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More Colombeau Products of Distributions

Blagovest Damyanov

Presented by V. Kiryakova

By 'Colombeau product of distributions' is meant the product of some distributions as they are embedded in Colombeau algebra, whenever the result can be evaluated in terms of distributions again. This paper, being a continuation of [3], is devoted to some further results on particular products in the Colombeau algebra for distributions with coinciding point singularities. Following a known result of Mikushinski, we also study in dimension one a variety of sums of distributional products existing as Colombeau product.

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1. Introduction and basic definitions

The differential $\mathbb{C}$-algebra $\mathcal{G}$ of generalized functions of J.-F. Colombeau [2] contains the distribution space $\mathcal{D}'$ and has a notion of 'association', that generalizes the equality of distributions. This is particularly useful for the evaluation of some products of distributions – as they are embedded in $\mathcal{G}$ – in terms of distributions again. Here we establish a result on the Colombeau product concerning functions having discontinuity at a point.

The following well known result in dimension one was obtained by Mikushinski in [10]:

$$x^{-1}.x^{-1} - \pi^2\delta^2(x) = x^{-2}, \quad x \in \mathbb{R}. \tag{1}$$

Here, none of the products on the left-hand side exists, yet their sum can be given a meaning in $\mathcal{D}'$. Another formula of that type in dimension one – in a nonstandard approach to distribution theory – was given in [12]:

$$H.\delta'(x) + \delta^2(x) \stackrel{*}{=} \delta'(x)/2. \tag{2}$$

H denotes here the Heaviside function, and '$\stackrel{*}{=}$' stands for equality up to an infinitesimal quantity; which can be thought of as a nonstandard analogue of Colombeau association.

In the same vein, we will derive formulas in dimension one, where the individual summands do not have associated distribution, yet their sum considered as a single entity admits such a distribution. We think it relevant to name them 'products of Mikushinski type'.

Let us now recall the basic definitions of Colombeau algebra $\mathcal{G}$, restricting ourselves to the one-dimensional case. If $\mathbb{N}_0$ stands for the nonnegative integers and q is in $\mathbb{N}_0$, we put $A_q(\mathbb{R}) = \{\varphi(x) \in \mathcal{D} : \int_{\mathbb{R}} x^j \varphi(x)\, dx = \delta_{0j}$ for $0 \le j \le q$, where $\delta_{00} = 1, \delta_{0j} = 0$ for $j > 0\}$. We denote $\varphi_\varepsilon = \varepsilon^{-1}\varphi(\varepsilon^{-1}x)$ for $\varphi \in A_q(\mathbb{R})$ and $\varepsilon > 0$. Now let $\mathcal{E}[\mathbb{R}]$ be the $\mathbb{C}$-algebra of functions $f(\varphi, x) : A_0(\mathbb{R}) \times \mathbb{R} \to \mathbb{C}$ that are C^∞-differentiable with respect to x by a fixed 'parameter' φ. Each generalized function of Colombeau is then an element of the quotient algebra $\mathcal{G} = \mathcal{E}_M[\mathbb{R}] \,/\, \mathcal{I}[\mathbb{R}]$. Here the subalgebra $\mathcal{E}_M[\mathbb{R}]$ and the ideal $\mathcal{I}[\mathbb{R}]$ of $\mathcal{E}_M[\mathbb{R}]$ are sets of functions $f(\varphi, x) \in \mathcal{E}[\mathbb{R}]$ such that the derivatives $\partial_x^p f(\varphi_\varepsilon, x)$ satisfy certain asymptotic evaluations, as $\varepsilon \to 0$ [2]. Similar, yet different schemes of 'new generalized functions' were introduced by Antonevich and Radyno [1] and by Egorov [5].

The algebra $\mathcal{G}$ contains the distributions, canonically embedded as a $\mathbb{C}$-vector subspace by the map $i : \mathcal{D}' \to \mathcal{G} : u \mapsto \widetilde{u} = [\widetilde{u}(\varphi, x)]$. The representatives here are $\widetilde{u}(\varphi, x) = (u * \check{\varphi})(x)$ where $\check{\varphi}(x) = \varphi(-x)$ and φ is running the set $A_q(\mathbb{R})$.

