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Boundedness of the Solutions of Impulsive Differential Equations with “Supremum”

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Sufficient conditions for uniform boundedness and for uniform-ultimate boundedness of solutions of impulsive differential equations with "supremum" are found with the help of piecewise continuous analogues of the Lyapunov functions.

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1. Introduction

One of the main problems in the qualitative theory of differential equations is finding sufficient conditions for boundedness of the solutions. The investigation of this problem can be conducted with the help of different methods one which is the direct Lyapunov method [3], [4], [7].

In the qualitative theory of impulsive differential equations the question of using Lyapunov functions for investigation of boundedness of their solutions arises also [5].

In the present paper we study the boundedness of the solutions of impulsive differential equations by means of Lyapunov‘s method.

Differential equations with "supremum" are adequate mathematical model of different real processes. They find application, for example, in the theory of automatic regulation [6]. As a simple example of mathematical simulation by means of such equations we will consider the system for regulation of the voltage of a generator of constant current. The object of regulation is a generator of constant current with parallel simulation and quantity regulated is voltage on the clamps on the generator feeding an electric circuit with different loads. The equation describing the work of the regulator has the form ([6])

$$T_0 u'(t) + u(t) + q \max_{s\in[t-h,t]} u(s) = f(t),$$

where T_0 and q are costants characterizing the object, $u(t)$ is the voltage regulated and $f(t)$ is the perturbed effect.

We note that in the case when the solution of the considered equation is continuous then the equation is called equation with "maxima" [1].

2. Preliminary notes and definitions

Consider the initial value problem for the nonlinear impulsive differential equations with "supremum" (IVP)

$$x' = f(t, x(t), \sup_{s\in[t-h,t]} x(s)) \quad \text{for} \quad t \geq t_0, \quad t \neq \tau_k(x(t)) \tag{1}$$

$$x(t + t_0) = \varphi(t) \quad \text{for} \quad t \in [-h, 0], \tag{2}$$

$$x(t + 0) = x(t - 0) + I_k(x(t)) \quad \text{for} \quad t = \tau_k(x(t)) \tag{3}$$

where $x \in \mathbb{R}^n$, $f : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\varphi : [-h,0] \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1,2,3,\ldots$, $t_0 \geq 0$, $h = const > 0$, $\tau_k : \mathbb{R}^n \to (0,\infty)$.

Introduce the following notation

$$\sigma_k = \{(t,x) \in (0,\infty) \times \mathbb{R}^n : t = \tau_k(x)\}, \quad k = 1,2,3,\ldots$$

We denote the solution of the IVP (1), (2), (3) by $x(t; t_0, \varphi)$ and by $J(t_0, \varphi)$ – the maximal interval in which $x(t; t_0, \varphi)$ is defined. We will make a description of the solution $x(t; t_0, \varphi)$ of the IVP (1), (2), (3):

(a) For $t \in [t_0 - h, t_0]$ the solution $x(t; t_0, \varphi)$ coincides with the function $\varphi(t - t_0)$.

(b) Let $t_1 < t_2 < \ldots < t_j < \ldots$ be the moments at which the integral curve of the solution of the IVP (1), (2), (3) meets the hyperfaces σ_k, $k = 1,2,\ldots$, i.e. t_l is a solution of the equation $t = \tau_{k_j}(x(t; 0, \varphi))$ for $l = 1,2,\ldots$.

Then for $t \in [t_0, t_1]$ the solution $x(t; t_0, \varphi)$ coincides with the solution $X_1(t; t_0)$ of the differential equations with "supremum"

$$x' = f(t, x(t), \sup_{s\in[t-h,t]} x(s))$$

with an initial condition

$$x(t + t_0) = \varphi(t) \quad \text{for } t \in [-h, 0].$$

For $t \in (t_1, t_2]$ the solution $x(t; t_0, \varphi)$ coincides with the solution $X_2(t; t_1)$ of the initial value problem

$$x' = f(t, x(t), \sup_{s\in[t-h,t]} x(s))$$

$$x(t+t_1) = \varphi_1(t) \quad \text{for} \quad t \in [-h, 0],$$

where $\varphi_1(t) = X_1(t+t_1; t_0)$ for $t \in [-h, 0)$ and $\varphi_1(0) = X_1(t_1; t_0) + I_1(X_1(t_1; t_0))$ and so on. Therefore the function $x(t; t_0, \varphi)$ is piecewise continuous in $J(t_0, \varphi)$.

