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GAREX(2) Models: Existence and Possibilities

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The general autoregressive time series of arbitrary but known finite order are of special importance. Meanwhile, special distributions, as the normal distribution, exponential, Laplace or some others, impose quite different existing conditions. This problem have to be solved for each distribution separately. The order of the time series is of great importance in specifying the existing conditions. GAREX(2) model and some of its special cases are considered in this paper. The correlation structure and the spectral density of the model are discussed.

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1. Introduction

According to the intention of generalizing the models of time series, Mališić ([4]) has suggested some type of generalization for the first order autoregressive time series with exponential $\varepsilon(\lambda)$ marginal distribution, the so called AREX(1). He has also proposed some new possible special cases like FAREX(1) and SAREX(1). The idea of expanding definitions of EAR(2) (Lawrance and Lewis [2]), NEAR(2) (Lawrance and Lewis [3]) and AREX(2) (Mališić [4]) time series to some general case with exponential marginal distribution has been accepted and the model *GAREX(2) - General AutoRegressive EXponentially distributed second order time series*, is defined and discussed in this paper.

We suggest the definition of the autoregressive time series with exponential $\varepsilon(\lambda)$ marginal distribution which is "pure" autoregression with some positive probabilities $p_1 > 0$, $p_2 > 0$ and positive coefficients of the autoregression $\alpha_1 > 0$, $\alpha_2 > 0$, respectively.

The above mentioned first order autoregressions have been treated by means of the random coefficient representation in Popović ([5], [6] and [7]). The appropriate random coefficient representation of GAREX(2) will be given in this paper.

2. Existence of the model

We shall consider time series $\{X_n,\ n \in Z\}$ and $\{\varepsilon_n,\ n \in Z\}$ that are the mixtures of distributions and are semi-independent (X_n and ε_t are independent iff $n < t$).

Definition 2.1. Let $\{E_n,\ n \in Z\}$ be the sequence of i.i.d. random variables with exponential $\varepsilon(\lambda)$ marginal distribution. Let α_1, α_2, β_1, β_2, p_0, p_1, p_2, q_1, and q_2 be nonnegative real numbers such that:

$$(A1) \qquad 0 \le p_0, p_1, p_2 \le 1,$$

$$(A2) \qquad 0 < q_1, q_2 \le 1,$$

$$(A3) \qquad p_0 + p_1 + p_2 + q_1 + q_2 = 1,$$

$$(A4) \qquad 0 < \alpha_1, \alpha_2, \beta_1, \beta_2 < 1,$$

$$(A5) \qquad \begin{aligned} b_1 &= \frac{u - \sqrt{u^2 - 4v(p_0 + q_1 + q_2)}}{2(p_0 + q_1 + q_2)}, \\ b_2 &= \frac{u + \sqrt{u^2 - 4v(p_0 + q_1 + q_2)}}{2(p_0 + q_1 + q_2)}, \end{aligned}$$

where $u = p_0\beta_1 + p_0\beta_2 + q_1\beta_2 + q_2\beta_1$, $v = p_0\beta_1\beta_2$, and if $p(x) = (u + p_1 + p_2 + p_1\alpha_2 + p_2\alpha_1 - \alpha_1 - \alpha_2)x + \alpha_1\alpha_2 - p_1\alpha_2 - p_2\alpha_1 - v$, then

$$(A6) \qquad p(b_1) > 0.$$

Let, also, exact one of the conditions (B1)-(B3) be satisfied:

$$(B1) \qquad \begin{gathered} b_1 < \min\{\alpha_1, \alpha_2\} < \min\{\beta_1, \beta_2\} < b_2 < \max\{\alpha_1, \alpha_2\} < \max\{\beta_1, \beta_2\}, \\ p(b_2) < 0; \end{gathered}$$

$$(B2) \qquad \begin{gathered} b_1 < \min\{\beta_1, \beta_2\} < b_2 < \alpha_1, \alpha_2 < \max\{\beta_1, \beta_2\}, \\ p(b_2) > 0; \end{gathered}$$

$$(B3) \qquad b_1 < \alpha_1, \alpha_2 < \min\{\beta_1, \beta_2\} < b_2 < \max\{\beta_1, \beta_2\}.$$

