Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Strengthened Cauchy Inequality for Bilinear Forms over Curved Domains. Cubic Case

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Presented by Bl. Sendov

The strengthened Cauchy - Buniakowskii - Schwarz inequality for elliptic bilinear forms over curved domains and 10-node 2-simplex elements is considered. It is proven that the inequality holds uniformly with respect to the finite element spaces. The results have applications in multilevel method for solving elliptic boundary-value problems. Upper bound for contraction number is found.

AMS Subj. Classification: 35J25
Key Words: Cauchy - Buniakowski - Schwarz inequality, elliptic bilinear forms, finite element spaces, elliptic boundary-value problems

1. Introduction

Let $U$, $V$ be two linear finite-dimensional spaces, $U \cap V = \{0\}$ and let there exists a constant $\gamma \in [0, 1]$ depending only on the spaces $U$ and $V$, but not dependent on the choice of the elements $u \in U$, and $v \in V$, such that

$$|(u, v)| \leq \gamma \sqrt{(u, u)} \sqrt{(v, v)}.$$  

The last inequality is the so-called strengthened Cauchy-Buniakowskii-Schwarz (C.B.S.) inequality. Among authors who have used the strengthened C.B.S. inequality in two-level method we mention Bank and Dupon [3], Braess [4,5], Maitre and Musy [10], Axelsson [1], and Axelsson and Gustafsson [2]. The inequality has been used in connection with the two-grid FAC-preconditioner by McCormick [12], McCormick and Thomas [13]. The C.B.S. inequality is applied in works of Bramble et al. [6] and Mandel and McCormick [11]. The role of the C.B.S. inequality in multilevel methods is considered in detail by Eijkhout and Vassilevski [9]. Computation of constants in the strengthened C.B.S. inequality we can find in [14]. Our goal is to study the behaviour of the
constant in the strengthened C.B.S. inequality for a class of 10-node curvilinear
triangle finite elements.

Let $H^1(\Omega)$ be the usual Sobolev’s space.

We consider an elliptic bilinear form $a(\cdot, \cdot)$:
\[
a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad u, v \in H^1(\Omega)
\]
with $u, v = 0$ on $\Gamma_D \subset \Gamma = \partial \Omega$, $\text{meas}(\Gamma_D) \neq 0$. The set $\Omega$ is open subset of
$\mathbb{R}^2$, with Lipschitz - continuous boundary. We assume that $\Gamma$ is piecewise $(P_3)^2$, where $P_k$ is the space of all polynomials of degree, not exceeding k-th.

We denote the point $(x_1, x_2)$ by $x$, and the vector with the same coordinates by $\underline{x}$.

Let $(\hat{T}, \hat{P}, \hat{\Sigma})$ be the 10-node 2-simplex finite element of reference (Fig. 1) defined as follows:
\[
\hat{T} = \{ (\hat{x}_1, \hat{x}_2) \mid \hat{x}_1 \geq 0, \hat{x}_2 \geq 0, \hat{x}_1 + \hat{x}_2 \leq 1 \} \text{ is the unit 2-simplex;}
\hat{P} = \hat{P}_3, \text{ where } \hat{P}_k \text{ is the space of all polynomials of degree, not exceeding}
\]
k-th on $\hat{T}$;
\[
\hat{\Sigma} = \{ (\hat{x}_1, \hat{x}_2) \mid \hat{x}_1 = \frac{i}{3}, \hat{x}_2 = \frac{j}{3}; i, j \leq 3; i, j \in \{0, 1, 2, 3\} \} \text{ is the set}
\]
of all Lagrangian interpolation nodes.

Let $a_{iT}(a_{1iT}, a_{2iT}), \; i = 1, 2, ..., 10$ be the nodes of the element $T$, $A_{F_T}$ be the matrix
\[
A_{F_T} = \begin{pmatrix}
a_{11T} & a_{12T} & a_{13T} & a_{14T} & a_{15T} & a_{16T} & a_{17T} & a_{18T} & a_{19T} & a_{110T} \\
a_{21T} & a_{22T} & a_{23T} & a_{24T} & a_{25T} & a_{26T} & a_{27T} & a_{28T} & a_{29T} & a_{210T}
\end{pmatrix},
\]
and
\[
\Phi(\hat{x}) = \left( \frac{1}{2} \hat{x}_1 (3\hat{x}_1 - 1)(3\hat{x}_1 - 2), \frac{1}{2} \hat{x}_2 (3\hat{x}_2 - 1)(3\hat{x}_2 - 2), \right. \\
\left. \frac{1}{2} \hat{x}_3 (3\hat{x}_3 - 1)(3\hat{x}_3 - 2), \frac{9}{2} \hat{x}_1 \hat{x}_2 (3\hat{x}_1 - 1), \frac{9}{2} \hat{x}_1 \hat{x}_2 (3\hat{x}_2 - 1), \frac{9}{2} \hat{x}_2 \hat{x}_3 (3\hat{x}_2 - 1), \\
\frac{9}{2} \hat{x}_2 \hat{x}_3 (3\hat{x}_3 - 1), \frac{9}{2} \hat{x}_1 \hat{x}_3 (3\hat{x}_1 - 1), \frac{9}{2} \hat{x}_1 \hat{x}_3 (3\hat{x}_1 - 1), 27 \hat{x}_1 \hat{x}_2 \hat{x}_3 \right) ^t,
\]
\[\hat{x}_3 = 1 - \hat{x}_1 - \hat{x}_2\] be the vector whose coordinates are the nodal basis functions of the element \( \hat{T} \), then we can write the cubic transformation:
\[
F_T = A_{F_T} \Phi(\hat{x}).
\]

An arbitrary 10-node 2-simplex element \((T, P_T, \Sigma_T)\) is defined by \(T = F_T(\hat{T})\), where \(F_T\) is invertible transformation.