Denote by $PC(X, Y)$ $(X \subset \mathbb{R}^n, Y \subset \mathbb{R}^n)$ the set of all functions $u : X \to Y$ which are piecewise continuous in X with points of discontinuity of the first kind at the points $t_k \in X$ and which are continuous from the left at the points $t_k \in X$, $u(t_k) = u(t_k - 0)$.

We denote by $PC^1(X, Y)$ the set of all functions $u \in PC(X, Y)$ which are continuously differentiable for $t \in X$, $t \neq t_k$.

Introduce the following conditions:

H1. $f \in C([0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$.

H2. The functions $I_k : R^n \to \mathbb{R}^n$, $k = 1, 2, \ldots$, are such that the inequality $\|x + I_k(x)\| < H$ holds if $\|x\| \leq H$ and $I_k(x) \neq 0$, where $H = const > 0$.

H3. $\tau_k \in C(\mathbb{R}^n, (0, \infty))$, $k = 1, 2, \ldots$ are such that $0 < \tau_1(x) < \tau_2(x) < \tau_3(x) < \ldots$ for $x \in \mathbb{R}^n$ and $\lim_{k \to \infty} \tau_k(x) = \infty$ uniformly in $x \in \mathbb{R}^n$.

H4. $J(t_0, \varphi) = [t_0 - h, \infty)$ for $t_0 > 0$ and $\varphi \in PC([-h, 0], \mathbb{R}^n)$.

We will note that for impulsive differential equations with variable impulsive perturbations it is possible that so called "beating" of the solution occurs, i.e. a phenomenon where the integral curve $(t, x(t; t_0, \varphi))$ meet several or infinitely many times one and the same hyperface. In the present paper we shall consider problem of type (1), (2), (3) for which "beating" of the solutions is absent. We note that sufficient conditions for absence of phenomenon "beating" of different type of impulsive differential equations are given in [2].

Introduce the following condition:

H5. The integral curve of each solution of IVP (1),(2), (3) meets for $t > t_0$ successively each one of the hyperfaces $\tau_k, k = 1, 2, \ldots$ not more than once.

Defintion 1. The solutions of the initial value problem (1),(2),(3) are said to be uniformly bounded, if for every $\alpha > 0$ and for any $t_0 \geq 0$, there exists $\beta = \beta(\alpha) > 0$ such that for each $\varphi \in PC([-h, 0], \mathbb{R}^n)$ such that $\sup\{\|\varphi(t)\| : t \in [-h, 0]\} < \alpha$ the inequality $\|x(t; t_0, \varphi)\| < \beta$ holds for $t > t_0$.

Definition 2. The solutions of the initial value problem (1), (2),(3) are said to be uniform-ultimately bounded, if there exists $B > 0$ such that for every $\alpha > 0$ and $t_0 \geq 0$ there exits $T = T(\alpha) > 0$ such that for each $\varphi \in PC([-h, 0], R^n)$ such that $\sup\{\|\varphi(t)\| : t \in [-h, 0]\} < \alpha$ the inequality $\|x(t; t_0, \varphi)\| < B$ holds for $t > t_0 + T$.

We will say that the function $V(t,x):[0,\infty)\times\mathbb{R}^n\to[0,\infty)$ belongs to the class W, if $V(t,x)$ is a continuous function and $V(t,x)$ is locally Lipschitz in x.

We define the derivative of the function $V(t,x)$ along the trajectory of the initial value problem (1),(2),(3) as follows

$$(4)\qquad V'(t,\psi(0))=\limsup_{\epsilon\to 0+}(1/\epsilon)\{V(t+\epsilon,x(t+\epsilon;t,\psi))-V(t,\psi(0))\},$$

where $\psi\in PC([-h,0],\mathbb{R}^n)$.

We will say that the function $a(t)$ belongs to the class K if $a\in C([0,\infty),[0,\infty))$, $a(0)=0$, and $a(t)$ is an increasing function.

3. Main results

We will obtain sufficient conditions for uniform boundedness and for uniform-ultimate boundedness of the solutions of the initial value problem (1),(2), (3).