Then the sequence $\{X_n,\ n \in Z\}$ which is defined as the mixture of distributions:

$$(2.1)\qquad X_n = \begin{cases} \varepsilon_n, & w.p.\quad p_0 \\ \alpha_1 X_{n-1}, & w.p.\quad p_1 \\ \alpha_2 X_{n-2}, & w.p.\quad p_2, \\ \beta_1 X_{n-1} + \varepsilon_n, & w.p.\quad q_1 \\ \beta_2 X_{n-2} + \varepsilon_n, & w.p.\quad q_2 \end{cases} \quad n \in Z,$$

where

$$(2.2)\qquad \varepsilon_n = \begin{cases} E_n, & w.p.\quad r_1 \\ \alpha_1 E_n, & w.p.\quad r_2 \\ \alpha_2 E_n, & w.p.\quad r_3 \\ b_1 E_n, & w.p.\quad r_4 \\ b_2 E_n, & w.p.\quad r_5 \end{cases}$$

$$(2.3)\qquad \begin{aligned} r_1 &= \frac{(1-\beta_1)(1-\beta_2)}{(p_0+q_1+q_2)(1-b_1)(1-b_2)}, \\ r_2 &= \frac{p_1(\beta_1-\alpha_1)(\alpha_1-\beta_2)}{(p_0+q_1+q_2)(\alpha_1-b_1)(\alpha_1-b_2)}, \\ r_3 &= \frac{p_2(\beta_1-\alpha_2)(\alpha_2-\beta_2)}{(p_0+q_1+q_2)(\alpha_2-b_1)(\alpha_2-b_2)}, \\ r_4 &= \frac{(b_1-\beta_1)(b_1-\beta_2)p(b_1)}{(p_0+q_1+q_2)(b_1-1)(b_1-\alpha_1)(b_1-\alpha_2)(b_1-b_2)}, \\ r_5 &= \frac{(b_2-\beta_1)(b_2-\beta_2)p(b_2)}{(p_0+q_1+q_2)(b_2-1)(b_2-\alpha_1)(b_2-\alpha_2)(b_2-b_1)}, \end{aligned}$$

is the wide sense stationary time series GAREX(2) with $\varepsilon(\lambda)$ marginal distribution. ■

Really, let $\Phi_X(s)$ and $\Phi_\varepsilon(s)$ be Laplace transforms corresponding to the random variables of two sequences $\{X_n\}$ and $\{\varepsilon_n\}$. As these sequences are wide sense stationary and semi-independent, it will be

$$\Phi_\varepsilon(s) = \frac{\Phi_X(s) - p_1\Phi_X(\alpha_1 s) - p_2\Phi_X(\alpha_2 s)}{p_0 + q_1\Phi_X(\beta_1 s) + q_2\Phi_X(\beta_2 s)} = \frac{\lambda(\lambda+\beta_1 s)(\lambda+\beta_2 s)P(s)}{(\lambda+s)(\lambda+\alpha_1 s)(\lambda+\alpha_2 s)Q(s)},$$

where

$$P(s) = (\lambda+\alpha_1 s)(\lambda+\alpha_2 s) - p_1(\lambda+s)(\lambda+\alpha_2 s) - p_2(\lambda+s)(\lambda+\alpha_1 s)$$

and

$$Q(s) = p_0\beta_1\beta_2 s^2 + (p_0\beta_1 + p_0\beta_2 + q_1\beta_2 + q_2\beta_1)\lambda s + (p_0+q_1+q_2)\lambda^2.$$

It follows that $Q(s)$ has two distinct real roots:

$$s_1 = \frac{-u - \sqrt{D}}{2v}\lambda, \quad s_2 = \frac{-u + \sqrt{D}}{2v}\lambda, \quad D = u^2 - 4v(p_0 + q_1 + q_2).$$

If we set

$$b_1 = -\frac{\lambda}{s_1}, \; b_2 = -\frac{\lambda}{s_2},$$

it will be $0 < b_1 < b_2 < 1$. Now $\Phi_\varepsilon(s)$ becomes

$$\Phi_\varepsilon(s) = \frac{\lambda(\lambda + \beta_1 s)(\lambda + \beta_2 s)P(s)}{(p_0 + q_1 + q_2)(\lambda + s)(\lambda + \alpha_1 s)(\lambda + \alpha_2 s)(\lambda + b_1 s)(\lambda + b_2 s)}.$$

Under the conditions (B1)-(B3), there are no multiplying roots of the last denominator. So,

$$\Phi_\varepsilon(s) = r_1\frac{\lambda}{\lambda + s} + r_2\frac{\lambda}{\lambda + \alpha_1 s} + r_3\frac{\lambda}{\lambda + \alpha_2 s} + r_4\frac{\lambda}{\lambda + b_1 s} + r_5\frac{\lambda}{\lambda + b_2 s},$$

where coefficients r_i, $i = 1, 2, \ldots, 5$ must be defined to be probabilities with sum one and $\Phi_\varepsilon(s)$ to satisfy (2.4). The solution of this problem is obvious because of the equalities

$$(b_1 - \beta_1)(b_2 - \beta_1) = \frac{q_1\beta_1(\beta_1 - \beta_2)}{p_0 + q_1 + q_2}$$

and

$$(b_1 - \beta_2)(b_2 - \beta_2) = \frac{q_2\beta_2(\beta_2 - \beta_1)}{p_0 + q_1 + q_2}$$

and the inequality $b_1 < b_2$.

