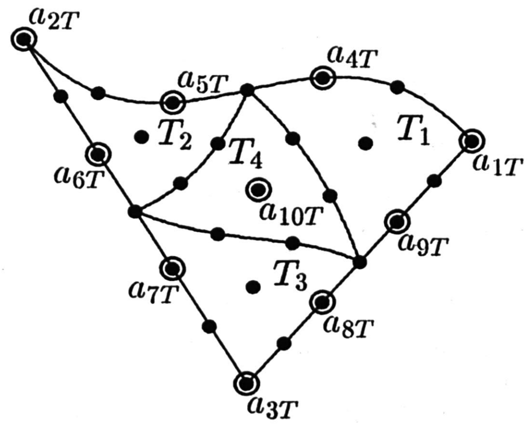

Figure 2: **Finite element $T \in \tau_h$ after refinement. We use the next simple legend: ○ - node from the coarse triangulation, • - node from the fine triangulation.**

2. Energy inequalities

We use not only straight elements but also isoparametric elements for getting an exact approximation of the boundary Γ. We represent the transformation F_T as a product of two transformations - $F_T = V_T \circ W_T$.

We define the transformation $W_T : \hat{T} \to \mathcal{T} \subset \mathbf{R}^2$ by:

$W_T(\hat{x}) = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \mathcal{A}_{W_T}\underline{\Phi}(\hat{x})$, where

$$\mathcal{A}_{W_T} = \begin{pmatrix} 1 & 0 & 0 & \alpha_{4T} & \alpha_{5T} & 0 & 0 & \frac{1}{3} & \frac{2}{3} & \alpha_{10,T} \\ 0 & 1 & 0 & \beta_{4T} & \beta_{5T} & \frac{2}{3} & \frac{1}{3} & 0 & 0 & \beta_{10,T} \end{pmatrix},$$

and the transformation $V_T : \mathcal{T} \to T$ by:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = V_T(X) = \begin{pmatrix} a_{13T} \\ a_{23T} \end{pmatrix} + \begin{pmatrix} a^T_{113} & a^T_{123} \\ a^T_{213} & a^T_{223} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

where we use denotations $a^T_{kij} = a_{kiT} - a_{kjT}$, $i, j, k \in \{1, 2, 3\}$.

The image T of the finite element of reference $\hat{T}$ by transformation $V_T \circ W_T$ represents element with only one curved side. The applicability of 10-node 2-simplex elements and the quality of approximations by such elements depend on the choice of the node a_{10T}. We determine the node a_{10T} by $\alpha_{10,T} = \frac{\alpha_{4T}}{2}$, $\beta_{10,T} = \frac{\beta_{5T}}{2}$. Thus we obtain transformation $\mathcal{W}_T : \hat{T} \to \mathcal{T}$ which is special case of the transformation W_T. The transformation $\mathcal{W}_T(\hat{x})$ can be described by:

$$X_1(\hat{x}) = \hat{x}_1 + \psi_1(\alpha_{4T}, \alpha_{5T})\hat{x}_1\hat{x}_2 + \psi_2(\alpha_{4T}, \alpha_{5T})\hat{x}_1\hat{x}_2^2,$$

$$X_2(\hat{x}) = \hat{x}_2 + \psi_1(\beta_{5T}, \beta_{4T})\hat{x}_1\hat{x}_2 + \psi_2(\beta_{5T}, \beta_{4T})\hat{x}_1^2\hat{x}_2,$$

$$\psi_1(x) = \frac{9}{2}(-1 + 2x_1 - x_2), \quad \psi_2(x) = \frac{9}{2}(1 - 3x_1 + 3x_2).$$

We make the next denotations: $\widetilde{T} = V_T(\hat{T})$, $a_{iT}^* = \mathcal{W}_T(\hat{a}_i)$, $\widetilde{a}_{iT} = V_T(\hat{a}_i)$, $i = 1, 2, ..., 10$, $h_T = diam(\widetilde{T})$, $\rho_T = diam$(inscribed spher of $\widetilde{T}$). We represent the triangulation τ_h in view of:

$$\tau_h = \{T = F_T(\hat{T}) \mid F_T = V_T \circ \mathcal{W}_T,\ diam(\widetilde{T}) < h\}.$$

We have an isoparametric family $(T \in \tau_h, P_T, \Sigma_T)$ of 10-node 2-simplex elements. Further, we consider only triangulations τ_h which satisfy:

(i) If the element $T \in \tau_h$ have less than two vertices over Γ then this element is a straight element;

(ii) There exists constant μ such that $\forall T \in \tau_h$, $\frac{h_T}{\rho_T} \le \mu$;

(iii) For all curved element $T \in \tau_h$ holds $\|\underline{a}_{iT} - \underline{\widetilde{a}}_{iT}\|_E = O(h_T^2)$, $i = 4, 5$, where $\|\cdot\|_E$ is Euclidean norm in $\mathbf{R}^2$.