Let \(\tau_h\) be an initial triangulation of the set \(\Omega\) by 10-node 2-simplex elements. Since the boundary is piecewise \((P_3)^2\) we can write \(\Omega = \bigcup_{T \in \tau_h} T\). We consider a family of finite-element spaces \((V_h)\):
\[
V_h = \{v_h \in H_0^1(\Omega) \mid v_{h|T} = p(x) : p = \hat{p} \circ F_T^{-1}, \hat{p} \in \hat{P}, T \in \tau_h\},
\]
where it is understood, that the parameter \(h\) is the defining parameter of the family and has limit zero.

We make hierarchical refinement of \(\tau_h\), dividing each element to four finite elements of the same class as shown in Fig. 2. Thus we obtain triangulation \(\tau_{h_1}\) of the domain \(\Omega\). The space \(V_{h_1}\) is finite element space associated with \(\tau_{h_1}\). We denote the set of the nodes of the triangulations \(\tau_h, \tau_{h_1}\) accordingly by \(N_h, N_{h_1}\). Let \(\{\varphi_i^{(1)}\}\) be the nodal basis in \(V_{h_1}\) associated with the set \(N_{h_1}\), excluding Dirichlet boundary points. We define the hierarchical space
\[
\tilde{V}_{h_1} = Span\{\varphi_i^{(1)}\}_{i : a_i \in N_{h_1} \setminus N_h}
\]
in addition to \(V_h \subset V_{h_1}\). As it is well-known [1, 9, 10] for polygonal domains holds the restricted strengthened C.B.S. inequality
\[
|a_T(v, w)| \leq \gamma_T \sqrt{a_T(v, v)} \sqrt{a_T(w, w)}, \quad \forall v \in V_h, \forall w \in \tilde{V}_{h_1},
\]
where
\[
a_T(u, v) = \int_T \nabla u \cdot \nabla v dx,
\]
is the restricted bilinear form.
2. Energy inequalities

We use not only straight elements but also isoparametric elements for getting an exact approximation of the boundary $\Gamma$. We represent the transformation $F_T$ as a product of two transformations - $F_T = V_T \circ W_T$.

We define the transformation $W_T : \hat{T} \rightarrow T \subset \mathbb{R}^2$ by:

$$W_T(\hat{x}) = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = A_{W_T} \Phi(\hat{x})$$,

and the transformation $V_T : T \rightarrow T$ by:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = V_T(X) = \begin{pmatrix} a_{13T} \\ a_{23T} \end{pmatrix} + \begin{pmatrix} a_{113}^T \\ a_{213}^T \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

where we use denotations $a_{kj}^T = a_{kiT} - a_{kjT}$, $i, j, k \in \{1, 2, 3\}$.

The image $T$ of the finite element of reference $\hat{T}$ by transformation $V_T \circ W_T$ represents element with only one curved side. The applicability of 10-node 2-simplex elements and the quality of approximations by such elements depend
on the choice of the node $a_{10T}$. We determine the node $a_{10T}$ by $\alpha_{10T} = \frac{\alpha_{4T}}{2}$, $\beta_{10T} = \frac{\beta_{3T}}{2}$. Thus we obtain transformation $\mathcal{W}_T : \hat{T} \rightarrow T$ which is special case of the transformation $W_T$. The transformation $\mathcal{W}_T(\hat{x})$ can be described by:

\[
X_1(\hat{x}) = \hat{x}_1 + \psi_1(\alpha_{4T}, \alpha_{5T}) \hat{x}_1 \hat{x}_2 + \psi_2(\alpha_{4T}, \alpha_{5T}) \hat{x}_1^2 \hat{x}_2^2,
\]
\[
X_2(\hat{x}) = \hat{x}_2 + \psi_1(\beta_{5T}, \beta_{4T}) \hat{x}_1 \hat{x}_2 + \psi_2(\beta_{5T}, \beta_{4T}) \hat{x}_1^2 \hat{x}_2^2,
\]
\[
\psi_1(x) = \frac{9}{2} (-1 + 2x_1 - x_2), \quad \psi_2(x) = \frac{9}{2} (1 - 3x_1 + 3x_2).
\]

We make the next denotations: $\tilde{T} = V_T(\hat{T})$, $a_{iT}^T = \mathcal{W}_T(\hat{a}_i)$, $\tilde{a}_{iT} = V_T(\hat{a}_i)$, $i = 1, 2, ..., 10$, $h_T = \text{diam}(\tilde{T})$, $\rho_T = \text{diam}(\text{inscribed spher of } \tilde{T})$. We represent the triangulation $\tau_h$ in view of:

\[
\tau_h = \{ T = F_T(\hat{T}) \mid F_T = V_T \circ \mathcal{W}_T, \text{diam}(\tilde{T}) < h \}.
\]

We have an isoparametric family ($T \in \tau_h, P_T, \Sigma_T$) of 10-node 2-simplex elements. Further, we consider only triangulations $\tau_h$ which satisfy:

(i) If the element $T \in \tau_h$ have less than two vertices over $\Gamma$ then this element is a straight element;

(ii) There exists constant $\mu$ such that $\forall T \in \tau_h$, $\frac{h_T}{\rho_T} \leq \mu$;

(iii) For all curved element $T \in \tau_h$ holds $\|a_{iT} - \tilde{a}_{iT}\|_E = O(h_T^2)$, $i = 4, 5$, where $\| \cdot \|_E$ is Euclidean norm in $\mathbb{R}^2$.