It follows that $b_1 < \min\{\beta_1, \beta_2\} < b_2 < \max\{\beta_1, \beta_2\}$. So, finally, we can see that r_2 will be greater than zero if $b_2 < \alpha_1 < \max\{\beta_1, \beta_2\}$ or $b_1 < \alpha_1 < \min\{\beta_1, \beta_2\}$, r_3 will be greater than zero if $b_2 < \alpha_2 < \max\{\beta_1, \beta_2\}$ or $b_1 < \alpha_2 < \min\{\beta_1, \beta_2\}$, and $r_4 > 0$ and $r_5 > 0$ if exact one condition of (B1)-(B3) is satisfied.

3. Correlation structure of the model

In spite of so many conditions for the existence of the model, the correlation structure follows the second order difference equation that characterizes usual second order autoregression.

Theorem 3.1. *Under the conditions of Definition 2.1, GAREX(2) time series has the real valued absolutely summable autocovariance function*

$$\begin{aligned} K_r &\equiv Cov(X_n, X_{n+r}) = \\ &= \frac{(1-u_2)R + u_1(1+u_2)}{2\lambda^2(1-u_2)R}\left(\frac{u_1+R}{2}\right)^{|r|} + \\ &+ \frac{(1-u_2)R - u_1(1+u_2)}{2\lambda^2(1-u_2)R}\left(\frac{u_1-R}{2}\right)^{|r|}\cos\pi|r|, \end{aligned} \tag{3.1}$$

where $u_1 = p_1\alpha_1 + q_1\beta_1$, $u_2 = p_2\alpha_2 + q_2\beta_2$ *and* $R = \sqrt{u_1^2 + 4u_2}$.

Proof. The autocovariance function K_r is the solution of the difference equation

$$K_r = u_1K_{r-1} + u_2K_{r-2}$$

corresponding to the difference equation of the autocorrelation function

$$\rho_r = u_1\rho_{r-1} + u_2\rho_{r-2},$$

for the initial conditions

$$\rho_0 = 1,\ \rho_1 = \frac{u_1}{1-u_2}$$

and the property of stationarity $K_{-r} = K_r$.

The absolute summability of the autocovariance function K_r follows from the fact that $-1 < (u_1 - R)/2 < 0 < (u_1 + R)/2 < 1$. ■

Corollary 3.1. *If the conditions of Definition 2.1 are satisfied, GAREX(2) has the real valued spectral density*

$$\begin{aligned} f(\tau) &= \frac{1}{4\pi\lambda^2(1-u_2)R}\times \\ &\times\left[\frac{(4-(u_1+R)^2)(u_1+R+u_2(u_1-R))}{4-4(u_1+R)\cos\tau+(u_1+R)^2}+\right. \\ &\left.+\frac{(4-(u_1-R)^2)(u_1-R+u_2(u_1+R))}{4-4(u_1-R)\cos\tau+(u_1-R)^2}\right]. \end{aligned}$$

If we consider the bispectrum of GAREX(2), we will find out that it will be represented by the second order nonhomogenious difference equation

$$\begin{aligned} C(r,s) &= E\left[(X_n-\lambda^{-1})(X_{n-r}-\lambda^{-1})(X_{n-s}-\lambda^{-1})\right] = \\ &= u_1C(r-1,s-1) + u_2C(r-2,s-2) + Q, \end{aligned}$$

where

$$\begin{aligned} Q &= u_1\lambda^{-1}\left[K_{r-1}+K_{s-1}+K_{r-s}+\lambda^{-2}\right]+u_2\lambda^{-1}\left[K_{r-2}+K_{s-2}+K_{r-s}+\right.\\ &+ \left.\lambda^{-2}\right]+(p_0+q_1+q_2)E(\varepsilon_n)\left[K_{r-s}+\lambda^{-2}\right]-\\ &- \lambda^{-1}\left[K_{r-s}+K_r+K_s\right]-\lambda^{-3}, \end{aligned}$$

with the same indicial equation as was followed by the spectrum.

Make a point that $u_2 = 0$ gives some special cases of the first order autoregressive time series. For instance, $p_2 = q_2 = 0$ will produce AREX(1) (Mališić [4]).

4. Random coefficient representation

The random coefficient representation gives linear form to the nonlinear models. This is the usual way of interpreting mixtures of distributions (Chan [1], Lawrance and Lewis [3], Popović [5], [6], [7] and others). The same can be done with GAREX(2).

It is easy to verify that the stochastic difference equation

$$X_n = U_1(n)X_{n-1} + U_2(n)X_{n-2} + V_nE_n$$

will represent autoregressive time series GAREX(2) (2.1)-(2.2) iff the conditions $(i)-(vii)$ are satisfied:

(i) $\{X_n,\ n \in Z\}$ is the sequence of identically distributed random variables with $\varepsilon(\lambda)$ marginal distribution.

(ii) $\{E_n,\ n \in Z\}$ is the sequence of i.i.d. random variables with $\varepsilon(\lambda)$ marginal distribution.