We will analyze, how the choice of the node a_{10T} influences over quality of interpolation by 10-node 2-simplex elements.

We define the interpolant Π on $H^3(T)$, $T \in \tau_h$ by

$$\Pi v = \sum_{i=1}^{10} v(a_{iT})\varphi_{iT}(\hat{x}).$$

Theorem 1. *Let $\|a_{iT} - \widetilde{a}_{iT}\|_E = O(h^2)$, $i = 4, 5, ..., 10$, $T \in \tau_h$ and $\hat{P}_2 \subset \hat{P}$ then we have $|v - \Pi v|_{m,T} = O(h^{3-m})$, $\forall v \in H^3(T)$ and $m = 0, 1, 2$.*

Theorem 1 is a special case of the fundamental result by Ciarlet and Raviart [8].

Theorem 2. *Let the triangulation τ_h fulfills the conditions $(i) - (iii)$, then we have*

(1) $$|v - \Pi v|_{m,T} = O(h^{3-m})$$

$\forall v \in H^3(T)$, $T \in \tau_h$ *and* $m = 0, 1, 2$.

P r o o f. For notational convenience, we shall drop the index T throughout the proof. Let $T \in \tau_h$ be a straight element. As the node a_{10} is barycenter of the element T, then (1) follows directly from Theorem 3.1.6 [7, p.124].

Let $T \in \tau_h$ be a curved element. The condition (iii) imposed over triangulation τ_h provides

$$\|\underline{a}_i - \underline{\widetilde{a}}_i\|_E = O(h^2) \quad i = 4, 5.$$

We will prove that $\|\underline{a}_{10} - \underline{\widetilde{a}}_{10}\|_E = O(h^2)$.

We denote $\underline{s}_i = \underline{a}_i - \underline{\widetilde{a}}_i$ and $\underline{s}_i^\star = \underline{a}_i^\star - \underline{\hat{a}}_i$, $i = 4, 5, 10$. We also denote the Fréchet derivative of the map (function) F(x) by $DF(x)$ and the matrix norm associated with Euclidean norm in $\mathbf{R}^2$ by $\|\cdot\|$.

We can write $\underline{s}_i = DV\underline{s}_i^\star$, $\underline{s}_i^\star = DV^{-1}\underline{s}_i$. The conditions $(i) - (iii)$ guarantee that there exist a constant C such that

$$\|DV\| \leq Ch, \quad \|DV^{-1}\| \leq \frac{C}{h}$$

[7, p.120], (as usual the same leter C stands for various constants). We obtain

$$\|\underline{s}_i^\star\|_E \leq \|DV^{-1}\| \, \|\underline{s}_i\|_E \leq Ch \quad i = 4, 5, \tag{2}$$

because $\|\underline{s}_i\|_E = O(h^2)$ $i = 4, 5$. Then

$$(\underline{s}_4^\star)^2 = \left(\alpha_4 - \frac{2}{3}\right)^2 + \left(\beta_4 - \frac{1}{3}\right)^2 \leq Ch^2,$$

$$(\underline{s}_5^\star)^2 = \left(\alpha_5 - \frac{1}{3}\right)^2 + \left(\beta_5 - \frac{2}{3}\right)^2 \leq Ch^2.$$

Adding the last two inequalities we obtain

$$Ch^2 \geq 2\left(\frac{\alpha_4}{2} - \frac{1}{3}\right)^2 + 2\left(\frac{\beta_5}{2} - \frac{1}{3}\right)^2 + \frac{1}{2}\left(\beta_4 - \frac{1}{3}\right)^2 + \frac{1}{2}\left(\alpha_5 - \frac{1}{3}\right)^2 \geq 2(\underline{s}_{10}^\star)^2,$$

and therefore $\|\underline{s}_{10}^\star\|_E \leq Ch$. Finally $\|\underline{s}_{10}\|_E \leq \|DV\| . \|\underline{s}_{10}^\star\|_E \leq Ch^2$. We can write $\|\underline{a}_i - \underline{\widetilde{a}}_i\|_E = O(h^2)$ $i = 4, 5, ..., 10$, consequently applying Theorem 1 we obtain (1) which completes the proof. ■

Let $\varphi_{iT}(x)$, $i = 1, 2, ..., 10$ be the nodal basis functions of the finite element T. We make the next denotations:

$$J_T(\hat{x}) = det(DF_T), \; J_{V_T} = det(DV_T), \; J_{\mathcal{W}_T}(\hat{x}) = det(D\mathcal{W}_T).$$

We choose such a numeration of the vertices a_{iT} $i = 1, 2, 3$ of the element T, that the determinant $J_{V_T} > 0$.

