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Approximate Values of the Integral Logarithm and the Exponential Integral Function

Roumen K. Krastev

Presented by V. Kiryakova

Described is the obtaining of approximate values of the exponential integral function and the integral logarithm: $\mathrm{li}(x) = \ln(r - px - qx^2)$, where $r = 1.0013, p = 0.259, q = 0.741$. The solutions are valid in particular intervals with errors less than 1%.. The formulae are applicable for instance to the analytical description (finding of parameters) of physical processes determined by these integrals.

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1. Introduction

The mathematical handbooks, for instance [1], give the following formulae for the exponential integral and the integral logarithm [2]:

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t} dt = C_0 + \ln|x| + \frac{x}{1.1!} + \frac{x^2}{2.2!} + \frac{x^3}{3.3!} + \cdots, \tag{1}$$

$$\mathrm{li}(x) = \int_{0}^{x} \frac{dt}{\ln t} = C_0 + \ln|\ln x| + \frac{\ln x}{1.1!} + \frac{(\ln x)^2}{2.2!} + \frac{(\ln x)^3}{3.3!} + \cdots, \tag{2}$$

where $C_0 = 0.5772...$ is the Euler constant. The series are convergent for $x < 0$ for (1) and for $0 < x < 1$ for (2). Analytical solutions of these integrals are described in this work.

2. Integral logarithm

The idea is to find an analytically approximate but enough exact solution by means of substitution of the subintegral function by another integrable one. In this interval the subintegral function can be approximated by the following function:

$$\frac{1}{\ln t} \approx 0.5792 \frac{t + 0.7270}{t - 1}. \tag{3}$$

Numerically, the two functions coincide for $0.35 < t < 1$. The solution of the new approximative integral is:

$$\mathrm{li}(t) \approx 0.5792\,(t - 1 + 1.7270 \ln |t - 1|) + C. \tag{4}$$

Since $t \in (0, 1)$, $(1 - t)$ can be written instead $|t - 1|$ in (4). Substituting t with $1 - t$ in (3) we obtain $t \approx (0.5792t - 1)\ln(1 - t)$, which is to be substituted in the first addend of (4). The integral is obtained in the form:

$$\mathrm{li}(t) \approx \ln(1 - t)^{0.355t + 0.4211} + C. \tag{5}$$

The sublogarithm function is approximated rather well with a second-order function. Then the approximate formula for the integral logarithm is:

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\ln t} \approx \ln(r - px - qx^2). \tag{6}$$

For the numerical check of (6) and for a more precise determination of the coefficients p, q and r, it is better the function (6) to be determined not by the coefficients in (5) but by means of data calculated according to (2) or by a numerical method. The relationship $z_1(x) = \exp(\mathrm{li}(x))$ is shown on **Fig.1**, $\mathrm{li}(x)$ is calculated according to (2). As seen from the plot of the derivative $z_1'(x) = dz_1/dx$, the dependance $z_1(x)$ can actually be approximated with a second-order function in a rather wide interval. From this plot the coefficients are determined as:

$$r = 1.0013; \qquad p = 0.259; \qquad q = 0.741. \tag{7}$$

Then (6) with coefficients (7) is an approximate solution valid in the interval $0.016 < x < 1$ with an error less than 5 %. The relationship $z_2 = r - px - qx^2$ is shown on **Fig.1**, too. The coefficients $r = 1, p = 0.259, q = 0.741$ give an approximate solution with error less 1 % in the interval $0.16 < x < 1$.

Consider also a function (6) with

$$r = 1; \qquad p = 1 - 2/e \approx 0.264, \qquad q = 2/e \approx 0.736, \tag{7'}$$

where e denotes the Neper number.

Does (6) with coefficients (7′) is the mathematically exact solution? Long precision of computation according to (1) and (2) does not result in the approximation of the plot $z_1(x)$ to the plot of the equation $z_2(x)$, so that answer is "No, it does not".

3. Inverse function of the integral logarithm

When $x = x(I)$ is to be found, it is suitable the formula to be represented as

$$I = \int_{x_0}^{x} \frac{dt}{\ln t} \approx \ln(a - b(x+c)^2) - I_0, \tag{6'}$$

where $a = r + p^2/(4q) \approx 1.022$, $b = q \approx 0.741$, $c = p/(2q) \approx 0.175$. To reduce the number of coefficients, the constant b can be taken out of the brackets and outside the logarithm and be added to the integration constant. Still, as we have a definite integral, the constant $\ln(b)$ is cancelled:

$$I \approx \ln(d - (x+c)^2) - I_0, \tag{6''}$$

where $d = a/b \approx 1.38$. The dependence to be found $x = x(I)$ is:

$$x \approx -c + \sqrt{d - \exp(I_0 + I)}\,. \tag{8}$$

The integration constant in (8) is defined by the initial conditions ($I = 0, x = x_0$), namely:

$$I_0 \approx \ln(d - (x_0 + c)^2). \tag{9}$$

4. Exponential integral function

The exponential integral function and its solutions (10), (11) are found by the substitution $t = e^y$ in (6) and (8), namely:

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^y\,dy}{y} \approx \ln(r - pe^x - qe^{2x}), \tag{10}$$

$$x \approx \ln(-c + \sqrt{d - \exp(I_0 + I)}), \tag{11}$$

where $I = \mathrm{Ei}(x) - \mathrm{Ei}(x_0)$. The integration constant $I_0(x_0)$ is determined as:

$$I_0 \approx \ln(d - (e^{x_0} + c)^2) \tag{12}$$

The errors in calculation and the valid intervals are shown on **Fig.2**.

References

[1] I. M. Vinogradov, *Mathematical Encyclopedia*, vol. **2** (In Russian), "Soviet Encyclopedia" Publ., Moscow (1979), p. 603.

[2] A. Erdélyi et al. (Ed-s), *Higher Transcendental Functions*, vol. **1** (In Russian), "Nauka", Moscow (1973), p. 254.

Central Laborathory of Physiko-Chemical Mechanics
Bulgarian Academy of Sciences
Sofia 1113, "Akad. G. Bonchev" Str., Block 1
e-mail: RKrastev@bgcict.acad.bg

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Fig. 1: Plots $z_1 = \exp(\mathrm{li}(x))$ and $z_2 = r - px - qx^2$.

Fig. 2: Errors in calculation of the exponential integral function and its inverse ((10), (11), (12), $p = 0.259, q = 0.741, r = 1.0013, c = 0.175, d = 1.381$). With the integral logarithm ((6), (8), (9)) the errors looks the same but $0.0025 < x < 1$ ($x_l = \ln(x_e)$).