(iii) $\{U_1(n),\ n \in Z\}$, $\{U_2(n),\ n \in Z\}$ and $\{V_n,\ n \in Z\}$ are the sequences of independent random variables with the following marginal distributions: $U_1(n)$ takes values α_1, β_1 and 0 with corresponding probabilities p_1, q_1 and $1-p_1-q_1$, $U_2(n)$ takes values α_2, β_2 and 0 with corresponding probabilities p_2, q_2 and $1-p_2-q_2$, V_n takes values $\alpha_1, \alpha_2, b_1, b_2$ and 1 with corresponding probabilities r_2, r_3, r_4, r_5 and r_1 for any integer n.

(iv) $\{\mathbf{Y}_n \equiv (U_1(n), U_2(n), V_n),\ n \in Z\}$ is the sequence of i.i.d. random vectors with the following distribution: $\mathbf{Y}_n$ takes values $(0,0,1)$, $(0,0,\alpha_1)$, $(0,0,\alpha_2)$, $(0,0,b_1)$, $(0,0,b_2)$, $(\alpha_1,0,0)$, $(\alpha_2,0,0)$, $(\beta_1,0,1)$, $(\beta_1,0,\alpha_1)$, $(\beta_1,0,\alpha_2)$, $(\beta_1,0,b_1)$, $(\beta_1,0,b_2)$, $(0,\beta_2,1)$, $(0,\beta_2,\alpha_1)$, $(0,\beta_2,\alpha_2)$, $(0,\beta_2,b_1)$ and $(0,\beta_2,b_2)$ with corresponding probabilities p_0r_1, p_0r_2, p_0r_3, p_0r_4, p_0r_5, p_1, p_2, q_1r_1, q_1r_2, q_1r_3, q_1r_4, q_1r_5, q_2r_1, q_2r_2, q_2r_3, q_2r_4 and q_2r_5 for all n.

(v) The random vector $\mathbf{Y}_n$ is independent of the random variable E_n for any integer n.

(vi) X_t and $\mathbf{Y}_n$ or E_n are independent iff $t < n$.

(vii) $U_1(n)$ and $U_2(t)$ are independent iff $n \neq t$.

The parameters α_i, β_i, b_i, p_i, q_i and r_i satisfy the conditions of Definition 2.1 except for the condition (A1) instead of which,

$$(A1') \qquad 0 < p_0, p_1, p_2 \leq 1$$

will be valid.

These conditions ensure $\varepsilon(\lambda)$ marginal distribution of the time series, id est the same certain distribution to all members of the sequence $\{X_n,\ n \in Z\}$. It is important to remark that the random variable $V_n E_n$ is dependent of the corresponding random coefficients $U_1(n)$ and $U_2(n)$.

The advantages of the random coefficient representation of GAREX(2) are many, but we will discuss only invertibility of the process.

Theorem 4.1. *Under the conditions* (i) – (vii), *difference equation (4.1) has unique σ_n-measurable, stationary, strictly stationary and ergodic solution of the form*

$$X_n = \sum_{j=0}^{\infty} A_{n-j} V_{n-j-1} E_{n-j-1} + V_n E_n, \tag{4.2}$$

where

$$A_n = U_1(n),\ A_{n-j} = A_{n-j+1} U_1(n-j) + B_{n-j+1},\ j = 1, 2, \ldots \tag{4.3}$$

$$B_n = U_2(n),\ B_{n-j} = A_{n-j+1} U_2(n-j),\ j = 1, 2, \ldots \tag{4.4}$$

and σ_n is the σ-field generated by the set of random vectors $\{(U_1(t), U_2(t), V_t, E_t),\ t \leq n\}$.

The prove of this theorem is given in Appendix.

It is important to remark that the sum (4.2) is finite almost sure, namely:

Theorem 4.2. *Under the conditions as in Theorem 4.1, the sum (4.2) will be finite (with random number of terms) with probability one.*

Proof. If we set $I_n = V_n E_n$, we shall have the solution

$$X_n = \sum_{j=0}^{\infty} A_{n-j} I_{n-j-1} + I_n,$$

where A_t and I_s are independent iff $t \neq s$. The sequence $\{I_n,\ n \in Z\}$ is the sequence of i.i.d. random variables. To complete the proof of the theorem, it will be enough to show that $\sum_{j=0}^{\infty} P\{A_{n-j} \neq 0\} < \infty$.

It is easy to verify that

$$P\{A_{n-j} \neq 0\} = P\{A_{n-j} > 0\} = \left(\frac{\theta}{2} + \frac{\theta^2 + 2\mu}{2\sqrt{\theta^2 + 4\mu}}\right) \cdot \left(\frac{\theta}{2} + \frac{\sqrt{\theta^2 + 4\mu}}{2}\right)^j +$$

$$(4.5) \qquad + \left(\frac{\theta}{2} - \frac{\theta^2 + 2\mu}{2\sqrt{\theta^2 + 4\mu}}\right) \cdot \left(\frac{\theta}{2} - \frac{\sqrt{\theta^2 + 4\mu}}{2}\right)^j, j = 0, 1, 2, \ldots$$

where $\theta = p_1 + q_1$, $\mu = p_2 + q_2$ and it is obvious that $0 < \frac{\theta}{2} + \frac{\sqrt{\theta^2+4\mu}}{2} < 1$, $-1 < \frac{\theta}{2} - \frac{\sqrt{\theta^2+4\mu}}{2} < 0$.