We shall show how the energy scalar products $a_T(\varphi_{iT}, \varphi_{jT})$ i, j =1,2,...,10 can be computed by integration over the finite element of reference. We start by:

$$a_T(\varphi_{iT}, \varphi_{jT}) = \int_T D\varphi_{iT}(x) \cdot D\varphi_{jT}(x)\,dx$$
$$= \int_T D\left(\hat{\varphi}_i \circ F_T^{-1}\right)(x) \cdot D\left(\hat{\varphi}_j \circ F_T^{-1}\right)(x)\,dx.$$

Applying the chain rule, we obtain

$$a_T(\varphi_{iT}, \varphi_{jT}) = \int_T \left[D\hat{\varphi}_i(F_T^{-1}(x))\right]^t DF_T^{-1}(x)\left[DF_T^{-1}(x)\right]^t D\hat{\varphi}_j(F_T^{-1}(x))\,dx$$
$$= \int_{\hat{T}} [D\hat{\varphi}_i(\hat{x})]^t[DF_T(\hat{x})]^{-1}\left[[DF_T(\hat{x})]^{-1}\right]^t D\hat{\varphi}_j(\hat{x}) J_T(\hat{x})\,d\hat{x}.$$

Applying the chain rule once again, we write

$$a_T(\varphi_{iT}, \varphi_{jT}) = \int_{\hat{T}} (\nabla\hat{\varphi}_i)^t[DV_T(X)D\mathcal{W}_T(\hat{x})]^{-1}$$
$$\times\left[[DV_T(X)D\mathcal{W}_T(\hat{x})]^{-1}\right]^t \nabla\hat{\varphi}_j J_T(\hat{x})\,d\hat{x} = \int_{\hat{T}} (\nabla\hat{\varphi}_i)^t[D\mathcal{W}_T]^{-1}[DV_T]^{-1}$$
$$\times\left[[D\mathcal{W}_T]^{-1}[DV_T]^{-1}\right]^t \nabla\hat{\varphi}_j J_T(\hat{x})\,d\hat{x}.$$

We denote the adjoint matrices of the matrices DV_T, $D\mathcal{W}_T$ accordingly by B_{V_T}, $B_{\mathcal{W}_T}$. Then

$$a_T(\varphi_{iT}, \varphi_{jT}) = \int_{\hat{T}} (\nabla\hat{\varphi}_i)^t \frac{B_{\mathcal{W}_T} B_{V_T} [B_{\mathcal{W}_T} B_{V_T}]^t}{J_T(\hat{x})} \nabla\hat{\varphi}_j\,d\hat{x}$$
$$= \int_{\hat{T}} (\nabla\hat{\varphi}_i)^t \frac{B_{\mathcal{W}_T} B_{V_T} B^t_{V_T} B^t_{\mathcal{W}_T}}{J_T(\hat{x})} \nabla\hat{\varphi}_j\,d\hat{x}.$$

Lemma 1. *Let M^* be the adjoint matrix of an (2×2) matrix M and let $det(M) > 0$. Then we have*

(3) $$cond(M^*) = \frac{\|M^*\|^2}{det(M)},$$

(4) $$cond(M^*) = \|M^{-1}\|^2 det(M),$$

(5) $$\lambda_{\min}[M^*(M^*)^t] = \frac{(det(M))^2}{\|M^*\|^2}.$$

Proof. Since the matrix M is (2×2) we have $\|M^*\| = \|M\|$ and $det(M^*) = det(M)$. Then

$$cond(M^*) = \|M^*\|.\|(M^*)^{-1}\| = \frac{\|M^*\|.\|(M^*)^*\|}{det(M^*)} = \frac{\|M^*\|^2}{det(M)}.$$

Thus we proved (3).

To prove (4) we continue with

$$cond(M^*) = \frac{\|M^*\|^2}{det(M)} = \left\|\frac{M^*}{det(M)}\right\|^2 det(M) = \|M^{-1}\|^2 det(M).$$

We will prove (5). We represent the second degree of the $det(M)$ as a product of the eigenvalues of the matrix $M^*(M^*)^t$:

$$(det(M))^2 = (det(M^*))^2 = det(M^*(M^*)^t)$$

$$= \lambda_{\max}[M^*(M^*)^t]\lambda_{\min}[M^*(M^*)^t] = \|M^*\|^2\lambda_{\min}[M^*(M^*)^t],$$

then

$$\lambda_{\min}[M^*(M^*)^t] = \frac{(det(M))^2}{\|M^*\|^2}.$$

The proof is ended. ∎

We define the functions

$$\omega_1(\varepsilon) = 1 - \frac{9}{2}(\sqrt{10}+3)\varepsilon - \frac{243}{16}(2\sqrt{10}+9)\varepsilon^2,$$

$$\omega_2(\varepsilon) = 1 + \left(81 + 13.5\sqrt{10}\right)\varepsilon + \left(2146.5 + 516.375\sqrt{10}\right)\varepsilon^2$$

$$\sigma_{\mathcal{W}}(\varepsilon) = \frac{\omega_2(\varepsilon)}{\omega_1(\varepsilon)},\quad \sigma_T(a_{1T}, a_{2T}, a_{3T}, \varepsilon) = \sigma_{\mathcal{W}}(\varepsilon)cond(B_V),$$

$$\varepsilon \in [0,\overline{\varepsilon}),\quad \overline{\varepsilon} = \frac{4}{9(9+2\sqrt{10})}.$$