Theorem 1. *Let the following conditions hold:*

1. The conditions (H) are fulfilled.

2. There exists a function $V\in W$ such that

(i) $a(\|x\|)\le V(t,x)\le b(\|x\|)$ for $(t,x)\in[0,\infty)\times\mathbb{R}^n$, where $a,b\in K$, $\lim_{s\to\infty}a(s)=\infty$;

(ii) There exists a function $p\in K$ such that for each function $\psi\in PC([-h,0],\mathbb{R}^n)$ for which $\|\psi(0)\|>H$ and for $t>0$, $t\ne\tau_k(\psi(0))$, $k=1,2,3\ldots$ and $p(V(t,\psi(0)))>\sup\{V(t+s,\psi(s)):s\in[-h,0]\}$, the inequality $V'(t,\psi(0))<0$ is valid;

(iii)$V(t_k+0,x+I_k(x))<V(t_k,x)$ for $t_k=\tau_k(x)$, $\|x\|>H$ and $I_k(x)\ne 0$.

Then the solutions of the IVP (1), (2), (3) are uniformly bounded.

Proof. Let $\alpha>0$ be an arbitrary constant, $t_0\ge 0$ be an arbitrary point and $\varphi\in PC([-h,0],\mathbb{R}^n)$ is such that $\sup\{\|\varphi(s)\|:s\in[-h,0]\}<\alpha$.

We will assume that the integral curve $(t,x(t;t_0,\varphi))$ of the solution of IVP (1), (2), (3) meets the hyperface σ_k, $k=1,2,3,\ldots$ at the point t_k, where $t_1<t_2<t_3<\ldots$, i.e. $(t_k,x(t_k;t_0,\varphi))\in\sigma_k$. From the condition H3 it follows that $\lim_{k\to\infty}t_k=\infty$.

Consider the following two cases:

Case A. Let $\alpha>H$.

Introduce the notations $x(t)=x(t;t_0,\varphi)$ and $V(t)=V(t,x(t))$. It follows from the properties of the functions $a(t)$ and $b(t)$ that there exists a constant $\beta=\beta(\alpha)>0$ such that $b(\alpha)<a(\beta)$, $\beta>\alpha$.

Let $t \in [t_0 - h, t_0]$. The condition (i), the initial condition (2) and the choice of the function φ imply the inequalities

$$a(\|x(t;t_0,\varphi)\|) = a(\|\varphi(t-t_0)\|) \leq V(t,\varphi(t-t_0)) \leq b(\|\varphi(t-t_0)\|) \leq b(\alpha) < a(\beta).$$

Therefore,

$$\|x(t;t_0,\varphi)\| < \beta \quad \text{for} \quad t \in [t_0 - h, t_0]. \tag{5}$$

We will prove that

$$V(t) \leq a(\beta) \quad \text{for} \quad t > t_0. \tag{6}$$

Suppose the contrary, i.e. there exists a point $t > t_0$ such that $V(t) > a(\beta)$. Introduce the notation

$$t^* = \inf\{t > t_0 : V(t) > a(\beta)\}.$$

The inequality $t^* \neq \tau_k(x(t^*;t_0,\varphi)$ for $k = 1,2,3,\ldots$ is valid or if there exists an integer number j such that $t^* = \tau_j(x(t^*;t_0,\varphi)$ then $I_j(x(t^*;t_0,\varphi) = 0$. Indeed, if we suppose that there exists a positive integer m such that $t^* = t_m$ and $I_m(x(t_m)) \neq 0$, then the following two cases can be distinguished:

Case A1. Let $\|x(t_m)\| > H$. Then it follows from condition (iii) that

$$V(t_m + 0) < V(t_m) \leq a(\beta). \tag{7}$$

The inequality (7) contradicts the choice of the point t^*.

Case A2. Let $\|x(t_m)\| \leq H$. We have from condition H2 of the theorem that

$$\|x(t_m) + I_m(x(t_m))\| < H. \tag{8}$$

From condition (i) we obtain that

$$V(t_m) \leq b(\|x(t_m)\|) \leq b(H) \leq b(\alpha) < a(\beta).$$

The choice of the point t_m implies the inequality $V(t_m+0) \geq a(\beta)$. From condition (i) we conclude that

$$b(\|x(t_m) + I_m(x(t_m))\|) \geq V(t_m + 0) \geq a(\beta) > b(\alpha). \tag{9}$$

Inequality (9) implies

$$\|x(t_m) + I_m(x(t_m)\| > \alpha \geq H$$

which is a contradiction with (8).