Really, by induction on k, we can prove that $P\{A_{n-k+1} B_{n-k+1} > 0\} = 0$. So,

$$P\{A_{n-j} > 0\} = P\{A_{n-j+1} U_1(n-j) > 0\} + P\{B_{n-j+1} > 0\},$$

and (4.5) will be the solution of the second order difference equation

$$\pi_j = \theta \pi_{j-1} + \mu \pi_{j-2}, \quad \pi_j = P\{A_{n-j} > 0\},$$

for the initial conditions $\pi_0 = \theta$, $\pi_1 = \theta^2 + \mu$. The Borel-Cantelli lemma implies that almost sure $A_{n-j} \neq 0$ finitely often. ∎

5. Some special cases

According to Definition 21, the well known models NEAR(2) and EAR(2) become special cases of GAREX(2).

Really, if $\alpha_1 = \alpha_2 = p_1 = p_2 = 0$ we shall have NEAR(2) and for $p_0 = p_1 = p_2 = \alpha_1 = \alpha_2 = 0$, $q_1 = 1 - \beta_2$ and $q_2 = \beta_2$ we shall have EAR(2).

The similar will be concerning FAREX(2):

Let $p_0 = 0$, $b = (q_1\beta_2 + q_2\beta_1)/(q_1 + q_2)$,

$$q(b) \equiv \left[(b - \alpha_1)(b - \alpha_2) - p_1(b-1)(b - \alpha_2) - p_2(b-1)(b - \alpha_1)\right]/b^2$$

and $\alpha_1\alpha_2 - p_1\alpha_2 - p_2\alpha_1 > 0$. Let, also, exact one of the conditions (C1)-(C.3) be satisfied:

$$(C1) \qquad \begin{array}{c} \min\{\alpha_1,\alpha_2\} < \min\{\beta_1,\beta_2\} < b < \max\{\alpha_1,\alpha_2\} < \max\{\beta_1,\beta_2\}, \\ q(b) < 0; \end{array}$$

$$(C2) \qquad \begin{array}{c} \min\{\beta_1,\beta_2\} < b < \alpha_1,\alpha_2 < \max\{\beta_1,\beta_2\}, \\ q(b) > 0; \end{array}$$

$$(C3) \qquad \alpha_1,\alpha_2 < \min\{\beta_1,\beta_2\} < b < \max\{\beta_1,\beta_2\}.$$

Then,

$$\varepsilon_n = \begin{cases} 0, & w.p. \quad \dfrac{\beta_1\beta_2(\alpha_1\alpha_2 - p_1\alpha_2 - p_2\alpha_1)}{\alpha_1\alpha_2(q_1+q_2)b}, \\ E_n, & w.p. \quad \dfrac{(1-\beta_1)(1-\beta_2)}{(q_1+q_2)(1-b)}, \\ \alpha_1 E_n, & w.p. \quad \dfrac{p_1(\beta_1-\alpha_1)(\alpha_1-\beta_2)}{(q_1+q_2)\alpha_1(\alpha_1-b)}, \\ \alpha_2 E_n, & w.p. \quad \dfrac{p_2(\beta_1-\alpha_2)(\alpha_2-\beta_2)}{(q_1+q_2)\alpha_2(\alpha_2-b)}, \\ bE_n, & w.p. \quad \dfrac{(b-\beta_1)(b-\beta_2)q(b)}{(q_1+q_2)(b-1)(b-\alpha_1)(b-\alpha_2)}. \end{cases}$$

Also the SAREX(2) model:

Let $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $p(b_1) > 0$ and $p(b_2) < 0$. Then

$$\varepsilon_n = \begin{cases} E_n, & w.p. \quad \dfrac{(1-\alpha_1)(1-\alpha_2)}{(p_0+q_1+q_2)(1-b_1)(1-b_2)}, \\ b_1 E_n, & w.p. \quad \dfrac{p(b_1)}{(p_0+q_1+q_2)(b_1-1)(b_1-b_2)}, \\ b_2 E_n, & w.p. \quad \dfrac{p(b_2)}{(p_0+q_1+q_2)(b_2-1)(b_2-b_1)}. \end{cases}$$