Definition. Let us assume that the matrices M_i $i = 1, 2$ have n rows and n columns. We will write $M_1 \leq M_2$ when the inequality $\underline{\xi}^T M_1 \underline{\xi} \leq \underline{\xi}^T M_2 \underline{\xi}$ holds $\forall \underline{\xi} \in \mathbf{R}^n$.

Theorem 3. *Let τ_h be triangulation which satisfies the conditions (i) – (iii) and the parameter h be so small that for all $T \in \tau_h$ we have*

$$\|\underline{s}_i^*\|_\infty \leq \varepsilon,\ \varepsilon \in [0,\overline{\varepsilon}),\ i = 4,5 \tag{6}$$

($\|\underline{x}\|_\infty = \max_{i=1,2} |x_i|$). Then for the element stiffness matrix A_T is valid the inequality

(7) $$\sigma_T^{-1} \hat{A} \leq A_T \leq \sigma_T \hat{A},$$

where the matrix $\hat{A}$ is the stiffness matrix for the finite element of reference.

Proof. We shall drop the index T throughout the proof as in Theorem 2. Let ε be a fixed number in the interval $[0, \bar{\varepsilon})$. We shall find upper and lower bounds for the positive definite matrix

$$Q = \frac{B_W B_V B_V^t B_W^t}{J(\hat{x})},$$

uniform with respect to $\hat{x}$.

Putting B_V instead of M^* in (5) we have the next result

$$\lambda_{\min}[B_V B_V^t] = \frac{J_V^2}{\|B_V\|^2}.$$

Replacing the eigenvalues of the product $B_V B_V^t$ in the inequality

$$\frac{\lambda_{\min}[B_V B_V^t]}{J_V J_W(\hat{x})} B_W B_W^t \leq Q \leq \frac{\lambda_{\max}[B_V B_V^t]}{J_V J_W(\hat{x})} B_W B_W^t,$$

we obtain

$$\left(\frac{\|B_V\|^2}{J_V J_W(\hat{x})}\right)^{-1} B_W B_W^t \leq Q \leq \frac{\|B_V\|^2}{J_V J_W(\hat{x})} B_W B_W^t.$$

Applying analogous reasonings for the product $B_W B_W^t$ we can write

$$\left(\frac{\|B_V\|^2 \|B_W\|^2}{J_V J_W(\hat{x})}\right)^{-1} I \leq Q \leq \frac{\|B_V\|^2 \|B_W\|^2}{J_V J_W(\hat{x})} I,$$

where I is the single matrix of order two. As a direct corollary of Lemma 1 we have

$$cond(B_V) = \frac{\|B_V\|^2}{J_V}, \quad cond(B_W(\hat{x})) = \frac{\|B_W(\hat{x})\|^2}{J_W(\hat{x})},$$

then we can write

(8) $$[cond(B_V) cond(B_W(\hat{x}))]^{-1} I \leq Q \leq cond(B_V) cond(B_W(\hat{x})) I.$$

Since $cond(B_V)$ is not dependent on $\hat{x}$ we search for uniform estimate with respect to $\hat{x}$ only for $cond(B_W(\hat{x}))$.

We begin with uniform lower bound for the Jacobian. It follows from (2), that there exists so small h_0, that $\forall h \leq h_0$ the inequality (6) is fulfilled. We obtain

$$\|\underline{s}_i^\star\|_E \leq \sqrt{2}\varepsilon,\ i = 4, 5,$$

$$\left(\alpha_4 - \frac{2}{3}\right)^2 + \left(\alpha_5 - \frac{1}{3}\right)^2 \leq 2\varepsilon^2,\ \left(\beta_4 - \frac{1}{3}\right) + \left(\beta_5 - \frac{2}{3}\right)^2 \leq 2\varepsilon^2$$

from (6). We calculate the Jacobian of the transformation $\mathcal{W}$:

$$J_{\mathcal{W}}(\hat{x}) = 1 + \psi_1(\beta_5, \beta_4)\hat{x}_1 + \psi_1(\alpha_4, \alpha_5)\hat{x}_2 + \psi_2(\beta_5, \beta_4)\hat{x}_1^2 + \psi_2(\alpha_4, \alpha_5)\hat{x}_2^2$$
$$-\psi_1(\alpha_4, \alpha_5)\psi_2(\beta_5, \beta_4)\hat{x}_1^2\hat{x}_2 - \psi_1(\beta_5, \beta_4)\psi_2(\alpha_4, \alpha_5)\hat{x}_1\hat{x}_2^2$$
$$-3\psi_2(\alpha_4, \alpha_5)\psi_2(\beta_5, \beta_4)\hat{x}_1^2\hat{x}_2^2.$$