Now a generalized function f in $\mathcal{G}$ is said to admit some u in $\mathcal{D}'$ as an 'associated distribution', which is denoted by $f \approx u$, if f has a representative $f(\varphi_\varepsilon, x)$ such that for each $\psi(x)$ in $\mathcal{D}$ there exists q in $\mathbb{N}_0$ such that, for all $\varphi(x)$ in $A_q(\mathbb{R})$,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} f(\varphi_\varepsilon, x)\psi(x)dx = \langle u, \psi \rangle. \tag{3}$$

This definition is independent of the representative chosen and the distribution associated, if it exists, is unique; the image in $\mathcal{G}$ of every distribution is associated with the latter [11, Ch.3]. The $\approx$-association is thus a generalization of the equality of distributions.

Then by 'Colombeau product of distributions' is meant the product of some distributions as they are embedded in Colombeau algebra $\mathcal{G}$ whenever the result admits an associated distribution (see [8] for comparison with other distribution products).

2. Results on Colombeau product of distributions

In what follows we shall need the next technical lemma.

Lemma 1. *For an arbitrary φ in $A_0(\mathbb{R})$, i.e. φ in $\mathcal{D}$ with $\int_{\mathbb{R}} \varphi(t)\,dt = 1$, suppose that* supp $\varphi \subseteq [a,b]$, *for some a, b in $\mathbb{R}$. Then, for any p in $\mathbb{N}_0$, it holds*

$$\int_a^b \varphi(t) \int_a^t (y-t)^p \varphi^{(p)}(y)\,dy\,dt = \frac{(-1)^p\, p!}{2}. \tag{4}$$

P r o o f. Consider first the term $J_p(t) := \int_a^t (y-t)^p \varphi^{(p)}(y)\,dy$. On multiple integrating by parts, we get:

$$\begin{aligned} J_p(t) &= (y-t)^p \varphi^{(p-1)}(y)\Big|_a^t - p \int_a^t \varphi^{(p-1)}(y)(y-t)^{p-1}\,dy \\ &= 0 + p(p-1) \int_a^t \varphi^{(p-2)}(y)(y-t)^{p-2}\,dy \\ &= 0 + (-1)^p\, p! \int_a^t \varphi(y)\,dy = (-1)^p\, p!\, J_0(t). \end{aligned}$$

Further we have, by assumption and the definition of $A_0(\mathbb{R})$,

$$\begin{aligned} \int_a^b \varphi(t)\, J_0(t)\,dt &= \int_a^b \varphi(t) \left(\int_a^t \varphi(y)\,dy \right) dt = \int_a^b \left(\int_a^t \varphi(y)\,dy \right) \\ &\times\ d\left(\int_a^t \varphi(y)\,dy \right) = \frac{1}{2} \left(\int_a^t \varphi(y)\,dy \right)^2 \Big|_a^b = \frac{1}{2}; \end{aligned}$$

Combining the two equations obtained, we thus get equation (4). ∎

We now proceed to the Colombeau product of functions having discontinuity at a point with the Dirac δ-function supported at that same point. Namely, suppose that the function $f(x)$ is continuous for all x in $\mathbb{R}$ except at the point $x = 0$, where it has discontinuity of first order, i.e. the values $f(0_+)$ and $f(0_-)$ exist but, in general, differ from each other. In distribution theory such functions can not be multiplied with a distribution having singular support that includes the point $x = 0$. Nonetheless, the Colombeau products of such functions with $\delta(x)$ exists, as shown by the following.

Theorem 1. *Let the function $f(x)$ be continuous for all x in $\mathbb{R}$ except at the point $x = 0$, where it has discontinuity of first order. Then the imbedding* *of $f(x)$ in the algebra $\mathcal{G}(R)$ satisfies*

$$\tilde{f}(x).\tilde{\delta}(x) \approx \frac{1}{2}[f(0_+)+f(0_-)]\,\delta(x). \tag{5}$$