6. Appendix

P r o o f of T h e o r e m 4.1. By repeated substitution of

$$X_{n-j} = U_1(n-j)X_{n-j-1} + U_2(n-j)X_{n-j-2} + V_{n-j}E_{n-j}, \; j = 1,2,\ldots k$$

into (4.1) and by induction on k, we obtain

$$X_n = A_{n-k}X_{n-k-1} + B_{n-k}X_{n-k-2} + \sum_{j=0}^{k-1} A_{n-j}V_{n-j-1}E_{n-j-1} + V_nE_n.$$

By means of the mean square convergence it will be

$$\begin{aligned} & \lim_{k\to\infty} E(X_n - \sum_{j=0}^{k-1} A_{n-j}V_{n-j-1}E_{n-j-1} - V_nE_n)^2 \\ = & \lim_{k\to\infty} E(A_{n-k}X_{n-k-1} + B_{n-k}X_{n-k-2})^2 = \lim_{k\to\infty} E(A^2_{n-k}X^2_{n-k-1}) \\ + & 2\lim_{k\to\infty} E(A_{n-k}B_{n-k}X_{n-k-1}X_{n-k-2}) + \lim_{k\to\infty} E(B^2_{n-k}X^2_{n-k-2}) = 0. \end{aligned}$$

Really, A_{n-k} and X_{n-k-1} and B_{n-k} and X_{n-k-2} are independent because of (vi). So,

$$\lim_{k\to\infty} E(A^2_{n-k}X^2_{n-k-1}) = 2\lambda^{-2}u_{11}\lim_{k\to\infty} E(A^2_{n-k+1}) + 2\lambda^{-2}\lim_{k\to\infty} E(B^2_{n-k+1})$$

and

$$\lim_{k\to\infty} E(B^2_{n-k}X^2_{n-k-2}) = 2\lambda^{-2}u_{22}\lim_{k\to\infty} E(A^2_{n-k+1}),$$

where $u_{11} = E(U_1^2(n))$ and $u_{22} = E(U_2^2(n))$.

It follows that

$$u_{11} + u_{22} < p_1 + q_1 + p_2 + q_2 \leq 1$$

and that the sequence $\{E(A^2_{n-j} + B^2_{n-j}),\ j = 0,1,2,\ldots\}$ is the decreasing one with a lower bound zero.

The existence of the solution

$$W_n \equiv \sum_{j=0}^{\infty} A_{n-j}V_{n-j-1}E_{n-j-1} + V_nE_n$$

has been proved.

The solution W_n is obviously σ_n-measurable. To prove that $\{W_n\}$ is stationary, it will be sufficient to prove that $E(W_n^2)$ exists. In order to do that, we shall set the following:

$$W_{n,k} = \sum_{j=0}^{k} A_{n-j}V_{n-j-1}E_{n-j-1} + V_nE_n.$$

Then

$$\begin{aligned} E(W_n^2) &= \lim_{k\to\infty} E(W_{n,k}^2) = \lim_{k\to\infty} E(W_{n,k} - X_n)^2 \\ &+ 2\lim_{k\to\infty}\sum_{j=0}^{k} E(X_n A_{n-j} V_{n-j-1} E_{n-j-1} + X_n V_n E_n) - \lim_{k\to\infty} E(X_n^2) \\ &= 2\lim_{k\to\infty}\sum_{j=0}^{k} E(X_n A_{n-j} V_{n-j-1} E_{n-j-1} + X_n V_n E_n) - 2\lambda^{-2}. \end{aligned}$$

Now, we can consider $E(X_n W_{n,k})$ only:

$$E(X_n W_{n,k}) = \sum_{j=0}^{k} E(X_n A_{n-j} V_{n-j-1} E_{n-j-1}) + E(X_n V_n E_n).$$

It is easy to verify that

$$E(X_n V_n E_n) = \frac{u_{13} + u_{23} + 2u_{33}}{\lambda^2}$$

where $u_{13} = E(U_1(n)V_n)$, $u_{23} = E(U_2(n)V_n)$, $u_{33} = E(V_n^2)$.

But, if we set $a_j = E(X_n A_{n-j} V_{n-j-1} E_{n-j-1})$, we can find out that

$$\begin{aligned} a_j &= \lambda^{-1} E(A_{n-j} A_{n-j-1} V_{n-j-1} X_{n-j-2}) \\ &+ \lambda^{-1} E(A_{n-j} B_{n-j-1} V_{n-j-1} X_{n-j-3}) \\ &+ \lambda^{-2} E(A_{n-j} V_{n-j-1} V_n) + \sum_{i=0}^{j} b_{ij} \end{aligned}$$

where $b_{ij} = E(A_{n-i} A_{n-j} V_{n-i-1} V_{n-j-1} E_{n-i-1} E_{n-j-1})$.