Therefore $t^* \neq t_k$, $k = 1, 2, 3, \ldots$ or the function $x(t)$ is continuous at the point t^*.

It follows from the continuity of the functions $x(t)$ and $V(t)$ that the function $V(t)$ possesses the following properties:

(P1) $V(t^*) = a(\beta)$;

(P2) $V(s) \leq a(\beta)$ for $t_0 - h \leq s < t^*$;

(P3) There exists a sequence $\{\gamma_m\}$ such that $\gamma_m > t^*$ and $\lim_{m\to\infty} \gamma_m = t^*$ and $V(\gamma_m) > a(\beta)$.

We conclude from the above properties of the function $V(t)$ that

$$V'(t^*) \geq 0. \tag{10}$$

Condition (*i*) and the property (*P1*) imply that

$$b(\|x(t^*)\|) \geq V(t^*) = a(\beta) > b(\alpha). \tag{11}$$

From the properties of the function $b(t)$ and the inequality (11) it follows the inequality

$$\|x(t^*)\| > \alpha \geq H. \tag{12}$$

According to properties (*P1*) and (*P2*) we get

$$V(t^*, x(t^*)) = V(t^*) = \sup\{V(s) : s \in [t^* - h, t^*]\}.$$

We define the function $\psi(t) \in PC([-h, 0], \mathbb{R}^n)$ by the equality $\psi(t) = x(t^* + t; t_0, \varphi)$ for $t \in [-h, 0]$.

From the inequality (12) it follows that $\|\psi(0)\| > H$ and $p(V(t, \psi(0))) > \sup\{V(t + s, \psi(s)) : s \in [-h, 0]\}$. Having in mind condition (*ii*) we conclude that

$$V'(t^*, \psi(0)) = V'(t^*, x(t^*; t_0, \varphi)) = V'(t^*) < 0. \tag{13}$$

The inequality (13) contradicts the inequality (10).

It follows that the inequality (6) is valid for $t > t_0$. From the inequality (6) and the condition (*i*) we conclude that $\|x(t)\| \leq \beta$ for $t > t_0$.

Case B. Let $\alpha < H$. Then we have the following two cases:

Case B1. Let $\|x(t; t_0, \varphi)\| < H$ for $t > t_0$. Then the solutions of the IVP (1),(2),(3) are uniformly bounded with a constant $\beta = H$.

Case B2. Let there exists a point $t > t_0$ such that $||x(t; t_0, \varphi)|| \geq H$. Denote $\eta = \inf\{t > t_0 : ||x(t; t_0, \varphi)|| \geq H\}$ and consider the solution $x(t; \eta, \psi)$ for $t > \eta$, where $\psi(t) = x(t + \eta; t_0, \varphi)$ for $t \in [-h, 0]$, $\sup\{||\psi(t)|| : t \in [-h, 0]\} \leq H$ and $||x(t; t_0, \varphi)|| < H$ for $t \in [t_0, \eta)$. It follows from Case A that for $\alpha = H$ and $\varphi = \psi$ there exists a constant $\beta = \beta(H) > 0$ such that $||x(t; \eta, \psi)|| < \beta$ for $t > \eta$. Therefore the solutions of the IVP (1),(2),(3) are uniformly bounded with a constant $H_1 = max(H, \beta(H))$. ■

We will obtain sufficient conditions for uniform-ultimate boundedness of the solutions of IVP (1),(2),(3).

Theorem 2. *Let the following conditions hold:*

1. The conditions (H) are fulfilled.

2. There exists a function $V \in W$ which satisfies condition (i) and (iii) of Theorem 1.

3. There exists a function $p \in K$ such that for each function $\psi \in PC([-h, 0], \mathbb{R}^n)$ for which $||\psi(0)|| > H$ and for $t > 0$, $t \neq \tau_k(\psi(0)), k = 1, 2, 3 \ldots$ and $p(V(t, \psi(0))) > \sup\{V(t+s, \psi(s)) : s \in [-h, 0]\}$, the inequality $V'(t, \psi(t)) < -c(||\psi||)$ is valid where $c \in K$.

Then the solutions of the IVP (1),(2),(3) are uniform-ultimately bounded.

P r o o f. From the properties of the function $a(t)$ it follows that there exists a constant $B > 0$ such that $a(B) = b(H)$.