We will analyze, how the choice of the node $a_{10T}$ influences over quality of interpolation by 10-node 2-simplex elements.

We define the interpolant $\Pi$ on $H^3(T)$, $T \in \tau_h$ by

\[
\Pi v = \sum_{i=1}^{10} v(a_{iT}) \varphi_i(T)(\hat{x}).
\]

**Theorem 1.** Let $\|a_{iT} - \tilde{a}_{iT}\|_E = O(h^2)$, $i = 4, 5, ..., 10$, $T \in \tau_h$ and $\hat{T}_2 \subset \hat{T}$ then we have $|v - \Pi v|_{m,T} = O(h^{3-m})$, $\forall v \in H^3(T)$ and $m = 0, 1, 2$.

Theorem 1 is a special case of the fundamental result by Ciarlet and Raviart [8].

**Theorem 2.** Let the triangulation $\tau_h$ fulfills the conditions (i) – (iii), then we have

\[
|v - \Pi v|_{m,T} = O(h^{3-m})
\]

$\forall v \in H^3(T)$, $T \in \tau_h$ and $m = 0, 1, 2$. 
Proof. For notational convenience, we shall drop the index $T$ throughout the proof. Let $T \in \tau_h$ be a straight element. As the node $a_{i0}$ is barycenter of the element $T$, then (1) follows directly from Theorem 3.1.6 [7, p.124].

Let $T \in \tau_h$ be a curved element. The condition (iii) imposed over triangulation $\tau_h$ provides

$$\|\tilde{a}_i - \tilde{a}_i\|_E = O(h^2) \quad i = 4, 5.$$  

We will prove that $\|\tilde{a}_{i0} - \tilde{a}_{i0}\|_E = O(h^2)$.

We denote $\tilde{a}_i = a_i - \tilde{a}_i$ and $\tilde{a}_i = a_i - \tilde{a}_i$, $i = 4, 5, 10$. We also denote the Fréchet derivative of the map (function) $F(x)$ by $DF(x)$ and the matrix norm associated with Euclidean norm in $\mathbb{R}^2$ by $\|\cdot\|$.

We can write $\tilde{a}_i = DV \tilde{a}_i^*, \tilde{a}_i^* =DV^{-1} \tilde{a}_i$. The conditions (i) – (iii) guarantee that there exist a constant $C$ such that

$$\|DV\| \leq Ch, \quad \|DV^{-1}\| \leq \frac{C}{h}$$  

[7, p.120], (as usual the same letter $C$ stands for various constants). We obtain

(2)  

$$\|\tilde{a}_i^*\|_E \leq \|DV^{-1}\| \|\tilde{a}_i\|_E \leq Ch \quad i = 4, 5,$$

because $\|\tilde{a}_i\|_E = O(h^2)$ $i = 4, 5$. Then

$$(\tilde{a}_i^*)^2 = \left(\alpha_4 - \frac{2}{3}\right)^2 + \left(\beta_4 - \frac{1}{3}\right)^2 \leq Ch^2,$$

$$(\tilde{a}_5^*)^2 = \left(\alpha_5 - \frac{1}{3}\right)^2 + \left(\beta_5 - \frac{2}{3}\right)^2 \leq Ch^2.$$  

Adding the last two inequalities we obtain

$$Ch^2 \geq 2 \left(\frac{\alpha_4}{2} - \frac{1}{3}\right)^2 + 2 \left(\frac{\beta_5}{2} - \frac{1}{3}\right)^2 + \frac{1}{2} \left(\beta_4 - \frac{1}{3}\right)^2 + \frac{1}{2} \left(\alpha_5 - \frac{1}{3}\right)^2 \geq 2(\tilde{a}_{i0}^*)^2,$$

and therefore $\|\tilde{a}_{i0}^*\|_E \leq Ch$. Finally $\|\tilde{a}_{i0}\|_E \leq \|DV\| \|\tilde{a}_{i0}^*\|_E \leq Ch^2$. We can write $\|\tilde{a}_i - \tilde{a}_i\|_E = O(h^2)$ $i = 4, 5, ..., 10$, consequently applying Theorem 1 we obtain (1) which completes the proof.

Let $\varphi_{iT}(x), \ i = 1, 2, ..., 10$ be the nodal basis functions of the finite element $T$. We make the next denotations:

$$J_T(\hat{x}) = det(DF_T), \ J_{VT} = det(DV_T), \ J_{WT}(\hat{x}) = det(DW_T).$$  

We choose such a numeration of the vertices $a_{iT} \ i = 1, 2, 3$ of the element $T$, that the determinant $J_{VT} > 0$. 