So, for $j = 0$

$$a_0 = \frac{u_{11}u_{13} + u_{11}v_1 + u_{11}u_{23} + 2u_{11}u_{22}}{\lambda^2},$$

and for $j > 0$, i.e. $j = 1, 2, \ldots, k$,

$$a_j = \frac{(u_{13} + u_{23})e_{j2}}{\lambda^2} + \frac{v_1^2 e_j}{\lambda^2} + \sum_{i=0}^{j} b_{ij},$$

where $v_1 = E(V_n)$, $e_j = E(A_{n-j})$ and $e_{j2} = E(A_{n-j}^2)$. Specially, b_{ij} will be as follows:

If $i = j$

$$b_{ii} = \frac{2u_{33}}{\lambda^2} e_{i2}.$$

If $j > i$, i.e. $j = i + l$ for some $l = 1, 2, \ldots, k$,

$$b_{i,i+l} = \frac{v_1^2}{\lambda^2} y_{il},$$

where

$$y_{il} = \frac{e_{i2}}{R} \left[\left(\frac{u_1 + R}{2} \right)^{l+1} - \left(\frac{u_1 - R}{2} \right)^{l+1} \right]$$

and $u_1 = E(U_1(n))$, $u_2 = E(U_2(n))$, $v_1 = E(V_n)$ for any integer n and $R = \sqrt{u_1^2 + 4u_2}$.

As it is the fact that

$$\left| \frac{u_1 - R}{2} \right| < 1$$

and

$$0 < \frac{u_1 + R}{2} < 1,$$

it follows that

$$\lim_{k \to \infty} \sum_{j=0}^{k} a_j = 0.$$

The existence of the second moment of W_n has been proved.

The solution W_n is strictly stationary. Really, the mean square convergence of the σ_n-measurable random variables $W_{n,k}$ with $k \to \infty$, implies probability convergence to the random variable W_n. Since the solution has the same functional form for each n, $\{W_n\}$ must be strictly stationary, as must be $\{X_n\}$.

The sequence of random vectors $\{\mathbf{Y}_n\}$ is an ergodic sequence since it is the sequence of i.i.d. random vectors. The σ-field $\sigma_{n,X}$, generated by $\{X_n, X_{n-1}, \ldots\}$, is such that $\sigma_{n,X} \subset \sigma_n$ if $\{X_n\}$ is a σ_n-measurable sequence. Let σ_X be the smallest σ-field containing $\lim_{n\to\infty} \sigma_{n,X}$ and σ be the smallest σ-field containing $\lim_{n\to\infty} \sigma_n$, then $\sigma_X \subset \sigma$ and it follows that $\{W_n\}$ is ergodic.

The last proposition we need to prove is the uniqueness of the stationary, σ_n-measurable solution W_n.

If we suppose the opposite, i.e. that there exist two different σ_n- measurable, stationary solutions W_n and G_n of the equation (4.1), we can set $H_n = W_n - G_n$. As it will be true

$$W_n = U_1(n) W_{n-1} + U_2(n) W_{n-2} + V_n E_n$$

and

$$G_n = U_1(n)G_{n-1} + U_2(n)G_{n-2} + V_nE_n,$$

then

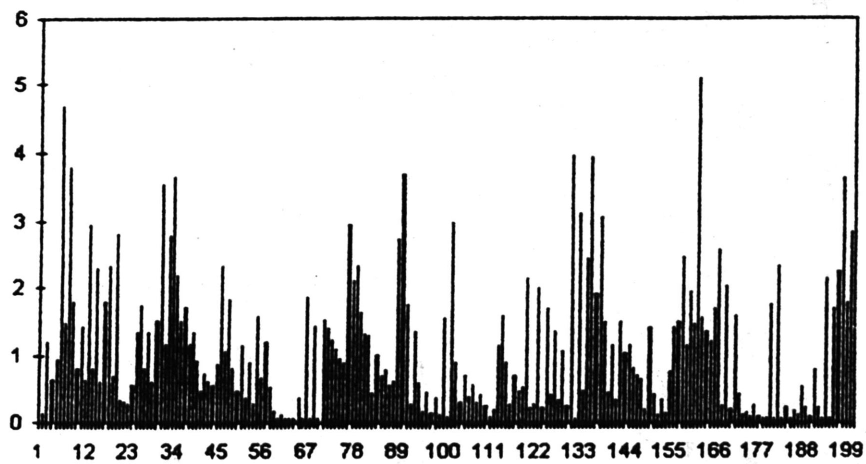
$$H_n = U_1(n)H_{n-1} + U_2(n)H_{n-2},$$