P r o o f. For an arbitrary φ in $A_0(\mathbb{R})$, suppose that $\operatorname{supp}\varphi(x) \subseteq [a,b]$ for some a, b in $\mathbb{R}$. Then the representative of f is given by

$$\begin{aligned}\tilde{f}(\varphi_\varepsilon, x) &= (f * \check{\varphi}_\varepsilon)(x) = \frac{1}{\varepsilon}\int_{\varepsilon a+x}^{0} f(y)\varphi\left(\frac{y-x}{\varepsilon}\right) dy + \frac{1}{\varepsilon}\int_{0}^{\varepsilon b+x} f(y) \\ &\times \varphi\left(\frac{y-x}{\varepsilon}\right) dy = \int_a^{-x/\varepsilon} f(\varepsilon t+x)\varphi(t)\,dt + \int_{-x/\varepsilon}^{b} f(\varepsilon t+x)\varphi(t)\,dt.\end{aligned}$$

Further, denoting $I_\varepsilon := \langle \tilde{f}(\varphi_\varepsilon, x)\delta(\varphi_\varepsilon, x), \psi(x)\rangle$, we obtain, for each test-function $\psi(x)$,

$$I_\varepsilon = \frac{1}{\varepsilon}\int_{-b\varepsilon}^{-a\varepsilon} \psi(x)\varphi\left(-\frac{x}{\varepsilon}\right) dx \left[\int_a^{-x/\varepsilon} f(\varepsilon t+x)\varphi(t)\,dt + \int_{-x/\varepsilon}^{b} f(\varepsilon t+x)\varphi(t)\,dt\right]$$

$$= \int_a^b \psi(-\varepsilon u)\varphi(u)\,du \left[\int_a^u f(\varepsilon(t-u))\,\varphi(t)\,dt + \int_u^b f(\varepsilon(t-u))\,\varphi(t)\,dt\right],$$

where the change $-x/\varepsilon = u$ is made. Hence, by assumption, we get

$$\lim_{\varepsilon\to 0} I_\varepsilon = \psi(0)\int_a^b \varphi(u)\,du \left[f(0_-)\int_a^u \varphi(t)\,dt + f(0_+)\int_u^b \varphi(t)\,dt\right]$$

$$= \psi(0)\int_a^b \varphi(u) \quad du \quad \left[f(0_-)\int_a^u \varphi(t)\,dt + f(0_+)\int_a^b \varphi(t)\,dt - f(0_+)\int_a^u \varphi(t)\,dt\right]$$

$$= \psi(0)\left[[f(0_-)-f(0_+)]\int_a^b \varphi(u)\,du \int_a^u \varphi(t)\,dt + f(0_+)\right].$$

It is taken into account here that $f(\varepsilon(t-u))$ tends to $f(0_+)$, respectively, to $f(0_-)$, as $\varepsilon \to 0$, whenever $t > u$, respectively, $t < u$, as well as that $\int_a^b \varphi(t)\,dt = 1$. Therefore Lemma 1, by $p = 0$, yields

$$\lim_{\varepsilon\to 0} I_\varepsilon = \psi(0)\,[f(0_-)-f(0_+)]\,\frac{1}{2} + f(0_+) = \frac{1}{2}[f(0_+)+f(0_-)]\,\langle\delta,\psi\rangle;$$

which gives equation (5). ■

R e m a r k. By the derivation of (5), we have used no auxiliary requirements on the mollifiers (the regularizing functions) $\varphi(x)$, such as to be even functions of x, as proceeded in the nonstandard approach to distributions developed in [12].

Next we recall several results given in [3] and [4] that will be needed below. The following particular Colombeau products of distributions were obtained in [3].

Proposition 1. *For an arbitrary p in $\mathbb{N}_0$, let $\widetilde{\delta^{(p)}}(x)$, $\widetilde{x_+^p}$ and $\widetilde{x_-^p}$ be the embeddings in $\mathcal{G}$ of the distributions $\delta^{(p)}(x)$, $x_+^p = \{x^p$ for $x \geq 0$, $= 0$ for $x < 0\}$ and $x_-^p = \{0$ for $x > 0, = (-x)^p$ for $x \leq 0)\}$. Then it holds*

$$\text{(6)} \qquad \widetilde{x_+^p} \,.\, \widetilde{\delta^{(p)}}(x) \approx \frac{(-1)^p p!}{2}\, \delta(x), \qquad \widetilde{x_-^p} \,.\, \widetilde{\delta^{(p)}}(x) \approx \frac{p!}{2}\, \delta(x).$$