We shall show how the energy scalar products $a_T(\varphi_{iT}, \varphi_{jT})$ $i, j = 1,2, ..., 10$ can be computed by integration over the finite element of reference. We start by:

$$a_T(\varphi_{iT}, \varphi_{jT}) = \int_T D\varphi_{iT}(x) \cdot D\varphi_{jT}(x) \, dx$$

$$= \int_T (D(\hat{\varphi}_i \circ F_T^{-1})(x) \cdot D(\hat{\varphi}_j \circ F_T^{-1})(x)) \, dx.$$  

Applying the chain rule, we obtain

$$a_T(\varphi_{iT}, \varphi_{jT}) = \int_T [D\hat{\varphi}_i(F_T^{-1}(x))]^t D F_T^{-1}(x) [D F_T^{-1}(x)]^t D\hat{\varphi}_j(F_T^{-1}(x)) \, dx$$

$$= \int_T [D\hat{\varphi}_i(\hat{x})]^t [D F_T(\hat{x})]^{-1} [D F_T(\hat{x})]^{-1} D\hat{\varphi}_j(\hat{x}) J_T(\hat{x}) \, d\hat{x}.$$  

Applying the chain rule once again, we write

$$a_T(\varphi_{iT}, \varphi_{jT}) = \int_T (\nabla \hat{\varphi}_i)^t [DV_T(X) DW_T(\hat{x})]^{-1}$$

$$\times [DV_T(X) DW_T(\hat{x})]^{-1} \nabla \hat{\varphi}_j J_T(\hat{x}) \, d\hat{x} = \int_T (\nabla \hat{\varphi}_i)^t [DW_T]^{-1} [DV_T]^{-1}$$

$$\times [DW_T]^{-1} [DV_T]^{-1} \nabla \hat{\varphi}_j J_T(\hat{x}) \, d\hat{x}.$$  

We denote the adjoint matrices of the matrices $DV_T$, $DW_T$ accordingly by $B_{VT}$, $B_{WT}$. Then

$$a_T(\varphi_{iT}, \varphi_{jT}) = \int_T (\nabla \hat{\varphi}_i)^t \frac{B_{WT} B_{VT} [B_{VT} B_{VT}]^t}{J_T(\hat{x})} \nabla \hat{\varphi}_j \, d\hat{x}$$

$$= \int_T (\nabla \hat{\varphi}_i)^t \frac{B_{WT} B_{VT} B_{WT}^t B_{VT}^t}{J_T(\hat{x})} \nabla \hat{\varphi}_j \, d\hat{x}.$$  

**Lemma 1.** Let $M^*$ be the adjoint matrix of an $(2 \times 2)$ matrix $M$ and let $\text{det}(M) > 0$. Then we have

\begin{align*}
(3) & \quad \text{cond}(M^*) = \frac{\|M^*\|^2}{\text{det}(M)}, \\
(4) & \quad \text{cond}(M^*) = \|M^{-1}\|^2 \text{det}(M), \\
(5) & \quad \lambda_{\min}[M^*(M^*)^t] = \left(\frac{\text{det}(M))}{\|M^*\|^2}\right)^2.
\end{align*}
Proof. Since the matrix $M$ is $(2 \times 2)$ we have $\|M^*\| = \|M\|$ and $\det(M^*) = \det(M)$. Then

$$\text{cond}(M^*) = \frac{\|M^*\| \cdot \|(M^*)^{-1}\|}{\det(M^*)} = \frac{\|M^*\| \cdot \|(M^*)^T\|}{\det(M^*)} = \frac{\|M^*\|^2}{\det(M)}.$$ 

Thus we proved (3).

To prove (4) we continue with

$$\text{cond}(M^*) = \frac{\|M^*\|^2}{\det(M)} = \left(\frac{M^*}{\det(M)}\right)^2 \det(M) = \|M^{-1}\|^2 \det(M).$$

We will prove (5). We represent the second degree of the $\det(M)$ as a product of the eigenvalues of the matrix $M^*(M^*)^t$:

$$(\det(M))^2 = (\det(M^*))^2 = \det(M^*(M^*)^t)$$

$$= \lambda_{\max}[M^*(M^*)^t] \lambda_{\min}[M^*(M^*)^t] = \|M^*\|^2 \lambda_{\min}[M^*(M^*)^t],$$

then

$$\lambda_{\min}[M^*(M^*)^t] = \frac{(\det(M))^2}{\|M^*\|^2}.$$ 

The proof is ended. 

We define the functions

$$\omega_1(\varepsilon) = 1 - \frac{9}{2}(\sqrt{10} + 3)\varepsilon - \frac{243}{16}(2\sqrt{10} + 9)\varepsilon^2,$$

$$\omega_2(\varepsilon) = 1 + \left(81 + 13.5\sqrt{10}\right)\varepsilon + \left(2146.5 + 516.375 \sqrt{10}\right)\varepsilon^2$$

$$\sigma_W(\varepsilon) = \frac{\omega_2(\varepsilon)}{\omega_1(\varepsilon)}, \quad \sigma_T(a_{1T}, a_{2T}, a_{3T}, \varepsilon) = \sigma_W(\varepsilon) \text{cond}(B_V),$$

$$\varepsilon \in [0, \bar{\varepsilon}), \quad \bar{\varepsilon} = \frac{4}{9(9 + 2\sqrt{10})}.$$ 

Definition. Let us assume that the matrices $M_i$ $i = 1, 2$ have $n$ rows and $n$ columns. We will write $M_1 \leq M_2$ when the inequality $\xi^T M_1 \xi \leq \xi^T M_2 \xi$ holds $\forall \xi \in \mathbb{R}^n$.