We estimate

$$J_{\mathcal{W}}(\hat{x}) \geq 1 - \|\psi_1\|_{\infty,K}(\hat{x}_1 + \hat{x}_2) - \|\psi_2\|_{\infty,K}(\hat{x}_1^2 + \hat{x}_2^2)$$
$$-\|\psi_1\|_{\infty,K}\|\psi_2\|_{\infty,K}\hat{x}_1\hat{x}_2(\hat{x}_1 + \hat{x}_2) - 3\|\psi_2\|_{\infty,K}^2(\hat{x}_1\hat{x}_2)^2,$$

where K is the circle:

$$K\ :\ \left(x_1 - \frac{2}{3}\right)^2 + \left(x_2 - \frac{1}{3}\right)^2 \leq 2\varepsilon^2.$$

We consider the function

$$\nu(\hat{x}) = \|\psi_1\|_{\infty,K}(\hat{x}_1 + \hat{x}_2) + \|\psi_2\|_{\infty,K}(\hat{x}_1^2 + \hat{x}_2^2)$$
$$+\|\psi_1\|_{\infty,K}\|\psi_2\|_{\infty,K}\hat{x}_1\hat{x}_2(\hat{x}_1 + \hat{x}_2) + 3\|\psi_2\|_{\infty,K}^2(\hat{x}_1\hat{x}_2)^2,\ \hat{x} \in \hat{T}.$$

We compute

$$\|\nu\|_{\infty,\hat{T}} = \|\psi_1\|_{\infty,K} + \frac{1}{2}\|\psi_2\|_{\infty,K} + \frac{1}{4}\|\psi_1\|_{\infty,K}\|\psi_2\|_{\infty,K} + \frac{3}{16}\|\psi_2\|_{\infty,K}^2,$$

$$\|\psi_1\|_{\infty,K} = \frac{9\sqrt{10}}{2}\varepsilon,\ \|\psi_2\|_{\infty,K} = 27\varepsilon.$$

Then

$$J_{\mathcal{W}}(\hat{x}) \geq 1 - \|\nu\|_{\infty,\hat{T}} = \omega_1(\varepsilon).$$

We establish the validity of the inequality $\omega_1(\varepsilon) > 0,\ \varepsilon \in [0, \bar{\varepsilon})$ with direct verification.

We calculate the matrix $B_{\mathcal{W}}B_{\mathcal{W}}^t = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$:

$$b_{11} = \left(1 + \psi_1(\beta_5,\beta_4)\hat{x}_1 + \psi_2(\beta_5,\beta_4)\hat{x}_1^2\right)^2 + (\psi_1(\alpha_4,\alpha_5)\hat{x}_1 + 2\psi_2(\alpha_4,\alpha_5)\hat{x}_1\hat{x}_2)^2,$$

$$b_{12} = -\left[(1 + \psi_1(\beta_5,\beta_4)\hat{x}_1 + \psi_2(\beta_5,\beta_4)\hat{x}_1^2)(\psi_1(\beta_5,\beta_4)\hat{x}_2 + 2\psi_2(\beta_5,\beta_4)\hat{x}_1\hat{x}_2)\right.$$
$$\left. + (1 + \psi_1(\alpha_4,\alpha_5)\hat{x}_2 + \psi_2(\alpha_4,\alpha_5)\hat{x}_2^2)(\psi_1(\alpha_4,\alpha_5)\hat{x}_1 + 2\psi_2(\alpha_4,\alpha_5)\hat{x}_1\hat{x}_2)\right],$$

$$b_{22} = \left(1 + \psi_1(\alpha_4,\alpha_5)\hat{x}_2 + \psi_2(\alpha_4,\alpha_5)\hat{x}_2^2\right)^2 + (\psi_1(\beta_5,\beta_4)\hat{x}_2 + 2\psi_2(\beta_5,\beta_4)\hat{x}_1\hat{x}_2)^2.$$