Remark. Equations (6) are known in distribution theory, although being only derived as regularized products by the particular choice of symmetric mollifiers. The equation

$$\text{(7)} \qquad x^p\, \delta^{(p+q)}(x) = \frac{(-1)^p\,(p+q)!}{q!}\, \delta^{(q)}(x) \quad (p, q \in \mathbb{N}_0),$$

can be shown to hold in $\mathcal{D}'$ [9, §3.3]. Now, since $x^p = x_+^p + (-1)^p x_-^p$ $(p \in \mathbb{N}_0)$, equations (6) are clearly consistent with (7) for $q = 0$.

The next result was proved in [4].

Proposition 2. *For an arbitrary p in $\mathbb{N}$, let $\widetilde{x^{-p}}$ and $\widetilde{\delta^{(p-1)}}(x)$ be the embeddings in $\mathcal{G}$ of the distributions x^{-p} and $\delta^{(p-1)}(x)$. Then it holds*

$$\text{(8)} \qquad \widetilde{x^{-p}} \,.\, \widetilde{\delta^{(p-1)}}(x) \approx \frac{(-1)^p (p-1)!}{2\,(2p-1)!}\, \delta^{(p-1)}(x).$$

Remark. This latter equation is derived in [6] and [7] under (different) additional requirements on the mollifiers as compared with the classical model product in $\mathcal{D}'$ [11, §2.7].

We recall further that the $\approx$-association is consistent with the C^∞–linear operations and the differentiation: $f \approx u$ implies $\partial^p f \approx \partial^p u$ and $a.f \approx a.u$ for an arbitrary p in $\mathbb{N}_0$ and a in $C^\infty(\mathbb{R})$ [11, §3.10]. However, only a weak variant of the Leibnitz rule can be applied to the derivative of Colombeau product of distributions, as shown by the next assertion proved in [4].

Theorem 2. *(a) Let the embeddings of the distributions u and v into $\mathcal{G}$ satisfy $\widetilde{u}\,.\,\widetilde{v} \approx w$, where w is in $\mathcal{D}'$. Then it holds*

$$(9) \qquad \widetilde{\partial u}\,.\,\widetilde{v} + \widetilde{u}\,.\,\widetilde{\partial v} \approx \partial w.$$

(b) If moreover $\widetilde{u}\,.\,\widetilde{\partial v}$ admits as associated distribution some w_1 in $\mathcal{D}'$, then the Colombeau product $\widetilde{\partial u}\,.\,\widetilde{v}$ exists and it holds

$$(10) \qquad \widetilde{\partial u}\,.\,\widetilde{v} \approx \partial w - w_1.$$

Remark. In general, only the sum on the left-hand side of (9) has an associated distribution, but not the individual summands in it.

3. Distributional products of Mikushinski type

We will now derive several formulas for distributional products, in which the individual summands do not have associated distribution, yet their sum considered as a single entity admits such a distribution, or else – products of Mikushinski type. If we apply the weak Leibnitz rule (9) to Colombeau product given by equation (8), we obtain the following.

Corollary 1. *For an arbitrary p in $\mathbb{N}$, the embeddings in $\mathcal{G}(\mathbb{R})$ of the distributions x^{-p} and $\delta^{(p)}(x)$ satisfy*

$$(11) \qquad \widetilde{x^{-p}}\,.\,\widetilde{\delta^{(p)}}(x) - p\,\widetilde{x^{-p-1}}\,.\,\widetilde{\delta^{(p-1)}}(x) \approx \frac{(-1)^p(p-1)!}{2(2p-1)!}\,\delta^{(2p)}(x).$$

In the particular case $p = 1$, it holds

$$(12) \qquad \widetilde{x^{-1}}\,.\,\widetilde{\delta}'(x) - \widetilde{x^{-2}}\,.\,\widetilde{\delta}(x) \approx -\frac{1}{2}\,\delta''(x).$$

Denoting now $\check{H} := H(-x)(= x_-^0)$, one easily checks that $(x_-)' = -\check{H}$ and $(\check{H})' = -\delta$. Then the result below follows again on applying the weak Leibnitz rule to equations (6).