Theorem 3. Let $\tau_h$ be triangulation which satisfies the conditions (i) – (iii) and the parameter $h$ be so small that for all $T \in \tau_h$ we have

$$\|\xi_T^i\|_{\infty} \leq \varepsilon, \varepsilon \in [0, \bar{\varepsilon}), \quad i = 4, 5$$
\begin{align*}
\|\xi\|_\infty &= \max_{i=1,2} |x_i|.
\text{Then for the element stiffness matrix } A_T \text{ is valid the inequality}
(7) \quad \sigma_T^{-1} \hat{A} \leq A_T \leq \sigma_T \hat{A},
\text{where the matrix } \hat{A} \text{ is the stiffness matrix for the finite element of reference.}
\end{align*}

\textbf{Proof.} We shall drop the index T throughout the proof as in Theorem 2. Let } \varepsilon \text{ be a fixed number in the interval } [0, \varepsilon). \text{ We shall find upper and lower bounds for the positive definite matrix}

\begin{align*}
Q &= \frac{B_W B_V B_V^t B_W^t}{J(\hat{x})},
\end{align*}

uniform with respect to } \hat{x}.

Putting } B_V \text{ instead of } M^* \text{ in (5) we have the next result}

\begin{align*}
\lambda_{\min}[B_V B_V^t] &= \frac{J_V^{\hat{x}}}{\|B_V\|^2}.
\end{align*}

Replacing the eigenvalues of the product } B_V B_V^t \text{ in the inequality}

\begin{align*}
\frac{\lambda_{\min}[B_V B_V^t]}{J_V J_W(\hat{x})} B_W B_V^t \leq Q \leq \frac{\lambda_{\max}[B_V B_V^t]}{J_V J_W(\hat{x})} B_W B_V^t,
\end{align*}

we obtain

\begin{align*}
\left(\frac{\|B_V\|^2}{J_V J_W(\hat{x})}\right)^{-1} B_W B_V^t \leq Q \leq \left(\frac{\|B_V\|^2}{J_V J_W(\hat{x})}\right) B_W B_V^t.
\end{align*}

Applying analogous reasonings for the product } B_W B_W^t \text{ we can write

\begin{align*}
\left(\frac{\|B_V\|^2 \|B_W\|^2}{J_V J_W(\hat{x})}\right)^{-1} I \leq Q \leq \left(\frac{\|B_V\|^2 \|B_W\|^2}{J_V J_W(\hat{x})}\right) I,
\end{align*}

where } I \text{ is the single matrix of order two. As a direct corollary of Lemma 1 we have}

\begin{align*}
\text{cond}(B_V) &= \frac{\|B_V\|^2}{J_V}, \quad \text{cond}(B_W(\hat{x})) = \frac{\|B_W(\hat{x})\|^2}{J_W(\hat{x})},
\end{align*}

then we can write

\begin{align*}
(8) \quad \left[\text{cond}(B_V)\text{cond}(B_W(\hat{x}))\right]^{-1} I \leq Q \leq \text{cond}(B_V)\text{cond}(B_W(\hat{x})) I.
\end{align*}