where H_n is also σ_n-measurable and stationary. It follows that

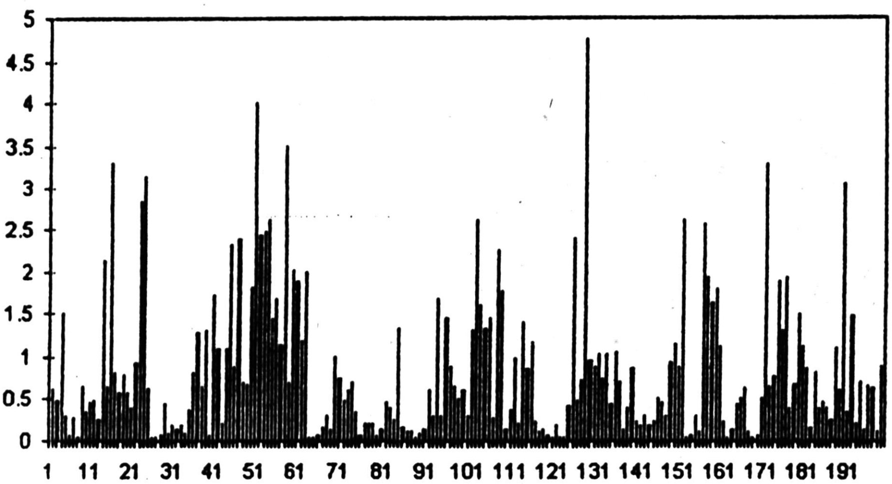
$$E(H_n^2) = E(U_1^2(n))E(H_{n-1}^2) + E(U_2^2(n))E(H_{n-2}^2).$$

Because of the stationarity of $\{H_n\}$, it will be $E(H_n^2) = E(H_{n-1}^2) = E(H_{n-2}^2) = h_{22}$. So, $h_{22}(1 - u_{11} - u_{22}) = 0$ and $1 - u_{11} - u_{22} > 0$. It means that $h_{22} = 0$ and consequently $P\{H_n = 0\} = 1$. ∎

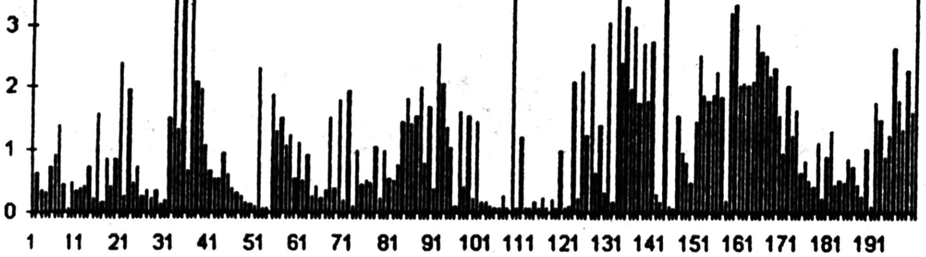
7. Some simulations



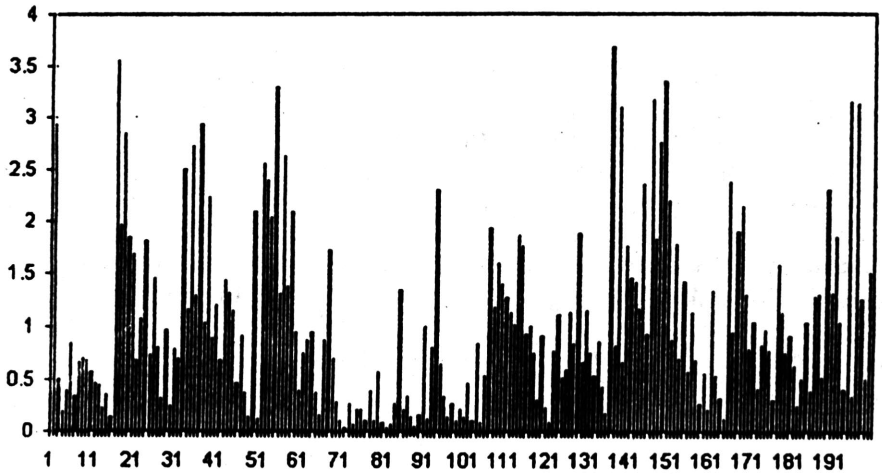
Simulated sample for the GAREX(2) process, where $p_0 = 0.1$, $p_1 = 0.1$, $p_2 = 0.4$, $q_1 = 0.2$, $q_2 = 0.2$, $\alpha_1 = 0.3$, $\alpha_2 = 0.78$, $\beta_1 = 0.4$, $\beta_2 = 0.8$



Simulated sample for the GAREX(2) process, where $p_0 = 0.1$, $p_1 = 0.1$, $p_2 = 0.1$, $q_1 = 0.3$, $q_2 = 0.4$, $\alpha_1 = 0.2$, $\alpha_2 = 0.58$, $\beta_1 = 0.6$, $\beta_2 = 0.8$



Simulated sample for the FAREX(2) process, where $p_1 = 0.04$, $p_2 = 0.26$, $q_1 = 0.3$, $q_2 = 0.4$, $\alpha_1 = 0.1$, $\alpha_2 = 0.5$, $\beta_1 = 0.2$, $\beta_2 = 0.8$



Simulated sample for the SAREX(2) process, where $p_0 = 0.1$, $p_1 = 0.2$, $p_2 = 0.3$, $q_1 = 0.2$, $q_2 = 0.2$, $\alpha_1 = 0.4$, $\alpha_2 = 0.8$

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