Corollary 2. *For an arbitrary p in $\mathbb{N}_0$, it holds in $\mathcal{G}(\mathbb{R})$:*

$$(13) \qquad \widetilde{x_+^p}\,.\,\widetilde{\delta^{(p+1)}}(x) + p\,\widetilde{x_+^{p-1}}\,.\,\widetilde{\delta^{(p)}}(x) \approx (-1)^p\,\frac{p!}{2}\,\delta'(x).$$

(14) $$\widetilde{x_-^p} \,.\, \widetilde{\delta^{(p+1)}}(x) \;-\; p\,\widetilde{x_-^{p-1}} \,.\, \widetilde{\delta^{(p)}}(x) \;\approx\; \frac{p!}{2}\,\delta'(x).$$

In the case $p = 0$ of (6), equation (9) yields:

(15) $$\widetilde{H}.\widetilde{\delta'}(x) \;+\; \widetilde{\delta}^{\,2}(x) \;\approx\; \frac{1}{2}\,\delta', \qquad \widetilde{H}.\widetilde{\delta'}(x) \;-\; \widetilde{\delta}^{\,2}(x) \;\approx\; \frac{1}{2}\,\delta'(x).$$

We can derive further formulas for Mikushinski products. Indeed, equations (13) and (14), for $p = 1$, read:

$$\widetilde{x_+} \,.\, \widetilde{\delta''}(x) \;+\; \widetilde{H} \,.\, \widetilde{\delta'}(x) \;\approx\; -\frac{1}{2}\,\delta'(x), \qquad \widetilde{x_-} \,.\, \widetilde{\delta''}(x) \;-\; \widetilde{H} \,.\, \widetilde{\delta'}(x) \;\approx\; \frac{1}{2}\,\delta'(x).$$

Combining these equations with the ones in (15), we get

(16) $$\widetilde{x_+} \,.\, \widetilde{\delta''}(x) \;-\widetilde{\delta}^{\,2}(x) \;\approx\; -\,\delta'(x), \qquad \widetilde{x_-} \,.\, \widetilde{\delta''}(x) \;-\widetilde{\delta}^{\,2}(x) \;\approx\; \delta'(x).$$

Now a direct calculation proves the next result that generalizes equations (15) and (16) for each p in $\mathbb{N}_0$.

Prposition 3. *For an arbitrary p in $\mathbb{N}_0$, the embeddings in $\mathcal{G}(\mathbb{R})$ of the distributions x_+^p and $\delta^{(p+1)}(x)$ satisfy*

(17) $$(-1)^p\,\widetilde{x_+^p} \,.\, \widetilde{\delta^{(p+1)}}(x) \;+\; p!\,\widetilde{\delta}^{\,2}(x) \;\approx\; \frac{(p+1)!}{2}\,\delta'(x).$$

P r o o f. First of all, for each $\psi(x)$ in $\mathcal{D}$, we obtain on the change $-x/\varepsilon = t$:

$$\langle\,\widetilde{\delta}^{\,2}(\varphi_\varepsilon, x), \psi(x)\,\rangle \;=\; \frac{1}{\varepsilon^2}\int_{-\varepsilon b}^{-\varepsilon a} \varphi^2\left(-\frac{x}{\varepsilon}\right)\,\psi(x)\,dx = \frac{1}{\varepsilon}\int_a^b \varphi^2(t)\,\psi(-\varepsilon t)\,dt$$

(18) $$=\; \frac{\psi(0)}{\varepsilon}\int_a^b \varphi^2(t)\,dt - \psi'(0)\int_a^b t\,\varphi^2(t)\,dt + O(\varepsilon).$$