Using the inequality

$$\|B_{\mathcal{W}}(\hat{x})\|^2 \le \frac{1}{2}(b_{11} + b_{22} + |b_{11} - b_{22}| + 2|b_{12}|), \tag{9}$$

we find uniform upper bound for $\|B_{\mathcal{W}}(\hat{x})\|^2$ with respect to $\hat{x}$. We estimate separately the addends in the right hand side of the inequality (9)

$$b_{11} + b_{22} \le 2 + 2\|\psi_1\|_{\infty,K}(\hat{x}_1 + \hat{x}_2) + 2(\|\psi_1\|_{\infty,K}^2 + \|\psi_2\|_{\infty,K})(\hat{x}_1^2 + \hat{x}_2^2)$$
$$+2\|\psi_1\|_{\infty,K}\|\psi_2\|_{\infty,K}(\hat{x}_1^3 + \hat{x}_2^3) + \|\psi_2\|_{\infty,K}^2(\hat{x}_1^4 + \hat{x}_2^4)$$
$$+4\|\psi_1\|_{\infty,K}\|\psi_2\|_{\infty,K}\hat{x}_1\hat{x}_2(\hat{x}_1 + \hat{x}_2) + 8\|\psi_2\|_{\infty,K}^2(\hat{x}_1\hat{x}_2)^2$$
$$\le 2 + 2(\|\psi_1\|_{\infty,K} + \|\psi_2\|_{\infty,K}) + 2\|\psi_1\|_{\infty,K}^2 + 3\|\psi_1\|_{\infty,K}\|\psi_2\|_{\infty,K} + \frac{3}{2}\|\psi_2\|_{\infty,K}^2,$$

$$|b_{11} - b_{22}| \le |\psi_1(\alpha_4,\alpha_5)\hat{x}_1 - \psi_1(\beta_5,\beta_4)\hat{x}_2 + 2(\psi_2(\alpha_4,\alpha_5) - \psi_2(\beta_5,\beta_4))\hat{x}_1\hat{x}_2| \times$$
$$|\psi_1(\alpha_4,\alpha_5)\hat{x}_1 + \psi_1(\beta_5,\beta_4)\hat{x}_2 + 2(\psi_2(\alpha_4,\alpha_5) + \psi_2(\beta_5,\beta_4))\hat{x}_1\hat{x}_2|$$
$$+\left|\psi_1(\beta_5,\beta_4)\hat{x}_1 - \psi_1(\alpha_4,\alpha_5)\hat{x}_2 + \psi_2(\beta_5,\beta_4)\hat{x}_1^2 - \psi_2(\alpha_4,\alpha_5)\hat{x}_2^2\right| \times$$
$$\left|2 + \psi_1(\beta_5,\beta_4)\hat{x}_1 + \psi_1(\alpha_4,\alpha_5)\hat{x}_2 + \psi_2(\beta_5,\beta_4)\hat{x}_1^2 + \psi_2(\alpha_4,\alpha_5)\hat{x}_2^2\right|.$$

The inequality

$$|b_{11} - b_{22}| \le \left(\|\psi_1\|_{\infty,K}(\hat{x}_1 + \hat{x}_2) + \|\psi_2\|_{\infty,K}(\hat{x}_1^2 + \hat{x}_2^2)\right) \times$$
$$\left(2 + \|\psi_1\|_{\infty,K}(\hat{x}_1 + \hat{x}_2) + \|\psi_2\|_{\infty,K}(\hat{x}_1^2 + \hat{x}_2^2)\right)$$
$$+\left(\|\psi_1\|_{\infty,K}(\hat{x}_1 + \hat{x}_2) + 4\|\psi_2\|_{\infty,K}\hat{x}_1\hat{x}_2\right)^2$$

is true because of

$$\|\psi_i\|_{\infty,K} = \left|\min_{x\in K}\psi_i(\hat{x})\right|, \quad i = 1,2.$$

Since $\hat{x}_1^i + \hat{x}_2^i \leq 1$ for $i = 1, 2, 3, \ldots$ we have

$$|b_{11} - b_{22}| \leq 2(\|\psi_1\|_{\infty,K} + \|\psi_2\|_{\infty,K})(1 + \|\psi_1\|_{\infty,K} + \|\psi_2\|_{\infty,K}).$$

For the last term in the right hand side of (9) we obtain

$$|b_{12}| \leq \|\psi_1\|_{\infty,K}(\hat{x}_1 + \hat{x}_2) + \Big[2\|\psi_1\|^2_{\infty,K} + 4\|\psi_2\|_{\infty,K}$$

$$+3\|\psi_1\|_{\infty,K}\|\psi_2\|_{\infty,K}(\hat{x}_1 + \hat{x}_2) + 2\|\psi_2\|^2_{\infty,K}(\hat{x}_1^2 + \hat{x}_2^2)\Big]\, \hat{x}_1\hat{x}_2$$

$$\leq \|\psi_1\|_{\infty,K} + \|\psi_2\|_{\infty,K} + \frac{3}{4}\|\psi_1\|_{\infty,K}\|\psi_2\|_{\infty,K} + \frac{1}{2}(\|\psi_1\|^2_{\infty,K} + \|\psi_2\|^2_{\infty,K}).$$

We estimate

$$\|B_{\mathcal{W}}(\hat{x})\|^2 \leq 1 + 3(\|\psi_1\|_{\infty,K} + \|\psi_2\|_{\infty,K})$$

$$+\frac{10\|\psi_1\|^2_{\infty,K} + 17\|\psi_1\|_{\infty,K}\|\psi_2\|_{\infty,K} + 9\|\psi_2\|^2_{\infty,K}}{4} = \omega_2(\varepsilon).$$