Let $\alpha > 0$ be an arbitrary number, $t_0 \geq 0$ be an arbitrary point and the function $\varphi \in C([-h, 0], \mathbb{R}^n)$ is such that $\sup\{||\varphi(s)|| : s \in [-h, 0]\} < \alpha$.

Consider the following two cases:

Case A. Let $\alpha \geq H$. Having in mind the proof of Theorem 1, it is easy to verify that there exists a constant $\beta = \beta(\alpha) > 0$ such that

$$V(t) < a(\beta) \quad \text{for } t \geq t_0. \tag{14}$$

Case A1. Let $\beta \leq B$. Then (14) and condition (i) imply that

$$a(||x(t; t_0, \varphi)||) \leq V(t, x(t; t_0, \varphi)) < a(\beta) \leq a(B), \qquad t \geq t_0. \tag{15}$$

We conclude from (15) and the properties of the function $a(t)$ that

$$||x(t; t_0, \varphi)|| < B \quad \text{for} \quad t \geq t_0. \tag{16}$$

The inequality (16) proves that the solutions of IVP (1), (2), (3) are uniform-ultimately bounded.

Case A2. Let $\beta > B$. Then $a(\beta) > a(B)$ and consider the interval $\Delta = [a(B), a(\beta)]$. Introduce the notation

$$A = \inf\{p(s) - s : s \in \Delta\} > 0. \tag{17}$$

Let N be the smallest integer such that $N \geq (a(\beta) - a(B))/A$. Define the points $\tau_j = t_0 + j(h + 2A/c(H))$, $j = 0, 1, \ldots, N$. We will prove the validity of the inequality

$$V(t) < a(B) + (N - j) \quad \text{for } t > \tau_j, \quad j = 0, 1, \ldots, N. \tag{18}$$

Let $j = 0$. Then $\tau_0 = t_0$ and the choice of N as well as the inequality (14) imply that

$$V(t) < a(\beta) \leq a(B) + NA = a(B) + (N - 0)A \quad \text{for } t > t_0. \tag{19}$$

Therefore the inequalitiy (18) is fulfilled for $j = 0$.

Suppose that (18) holds true for $j = 0, 1, \ldots, l$ $(l < N)$. We will prove that it is also fulfilled for $j = l + 1$.

First we will show that there exists a point $\eta \in [\tau_l + h, \tau_{l+1}]$ at which the inequality

$$V(t) < a(B) + (N - l - 1)A \tag{20}$$

is valid.

Suppose the contrary, i.e. that the inequalities

$$a(B) + (N - l - 1)A \leq V(t) < a(B) + (N - l)A \tag{21}$$

are fulfilled for $t \in [\tau_l + h, \tau_{l+1}]$.

From condition (i), inequalities (21) and the choice of B, we get that

$$b(\|x(t; t_0, \varphi\|)) \geq V(t) > a(B) + (N - l - 1)A \geq a(B) = b(H). \tag{22}$$

It follows from the inequality (22) that

$$\|x(t; t_0, \varphi)\| > H \quad \text{for} \quad t \in [\tau_l + h, \tau_{l+1}]. \tag{23}$$

The properties of the function $p(t)$, the choice of the number A, (21) and the inequalities $a(B) < V(t) < a(\beta)$ imply that

$$\begin{aligned} p(V(t)) &\geq V(t) + A > a(B) + (N - l)A \\ &\geq \sup\{V(t + s) : s \in [-h, 0]\} \quad \text{for} \quad t \in [\tau_l + h, \tau_{l+1}]. \end{aligned} \tag{24}$$

From condition 3 of Theorem 2, inequalities (23), (24) and the properties of the functions of the class K, it follows that

$$V'(t) \le -c(\|x(t;t_0,\varphi)\|) < -c(H) \quad \text{for } t \in [\tau_l + h, \tau_{l+1}]. \tag{25}$$

Choose a point $\xi \in [\tau_l + h + \beta, \tau_{l+1}]$, where $\beta = (\tau_{l+1} - \tau_l - h)/2 = A/c(H)$.We have from (21),(22), condition (ii) and the inequality $\xi - \tau_l - h > \beta$ that

$$\begin{aligned} V(\xi) &= V'(\gamma)(\xi - \tau_l - h) + V(\tau_l + h) \\ &< -c(H)(\xi - \tau_l - h) + V(\tau_l + h) \le a(B) + (N - l - 1)A, \end{aligned} \tag{26}$$

where $\gamma \in (\tau_l + h, \xi)$.