Since } \text{cond}(B_V) \text{ is not dependent on } \hat{x} \text{ we search for uniform estimate with respect to } \hat{x} \text{ only for } \text{cond}(B_W(\hat{x})).
We begin with uniform lower bound for the Jacobian. It follows from (2), that there exists so small $h_0$, that $\forall h \leq h_0$ the inequality (6) is fulfilled. We obtain
\[
\|\xi_i^2\|_{E} \leq \sqrt{2} \varepsilon, \quad i = 4, 5,
\]
\[
\left(\alpha_4 - \frac{2}{3}\right)^2 + \left(\alpha_5 - \frac{1}{3}\right)^2 \leq 2\varepsilon^2, \quad \left(\beta_4 - \frac{1}{3}\right) + \left(\beta_5 - \frac{2}{3}\right)^2 \leq 2\varepsilon^2
\]
from (6). We calculate the Jacobian of the transformation $\mathcal{W}$:
\[
J_{\mathcal{W}}(\hat{x}) = 1 + \psi_1(\beta_5, \beta_4)\hat{x}_1 + \psi_1(\alpha_4, \alpha_5)\hat{x}_2 + \psi_2(\beta_5, \beta_4)\hat{x}_1^2 + \psi_2(\alpha_4, \alpha_5)\hat{x}_2^2
\]
\[
-\psi_1(\alpha_4, \alpha_5)\psi_2(\beta_5, \beta_4)\hat{x}_1\hat{x}_2 - \psi_1(\beta_5, \beta_4)\psi_2(\alpha_4, \alpha_5)\hat{x}_1^2\hat{x}_2 - 3\psi_2(\alpha_4, \alpha_5)\psi_2(\beta_5, \beta_4)\hat{x}_1^2\hat{x}_2^2.
\]
We estimate
\[
J_{\mathcal{W}}(\hat{x}) \geq 1 - \|\psi_1\|_{\infty, K}(\hat{x}_1 + \hat{x}_2) - \|\psi_2\|_{\infty, K}(\hat{x}_1^2 + \hat{x}_2^2)
\]
\[
-\|\psi_1\|_{\infty, K}\|\psi_2\|_{\infty, K}\hat{x}_1\hat{x}_2(\hat{x}_1 + \hat{x}_2) - 3\|\psi_2\|_{\infty, K}(\hat{x}_1\hat{x}_2)^2,
\]
where $K$ is the circle:
\[
K : \left(x_1 - \frac{2}{3}\right)^2 + \left(x_2 - \frac{1}{3}\right)^2 \leq 2\varepsilon^2.
\]
We consider the function
\[
\nu(\hat{x}) = \|\psi_1\|_{\infty, K}(\hat{x}_1 + \hat{x}_2) + \|\psi_2\|_{\infty, K}(\hat{x}_1^2 + \hat{x}_2^2)
\]
\[
+\|\psi_1\|_{\infty, K}\|\psi_2\|_{\infty, K}\hat{x}_1\hat{x}_2(\hat{x}_1 + \hat{x}_2) + 3\|\psi_2\|_{\infty, K}(\hat{x}_1\hat{x}_2)^2, \hat{x} \in \hat{T}.
\]
We compute
\[
\|\nu\|_{\infty, \hat{T}} = \|\psi_1\|_{\infty, K} + \frac{1}{2}\|\psi_2\|_{\infty, K} + \frac{1}{4}\|\psi_1\|_{\infty, K}\|\psi_2\|_{\infty, K} + \frac{3}{10}\|\psi_2\|_{\infty, K}^2,
\]
\[
\|\psi_1\|_{\infty, K} = \frac{9\sqrt{10}}{2}\varepsilon, \quad \|\psi_2\|_{\infty, K} = 2\varepsilon.
\]
Then
\[
J_{\mathcal{W}}(\hat{x}) \geq 1 - \|\nu\|_{\infty, \hat{T}} = \omega_1(\varepsilon).
\]
We establish the validity of the inequality $\omega_1(\varepsilon) > 0$, $\varepsilon \in [0, \varepsilon)$ with direct verification.
We calculate the matrix $B_W B_W^t = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$:

\[
b_{11} = (1 + \psi_1 (\beta_5, \beta_4) \hat{x}_1 + \psi_2 (\beta_5, \beta_4) \hat{x}_2)^2 + (\psi_1 (\alpha_4, \alpha_5) \hat{x}_1 + 2 \psi_2 (\alpha_4, \alpha_5) \hat{x}_1 \hat{x}_2)^2,
\]
\[
b_{12} = -[(1 + \psi_1 (\beta_5, \beta_4) \hat{x}_1 + \psi_2 (\beta_5, \beta_4) \hat{x}_2)(\psi_1 (\beta_5, \beta_4) \hat{x}_1 + 2 \psi_2 (\beta_5, \beta_4) \hat{x}_1 \hat{x}_2) + (1 + \psi_1 (\alpha_4, \alpha_5) \hat{x}_2 + \psi_2 (\alpha_4, \alpha_5) \hat{x}_2)(\psi_1 (\alpha_4, \alpha_5) \hat{x}_1 + 2 \psi_2 (\alpha_4, \alpha_5) \hat{x}_1 \hat{x}_2)],
\]
\[
b_{22} = (1 + \psi_1 (\alpha_4, \alpha_5) \hat{x}_2 + \psi_2 (\alpha_4, \alpha_5) \hat{x}_2)^2 + (\psi_1 (\beta_5, \beta_4) \hat{x}_2 + 2 \psi_2 (\beta_5, \beta_4) \hat{x}_1 \hat{x}_2)^2.
\]

Using the inequality

\[(9) \quad \|B_W (\hat{x})\|^2 \leq \frac{1}{2} (b_{11} + b_{22} + |b_{11} - b_{22}| + 2|b_{12}|),\]

we find uniform upper bound for $\|B_W (\hat{x})\|^2$ with respect to $\hat{x}$. We estimate separately the addends on the right hand side of the inequality (9)

\[
b_{11} + b_{22} \leq 2 + 2\|\psi_1\|_{\infty, K} (\hat{x}_1 + 2\hat{x}_2) + 2(\|\psi_1\|_{\infty, K} + \|\psi_2\|_{\infty, K})(\hat{x}_1^2 + \hat{x}_2^2)
\]
\[
+ 4\|\psi_1\|_{\infty, K} \|\psi_2\|_{\infty, K} (\hat{x}_1 \hat{x}_1 + \hat{x}_2 \hat{x}_2) + 8\|\psi_2\|_{\infty, K} (\hat{x}_1 \hat{x}_2)^2
\]
\[
\leq 2 + 2(\|\psi_1\|_{\infty, K} + \|\psi_2\|_{\infty, K}) + 2\|\psi_1\|_{\infty, K} + 3\|\psi_1\|_{\infty, K} \|\psi_2\|_{\infty, K} + \frac{3}{2} \|\psi_2\|_{\infty, K},
\]
\[
|b_{11} - b_{22}| \leq |\psi_1 (\alpha_4, \alpha_5) \hat{x}_1 - \psi_1 (\beta_5, \beta_4) \hat{x}_2 + 2 (\psi_2 (\alpha_4, \alpha_5) - \psi_2 (\beta_5, \beta_4)) \hat{x}_1 \hat{x}_2| 
\]
\[
+ |\psi_1 (\alpha_4, \alpha_5) \hat{x}_1 + \psi_1 (\beta_5, \beta_4) \hat{x}_2 + 2 (\psi_2 (\alpha_4, \alpha_5) + \psi_2 (\beta_5, \beta_4)) \hat{x}_1 \hat{x}_2| 
\]
\[
+ |2 + \psi_1 (\beta_5, \beta_4) \hat{x}_1 + \psi_1 (\alpha_4, \alpha_5) \hat{x}_2 + \psi_2 (\beta_5, \beta_4) \hat{x}_1^2 + 2 \psi_2 (\alpha_4, \alpha_5) \hat{x}_1 \hat{x}_2|.
\]