The inequality

(10) $$\sigma_T^{-1} I \leq Q \leq \sigma_T I$$

follows from the inequality $cond(B_{\mathcal{W}}(\hat{x})) \leq \sigma_{\mathcal{W}}(\varepsilon)$ and (8). Now we can estimate the matrix A_T

$$\underline{\xi}^t A_T \underline{\xi} = \underline{\xi}^t \left\| \int_{\hat{T}} (\nabla\hat{\varphi}_i)^t Q \nabla\hat{\varphi}_j \, d\hat{x} \right\|_{i,j=1,2,\ldots,10} \underline{\xi} = \int_{\hat{T}} \sum_{i,j=1}^{10} \xi_i \left((\nabla\hat{\varphi}_i)^t Q \nabla\hat{\varphi}_j\right) \xi_j \, d\hat{x}$$

$$= \int_{\hat{T}} \left(\sum_{i=1}^{10} \xi_i \nabla\hat{\varphi}_i\right)^t Q \left(\sum_{j=1}^{10} \xi_j \nabla\hat{\varphi}_j\right) d\hat{x}, \quad \forall \underline{\xi} \in \mathbf{R}^{10}.$$

The inequalities

$$\underline{\xi}^t A_T \underline{\xi} \leq \sigma_T \int_{\hat{T}} \left(\sum_{i=1}^{10} \xi_i \nabla\hat{\varphi}_i\right)^t I \left(\sum_{j=1}^{10} \xi_j \nabla\hat{\varphi}_j\right) d\hat{x} = \sigma_T \underline{\xi}^t \hat{A} \underline{\xi},$$

$$\sigma_T^{-1} \underline{\xi}^t \hat{A} \underline{\xi} \leq \underline{\xi}^t A_T \underline{\xi},$$

follow from (10). The last results mean that (7) is fulfilled. ■

We make hierarchical refinement for the finite element of reference (Fig. 3). We obtain four curved elements T_i $i = 1, 2, 3, 4$ (Fig. 4) after refinement of an arbitrary curved element $T \in \tau_h$. If we need continue the refinement process

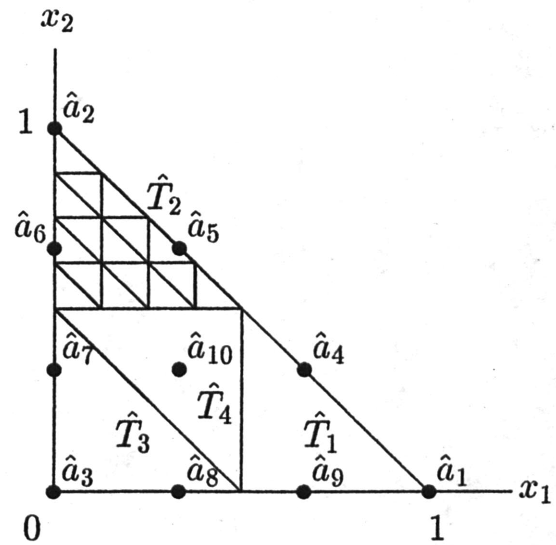

Figure 3: Hierarchical refinement of the finite element of reference. Local refinement of the element $\hat{T}_2$.

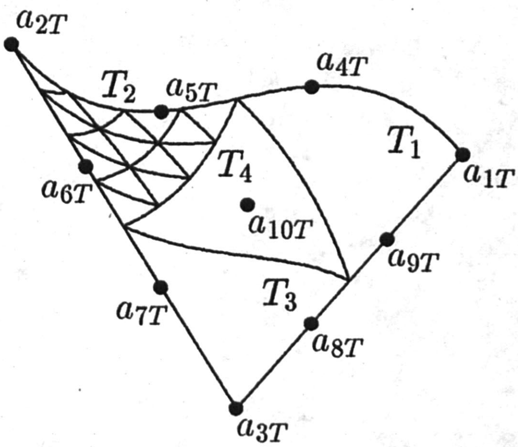

Figure 4: Hierarchical refinement of the finite element T. Local refinement of the element T_2.

we have to refine curved elements with more than one curved side. It does not lead to difficulties since for the refinement of the curved element T_2 for example, we need only transformation F_T and local refinement of the element $\hat{T}_2$.