Denoting $V_p := \langle\,\widetilde{x_+^p}(\varphi_\varepsilon, x) \,.\, \widetilde{\delta^{(p+1)}}(\varphi_\varepsilon, x),\, \psi(x)\,\rangle$, we get further

$$V_p \;=\; \frac{(-1)^{p+1}}{\varepsilon^{p+2}}\int_{-b\varepsilon}^{-a\varepsilon}\left(\int_{-x/\varepsilon}^{b}(x+\varepsilon t)^p\,\varphi(t)dt\right)\,\varphi^{(p+1)}\!\left(-\frac{x}{\varepsilon}\right)\psi(x)\,dx$$

$$=\; \frac{-1}{\varepsilon}\int_a^b \psi(-\varepsilon y)\,\varphi^{(p+1)}(y)dy\int_y^b (y-t)^p\,\varphi(t)dt$$

$$
\begin{aligned}
&= -\frac{\psi(0)}{\varepsilon}\int_a^b \varphi(t)dt \int_a^t (y-t)^p\, \varphi^{(p+1)}(y)dy \\
&+ \psi'(0)\int_a^b \varphi(t)dt \int_a^t y\,(y-t)^p\, \varphi^{(p+1)}(y)dy \;+\; O(\varepsilon) \\
&=: \; -\frac{\psi(0)}{\varepsilon}J_1 \;+\; \psi'(0)\,J_2 \;+\; O(\varepsilon).
\end{aligned}
$$

On a multiple integration by parts, the integrated term being zero each time, we get

$$
J_1 = (-1)^p\, p! \int_a^b \varphi\, dt \int_a^t \varphi'(y)\, dy \;=\; (-1)^p\, p! \int_a^b \varphi^2(t)\, dt.
$$

Now Lemma 1 and a multiple integration by parts give

$$
\begin{aligned}
J_2 &= \int_a^b \varphi(t)dt \int_a^t (y-t)^{p+1}\varphi^{(p+1)}(y)dy + \int_a^b t\varphi(t)dt \int_a^t (y-t)^p\varphi^{(p+1)}(y)dy \\
&= (-1)^{p+1}\,(p+1)!\,\frac{1}{2} \;+\; (-1)^p\, p! \int_a^b t\,\varphi^2(t)dt.
\end{aligned}
$$

Hence,

$$
(-1)^p\, V_p = -p!\left(\frac{\psi(0)}{\varepsilon}\int_a^b \varphi^2(t)\, dt - \psi'(0)\int_a^b t\,\varphi^2(t)\, dt\right) - \frac{1}{2}(p+1)!\,\psi'(0) + O(\varepsilon).
$$

Taking now into account equation (18), we get

$$
\begin{aligned}
(-1)^p\,\langle\, \widetilde{x_+^p}(\varphi_\varepsilon, x)\,.\,\widetilde{\delta^{(p+1)}}(\varphi_\varepsilon, x),\; \psi(x)\,\rangle \;+\; p!\,\langle\, \widetilde{\delta}^2(\varphi_\varepsilon, x), \psi(x)\,\rangle \\
= \frac{(p+1)!}{2}\langle\, \delta'(x), \psi(x)\,\rangle \;+\; O(\varepsilon).
\end{aligned}
$$

Therefore passing to the limit, as $\varepsilon \to 0$, we obtain equation (17) for each p in $\mathbb{N}$. ■

Proposition 4. *For an arbitrary p in $\mathbb{N}_0$, the embeddings in $\mathcal{G}(\mathbb{R})$ of the distributions x_-^p and $\delta^{(p+1)}(x)$ satisfy*

$$
\widetilde{x_-^p}\,.\,\widetilde{\delta^{(p+1)}}(x) \;-\; p!\,\widetilde{\delta}^2(x) \;\approx\; \frac{(p+1)!}{2}\,\delta'(x). \tag{19}
$$

Proof. Recall that $x_-^p = (-x)_+^p$ for all p in $\mathbb{N}_0$. Thus, replacing $x \to -x$ in (17) and taking into account that $\delta^{(p+1)}(-x) = (-1)^{p+1}\delta^{(p+1)}(x)$, we obtain equation (19). ∎

Remark. Equations (17) and (19) in the algebra $\mathcal{G}(\mathbb{R})$ are easily shown to be consistent with equation (7) for $q = 1$, which holds in the distribution space $\mathcal{D}'$.

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Bulgarian Academy of Sciences
INRNE Theory Group
72, Tsarigradsko Shosse
1784 Sofia, BULGARIA
e-mail: damyanov@bg400.bg

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