The inequality

\[
|b_{11} - b_{22}| \leq \left(\|\psi_1\|_{\infty, K} (\hat{x}_1 + \hat{x}_2) + \|\psi_2\|_{\infty, K} (\hat{x}_1^2 + \hat{x}_2^2)\right) 
\]
\[
(2 + \|\psi_1\|_{\infty, K} (\hat{x}_1 + \hat{x}_2) + \|\psi_2\|_{\infty, K} (\hat{x}_1^2 + \hat{x}_2^2))
\]
\[
+ \left(\|\psi_1\|_{\infty, K} (\hat{x}_1 + \hat{x}_2) + 4\|\psi_2\|_{\infty, K} \hat{x}_1 \hat{x}_2\right)^2
\]

is true because of

\[
\|\psi_i\|_{\infty, K} = \min_{x \in K} \psi_i (\hat{x}), \quad i = 1, 2.
\]
Since $\hat{x}_1^i + \hat{x}_2^i \leq 1$ for $i = 1, 2, 3, \ldots$ we have

$$|b_{11} - b_{22}| \leq 2(||\psi_1||_{\infty,K} + ||\psi_2||_{\infty,K})(1 + ||\psi_1||_{\infty,K} + ||\psi_2||_{\infty,K}).$$

For the last term in the right hand side of (9) we obtain

$$|b_{12}| \leq ||\psi_1||_{\infty,K}(\hat{x}_1 + \hat{x}_2) + \left[2||\psi_1||_{\infty,K}^2 + 4||\psi_2||_{\infty,K}\right] \hat{x}_1 \hat{x}_2$$

$$+ 3||\psi_1||_{\infty,K} ||\psi_2||_{\infty,K}(\hat{x}_1 + \hat{x}_2) + 2||\psi_2||_{\infty,K}^2 (\hat{x}_1^2 + \hat{x}_2^2)$$

$$\leq ||\psi_1||_{\infty,K} + ||\psi_2||_{\infty,K} + \frac{3}{4} ||\psi_1||_{\infty,K} ||\psi_2||_{\infty,K} + \frac{1}{2} (||\psi_1||_{\infty,K}^2 + ||\psi_2||_{\infty,K}^2).$$

We estimate

$$||B_W(\hat{x})||^2 \leq 1 + 3(||\psi_1||_{\infty,K} + ||\psi_2||_{\infty,K})$$

$$+ \frac{10||\psi_1||_{\infty,K}^2 + 17||\psi_1||_{\infty,K} ||\psi_2||_{\infty,K} + 9||\psi_2||_{\infty,K}^2}{4} = \omega_2(\varepsilon).$$

The inequality

$$\sigma_T^{-1} I \leq Q \leq \sigma_T I$$

follows from the inequality $\text{cond}(B_W(\hat{x})) \leq \sigma_W(\varepsilon)$ and (8). Now we can estimate the matrix $A_T$

$$\xi^t A_T \xi = \xi^t \left| \int_T (\nabla \phi_i)^t Q \nabla \phi_j \, d\hat{x} \right|_{i,j=1,2,\ldots,10} \xi = \int_T \left( \sum_{i,j=1}^{10} \xi_i (\nabla \phi_i)^t Q \nabla \phi_j \right) \xi_j \, d\hat{x}$$

$$= \int_T \left( \sum_{i=1}^{10} \xi_i \nabla \phi_i \right)^t Q \left( \sum_{j=1}^{10} \xi_j \nabla \phi_j \right) \, d\hat{x}, \quad \forall \xi \in \mathbb{R}^{10}.$$

The inequalities

$$\xi^t A_T \xi \leq \sigma_T \int_T \left( \sum_{i=1}^{10} \xi_i \nabla \phi_i \right)^t I \left( \sum_{j=1}^{10} \xi_j \nabla \phi_j \right) \, d\hat{x} = \sigma_T \xi^t \hat{\Lambda} \xi,$$

$$\sigma_T^{-1} \xi^t \hat{\Lambda} \xi \leq \xi^t A_T \xi,$$

follow from (10). The last results mean that (7) is fulfilled.

We make hierarchical refinement for the finite element of reference (Fig. 3). We obtain four curved elements $T_i, \quad i = 1, 2, 3, 4$ (Fig. 4) after refinement of an arbitrary curved element $T \in \tau_h$. If we need continue the refinement process
Figure 3: Hierarchical refinement of the finite element of reference. Local refinement of the element $\hat{T}_2$.