Inequality (26) contradicts (21). Therefore there exists a point $\eta \in [\tau_l + h, \tau_{l+1}]$ at which (20) holds true. We will prove that (20) is valid for $t > \eta$. Suppose the contrary and denote

$$t^{**} = \inf\{t : t \ge \eta, V(t) \ge a(B) + (N - l - 1)A\}.$$

Having in mind the choice of the points t^* and t^{**} as well as the continuity of $V(t)$ we conclude that the function $V(t)$ possesses the following properties:

$(P1)$ $V(t^{**}) = a(B) + (N - l - 1)A$

$(P2)$ $V(s) < a(B) + (N - l - 1)A$ for $\eta \le s < t^{**}$.

$(P3)$ There exists a sequence $\{\eta_m\}$ such that $\eta_m > t^{**}$ and $\lim_{m\to\infty} \eta_m = t^{**}$ and $V(\eta_m) \ge a(B) + (N - l - 1)A$.

It follows from the above properties of the function $V(t)$ that

$$V'(t^{**}) \ge 0. \tag{27}$$

We have from conditions (i) and the choice of the point t^{**} and the constant β that

$$b(\|x(t^{**};t_0,\varphi)\|) \ge V(t^{**}) \ge a(B) + (N - l - 1)A > a(\beta) = b(H). \tag{28}$$

The inequality (28) and the monotonicity of $b(t)$ imply that

$$\|x(t^{**};t_0,\varphi)\| > H. \tag{29}$$

Then from the properties (P1) and (P2) of the function $V(t)$ and the properties of $p(t)$ we get that

$$p(V(t^{**})) > V(t^{**}) = a(B) + (N - l - 1)A > V(s) \quad \text{for } s \in [\eta, t^{**}]. \tag{30}$$

Inequalities (30), (31) and the condition 3 lead to

$$V'(\|x(t^{**}; t_0, \varphi)\|) \leq -c(\|x(t^{**}; t_0, \varphi)\|) < 0. \tag{31}$$

Inequality (31) contradicts (27).

The contradiction obtained shows that the inequality (20) is valid for $t > \eta$ and therefore for $t > \tau_{l+1}$.

Thus, we proved the validity of the inequalitiy (18).

Set $T = \tau_N - \tau_0 = N(h + 2A/c(H)) > 0$. Note that T depends only on α and not on t_0 .

We have from condition (i) and inequality (18) for $j = N$ that

$$a(\|x(t; t_0, \varphi)\|) \leq V(t) < a(B) \quad \text{for } t > \tau_N = t_0 + T. \tag{32}$$

Therefore $\|x(t; t_0, \varphi)\| < B$ for $t > t_0 + T$ which proves that the solution of IVP (1), (2), (3) are uniform-ultimately bounded.

Case B. Let $\alpha < H$. The proof of this case is analogous to the proof of Case B of Theorem 1. ■

As a consequence of Theorem 2 we obtain the following result

Theorem 3. *Let the following conditions hold:*

1. The conditions (H) are fulfilled.

2. There exist a function $V \in W$ which satisfies the conditions (i) and (iii) of Theorem 1.

3. There exists a function $p \in K$ such that for each function $\psi \in PC([-h, 0], \mathbb{R}^n)$ for which $\|\psi(0)\| > Q$ and for $t > 0$, $t \neq \tau_k(\psi(0))$, $k = 1, 2, 3, \ldots$ and $p(V(t, \psi(0))) > \sup\{V(t + s, \psi(s)) : s \in [-h, 0]\}$, the inequality $V'(t, \psi(t)) < M - d(\|\psi(0)\|)$ is valid where $d \in K$, $Q = const > 0$, $M = const > 0$, $\lim_{s \to \infty} d(s) = \infty$.

Then the solutions of the IVP (1), (2), (3) are uniform-ultimately bounded.

Proof. It follows from the properties of the function $d(s)$ that there exists a constant $C = C(M) > 0$ such that $d(s) - M > 0$ for $s > C$.