We denote the restrictions of the spaces $\mathbf{V}_h$, $\widetilde{\mathbf{V}}_{h_1}$, $\mathbf{V}_{h_1}$ over the element T respectively by $\mathbf{V}_{h,T}$, $\widetilde{\mathbf{V}}_{h_1,T}$, $\mathbf{V}_{h_1,T}$. We write the so-called two-level hierarchical basis element stiffness matrix

$$\mathcal{A}_T = \begin{pmatrix} \mathcal{A}_{T;11} & \mathcal{A}_{T;12} \\ \mathcal{A}_{T;21} & \mathcal{A}_{T;22} \end{pmatrix}, \quad \forall T \in \tau_h.$$

We consider the generalized eigenvalue problem

$$\lambda A_T \underline{\xi} = S_T \underline{\xi}, \tag{11}$$

over $T \in \tau_h$, where

$$S_T = \mathcal{A}_{T;22} - \mathcal{A}_{T;21}\mathcal{A}_{T;11}^{-1}\mathcal{A}_{T;12}$$

is the element Schur complement. The quantity $\lambda_{T,\min}$ is the smallest solution for the problem (11).

The next theorem states that the strengthened Cauchy - Buniakowskii - Schwarz inequality is valid over curved domains Ω uniformly with respect to h, when the corresponding triangulations τ_h satisfies some conditions.

Theorem 4. *Let the conditions of Theorem 3 hold. Then there exists a constant $\gamma \in [0,1)$ depending only on the geometry of the initial triangulation τ_h, such that*

$$|a(v,w)| \le \gamma\sqrt{a(v,v)}\sqrt{a(w,w)}$$

for all $v \in \mathbf{V}_h$ and $w \in \widetilde{\mathbf{V}}_{h_1}$.

Proof. First, we shall prove that σ_T is independent on h. The functions ω_i $i = 1,2$ depend only on ε, hence it is necessery merely to prove that $cond(B_V)$ is independent on h. Putting B_V instead of M^* in (4) and using

$$\|DV_T^{-1}\|^2 = O(h_T^{-2}) \text{ and } J_{V_T} = O(h_T^2),$$

we have

$$cond(B_{V_T}) = \|DV_T^{-1}\|^2 J_{V_T} = O(1).$$

Consequently σ_T is independent on h.

Since the spaces $\mathbf{V}_{h,T}$, $\widetilde{\mathbf{V}}_{h_1,T}$ are finite dimensional and $\mathbf{V}_{h,T} \bigcap \widetilde{\mathbf{V}}_{h_1,T} = \{0\}$ there exists a constant

$$\gamma_T = \gamma_T(\mathbf{V}_{h,T}, \widetilde{\mathbf{V}}_{h_1,T}) \in [0,1)$$

such that

$$|a_T(v,w)| \le \gamma_T(\mathbf{V}_{h,T}, \widetilde{\mathbf{V}}_{h_1,T})\sqrt{a_T(v,v)}\sqrt{a_T(w,w)}, \quad \forall v \in \mathbf{V}_h,\ \forall w \in \widetilde{\mathbf{V}}_{h_1},$$

(see [9]).

We shall find upper bound for γ_T estimating the eigenvalue $\lambda_{T,\min}$ by $\hat{\lambda}_{\max} = \lambda_{\max}[\hat{A}]$. Since $S_T \le A_T$ [9] and $A_T \le \sigma_T \hat{A}$, we have

$$\lambda_{T,\min} = \lambda_{\min}[A_T^{-1} S_T] \ge \lambda_{\min}[A_T^{-2}]$$

$$\ge \left(\lambda_{\max}[A_T^2]\right)^{-1} \ge (\lambda_{\max}[A_T])^{-2} \ge \left(\sigma_T \hat{\lambda}_{\max}\right)^{-2}.$$

Then $\gamma_T \le \sqrt{1 - \left(\sigma_T \hat{\lambda}_{\max}\right)^{-2}}$.

Further we prove the global strengthened C. B. S. inequality

$$|a(v,w)| \le \sum_{T\in\tau_h} |a_T(v,w)| \le \sum_{T\in\tau_h} \gamma_T \sqrt{a_T(v,v)}\sqrt{a_T(w,w)}$$

$$\le \sum_{T\in\tau_h} \sqrt{1 - \hat{\lambda}_{\min}\sigma_T^{-1}}\sqrt{a_T(v,v)}\sqrt{a_T(w,w)}$$

$$\le \sqrt{1 - \hat{\lambda}_{\min}\sigma^{-1}} \sum_{T\in\tau_h} \sqrt{a_T(v,v)}\sqrt{a_T(w,w)},$$

where $\sigma = \max_{T\in\tau_h} \sigma_T$. We put

$$\gamma = \sqrt{1 - \left(\hat{\lambda}_{\max}\sigma\right)^{-2}}$$

and we obtain

$$|a(v,w)| \le \gamma \left(\sum_{T\in\tau_h} a_T(v,v)\right)^{\frac{1}{2}} \left(\sum_{T\in\tau_h} a_T(w,w)\right)^{\frac{1}{2}} \le \gamma\sqrt{a(v,v)}\sqrt{a(w,w)}.$$

The proof is completed. ■

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