Figure 4: Hierarchical refinement of the finite element $T$. Local refinement of the element $T_2$. 
we have to refine curved elements with more than one curved side. It does not lead to difficulties since for the refinement of the curved element \( T_2 \) for example, we need only transformation \( F_T \) and local refinement of the element \( \tilde{T}_2 \).

We denote the restrictions of the spaces \( V_{h, T}, \tilde{V}_{h_{1}, T}, V_{h_{1}, T} \) over the element \( T \) respectively by \( V_{h, T}, \tilde{V}_{h_{1, T}}, V_{h_{1, T}} \). We write the so-called two-level hierarchical basis element stiffness matrix

\[
A_T = \begin{pmatrix}
A_{T;11} & A_{T;12} \\
A_{T;21} & A_{T;22}
\end{pmatrix}, \quad \forall T \in \tau_h.
\]

We consider the generalized eigenvalue problem

\[(11) \quad \lambda A_T \xi = S_T \xi,
\]

over \( T \in \tau_h \), where

\[
S_T = A_{T;22} - A_{T;21} A_{T;11}^{-1} A_{T;12}
\]

is the element Schur complement. The quantity \( \lambda_{T, \text{min}} \) is the smallest solution for the problem \((11)\).

The next theorem states that the strengthened Cauchy - Buninakowskii - Schwarz inequality is valid over curved domains \( \Omega \) uniformly with respect to \( h \), when the corresponding triangulations \( \tau_h \) satisfies some conditions.

**Theorem 4.** Let the conditions of Theorem 3 hold. Then there exists a constant \( \gamma \in [0, 1) \) depending only on the geometry of the initial triangulation \( \tau_h \), such that

\[
|a(v, w)| \leq \gamma \sqrt{a(v, v)} \sqrt{a(w, w)}
\]

for all \( v \in V_h \) and \( w \in \tilde{V}_{h_1} \).

**Proof.** First, we shall prove that \( \sigma_T \) is independent on \( h \). The functions \( \omega_i, i = 1, 2 \) depend only on \( \varepsilon \), hence it is necessary merely to prove that \( \text{cond}(B_V) \) is independent on \( h \). Putting \( B_V \) instead of \( M^* \) in \((4)\) and using

\[
\|DV_T^{-1}\|^2 = O(h_T^{-2}) \quad \text{and} \quad J_{VT} = O(h_T^2),
\]

we have

\[
\text{cond}(B_{VT}) = \|DV_T^{-1}\|^2 J_{VT} = O(1).
\]

Consequently \( \sigma_T \) is independent on \( h \).

Since the spaces \( V_{h,T}, \tilde{V}_{h,1,T} \) are finite dimensional and \( V_{h,T} \cap \tilde{V}_{h,1,T} = \{0\} \) there exists a constant

\[
\gamma_T = \gamma_T(V_{h,T}, \tilde{V}_{h,1,T}) \in [0, 1)
\]
such that

$$|a_T(v, w)| \leq \gamma_T(V_{h,T}, \tilde{V}_{h_1,T}) \sqrt{a_T(v,v)} \sqrt{a_T(w,w)}, \ \forall v \in V_h, \forall w \in \tilde{V}_{h_1},$$

(see [9]).

We shall find upper bound for $\gamma_T$ estimating the eigenvalue $\lambda_{T,\min}$ by $\hat{\lambda}_{\max} = \lambda_{\max}[\hat{A}]$. Since $S_T \leq A_T$ [9] and $A_T \leq \sigma_T \hat{A}$, we have

$$\lambda_{T,\min} = \lambda_{\min}[A_T^{-1} S_T] \geq \lambda_{\min}[A_T^{-2}]$$

$$\geq (\lambda_{\max}[A_T^2])^{-1} \geq (\lambda_{\max}[A_T])^{-2} \geq \left(\sigma_T \hat{\lambda}_{\max}\right)^{-2}.$$ 

Then $\gamma_T \leq \sqrt{1 - \left(\sigma_T \hat{\lambda}_{\max}\right)^{-2}}$.

Further we prove the global strengthened C. B. S. inequality

$$|a(v, w)| \leq \sum_{T \in \tau_h} |a_T(v, w)| \leq \sum_{T \in \tau_h} \gamma_T \sqrt{a_T(v,v)} \sqrt{a_T(w,w)}$$

$$\leq \sum_{T \in \tau_h} \sqrt{1 - \hat{\lambda}_{\min}^{-1} \sigma_T^{-1}} \sqrt{a_T(v,v)} \sqrt{a_T(w,w)}$$

$$\leq \sqrt{1 - \hat{\lambda}_{\min}^{-1}} \sum_{T \in \tau_h} \sqrt{a_T(v,v)} \sqrt{a_T(w,w)},$$

where $\sigma = \max_{T \in \tau_h} \sigma_T$. We put

$$\gamma = \sqrt{1 - \left(\hat{\lambda}_{\max} \sigma\right)^{-2}}$$

and we obtain

$$|a(v, w)| \leq \gamma \left(\sum_{T \in \tau_h} a_T(v,v)\right)^{\frac{1}{2}} \left(\sum_{T \in \tau_h} a_T(w,w)\right)^{\frac{1}{2}} \leq \gamma \sqrt{a(v,v)} \sqrt{a(w,w)}.$$ 

The proof is completed.

Acknowledgement. This work is partially supported by the Bulgarian Ministry of Science and Technologies, under Contract MM-524/95.
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Received: 31.07.1998