Define the function $c : [0, \infty) \to [0, \infty)$ as follows $c(s) = d(s) - M$ for $s > C$ and $c(s) = (d(s) - M)s/C$ for $0 \leq s \leq C$. The function $c(s) \in K$. Then the conditions of Theorem 2 are fulfilled with a constant $H = max(C, Q)$ and therefore the solutions of IVP(1), (2), (3) are uniform-ultimately bounded. ■

Remark 1. We will note that in the case when $I_k(x) = 0$ for $k = 1, 2, 3, \ldots$ the IVP (1), (2), (3) is an initial value problem for differential equations with "maxima"

$$x'(t) = f(t, x(t), \max_{s\in[t-h,t]} x(s)) \quad \text{for } t \geq 0$$

$$x(t + t_0) = \varphi(t) \quad \text{for } t \in [-h, 0]$$

and Theorems 1, 2 and 3 give sufficient conditions for boundedness of the solutions which are continuous functions.

4. Example

Consider the initial value problem for the following scalar impulsive differential equation with "supremum"

(33) $$\,(t)x'(t) = -f(t)x^2(t) + g(t)x(t) \sup_{s\in[t-h,t]} x(s) + h(t) \quad \text{for } t \geq t_0,\ t \neq t_k$$

(34) $$x(t + t_0) = \varphi(t) \quad \text{for } t \in [-h, 0]$$

(35) $$x(t_k + 0) = (1 + c_k)x(t_k),$$

where $x \in R$, $0 < t_1 < t_2 < \ldots$, $c_k = const$, $k = 1, 2, 3, \ldots$.

Let the following conditions hold:

A1. $f \in C([0, \infty), (0, \infty))$.

A2. $g \in C([0, \infty), \mathbb{R})$ and there exist constants $L > 0$ and $q > 1$ such that $f(t) - L \geq q|g(t)|$ for $t \geq 0$.

A3. $h \in C([0, \infty), \mathbb{R})$ and there exists a constant $M > 0$ such that $|h(t)| \leq M$ for $t \geq 0$.

A4. $\lim_{k\to\infty} t_k = \infty$.

A5. $-2 < c_k < 0$, $k = 1, 2, 3, \ldots$.

A6. $\varphi \in C([-h, 0], R)$.

Consider the function $V(t, x) = x^2/2$ which is of the class W.

Define the functions $a(s) = s^2/4$, $b(s) = s^2$, $p(s) = q^2$ s and $d(s) = Ls^2$.It is easy to see that the condition (i) of Theorem 2 is fulfilled.

Let $t \geq 0$ be an arbitrary point, the function $\psi \in C([-h, 0], \mathbb{R})$, be such that $q|\psi(0)| > \sup\{|\psi(s)| : s \in [-h, 0]\}$ and $|\psi(0)| > H$. Then for $s \in [-h, 0]$ $p(V(t, \psi(0)) = q^2V(t, \psi(0)) = q^2\psi^2(0)/2 > \psi^2(s)/2 = V(s, \psi(s))$.

We have for the derivative along the trajectory of solution of IVP (33), (34), (35) that

$$
\begin{aligned}
V'(t,\psi(0)) &= -f(t)\psi^2(0) + g(t)\psi(0)\sup\nolimits_{s\in[-h,0]}\psi(s) + h(t)\\
&\le -f(t)\psi^2(0) + |g(t)|.|\psi(0)|.|\sup\nolimits_{s\in[-h,0]}\psi(s)| + |h(t)|\\
&\le -f(t)\psi^2(0) + q|g(t)|\psi^2(0) + |h(t)|\\
&\le M - L\psi^2(0) = M - d(|\psi(0)|).
\end{aligned}
\tag{36}
$$

Inequality (36) shows the validity of condition 3 of Theorem 3. From condition A5 the inequality

$$V(t_k+0, x + c_k x) = x^2(1+c_k)^2/2 < x^2/2, \quad x \neq 0.$$

Let $|x| \le H$. Then the following inequalities are fulfilled

$$|x + c_k x| = |x|.|1+c_k| < H.|1+c_k| < H.$$

Therefore the conditions H2 are fulfilled.

We conclude by Theorem 3 that if conditions (A) are fulfilled, then the solutions of the IVP (33), (34), (35) are uniform-ultimately bounded.

It follows from the inequality (36) that if $H > (M/L)^{1/2}$ then all conditions of Theorem 1 hold and therefore the solutions of IVP (33), (34), (35) are uniformly bounded.

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