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Zero-Range Interaction and Flame Front Equation

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Zero-range interaction and flame front equation are described. The approach is based on the theory of self-adjoint extensions of symmetric operators and is analogous to the zero-range potentials in quantuum mechanics.

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1. Introduction

As early as in the mid-1940s, Zel'dovich [15] proposed a qualitative explanation for why cellular flames tend to form in mixture that a deficient in the light reactant. In the framework of this model, linear analysis of the stability of a plane flame front to long-wave disturbances yields the following dispersion relation [2]:

$$\sigma = D_{th}[2^{-1}\beta(1 - Le) - 1]k^2,$$

where $\beta = E(T_b - T_u)/RT_b^2$, $Le = D_{th}/D_{mol}$ is the Lewis number of the limiting reactant, assumed to be strongly deficient, σ is the rate-of-instability parameter, $\overrightarrow{k}$ is the wave vector of the disturbance of the flame front, $k = |\overrightarrow{k}|, F \cong \exp(\sigma t + ikr)$, E is a constant, specific to the reaction and called its activation energy, R is the universal gas constant, T_u is the temperature of the unburned cold mixture, at which the reaction rate is negligibly small, T_b is the temperature of the burned gas, usually 5 to 10 times T_u; D_{mol} is the molecular diffusivity, D_{th} is the thermal diffusivity of the mixture. The flame is stable only if the mobility of the limiting reactant is sufficiently low $(Le > Le = 1 - 2/\beta)$. At $Le < Le_c$ the flame is unstable. In a typical flame, $\beta \cong 15$, and so $Le_c \cong 0.87$.

However, as it was pointed out later [13], a flame, though possibly unstable to long wave disturbances, is nevertheless always stable to short-wave disturbances. At $Le \cong Le_c$ the dispersion relation incorporating the relaxation effect of short wave disturbances is

$$\sigma = D_{th}[2^{-1}\beta(1 - Lc) - 1]\, k^2 - 4D_{th}L_{th}^2 k^4, \tag{1}$$

where L_{th} is the thermal thickness of the flame defined as D_{th}/U_u, where U_u is the propagation speed of the flame relative to the unburned gas. Hence, we obtain the following equation (in a nondimensional parameter-free form) for the function $z = F(x, y, t)$ describing the curved flame front [14]:

$$\sigma F + \triangle F + 4\,\triangle^2\, F = 0.$$

R e m a r k. This equation may be obtained from the Kuramoto-Sivashinsky one [14, 5]:

$$\sigma F + 2^{-1}(\bigtriangledown F)^2 + \triangle F + 4\,\triangle^2\, F = 0,$$

when the non-linear term is omitted.

To simulate a small obstacle for the flame or a small source, it is possible to use the approach analogous to the zero-range potential method in atomic physics [7]. Thirty years ago it was understood that to introduce a zero-range potential means to construct a self-adjoint extension of a symmetric operator [1, 9]. The mathematical approach reveals general features of different physical phenomena and leads to the expansion of the range of applications of the method (see, e.g. [10, 11, 12, 9] and references therein). That is why the consideration of laminar flame from the point of viev of the operator extension theory seems to be useful.

2. Description of the model

Let us describe a zero-range model for the equation (1). Let $l = -\triangle -4\triangle^2$, and L_0 be the corresponding operator defined on the set of smooth functions satisfying the condition:

$$D(L_0) = \{u : u \in L_2(\mathbb{R}^2),\, lu \in L^2(\mathbb{R}^2),\, u(r_0) = u'^{-}_{x_i}(r_0) = u''_{x_j x_i}(r_0) = 0\}.$$

R e m a r k. One can note that the functions from $D(L_0)$ are continuous, and their derivatives (of first or second order) are continuous too ($u \in C^2_{2,loc}$) in accordance with the imbedding theorems, that is, the boundary conditions at the point $x = 0$ are correct.

Let J_ν be the Bessel transformation:

$$\widetilde{f}_\nu(u) = (J_\nu f)(u) = \int_0^\infty \sqrt{t} f(t) \sqrt{tu} J_\nu(tu) dt.$$

Lemma 1. ([6]) *If* $f(t) = O(t^{\alpha+1/2})$ *for* $t \to 0$, $\alpha + \nu + 5/2 > 0, f(t) = O(t^{\beta+1/2})$ *for* $t \to \infty$, $\beta + 2 < 0$, *then*

$$\widetilde{f}_\nu(u) \quad (\nu > -1, u > 0)$$

exists and

$$\widetilde{f}_\nu(u) = O(u^{\alpha'+1/2}), \quad u \to 0, \alpha' + 1/2 \geq \min\{\nu, -\beta - 5/2\}.$$

$$\widetilde{f}_\nu(u) = O(u^{\beta'+1/2}), \quad u \to 0, \beta + 1/2 \geq \max\{-1/2, -\alpha - 5/2\}.$$

Lemma 2. *The closure* $\overline{L_0}$ *of the operator* L_0 *is a symmetric one with the deficiency indices (6,6).*

Proof. The fact that the operator $\overline{L_0}$ is a symmetric one is evident. To determine the deficiency indices, it is necessary to find the fundamental solution φ:

$$\Delta\varphi + 4\Delta^2\varphi + \lambda\varphi = \delta(r - r_0),$$

corresponding to a regular point λ ($k^2 = \lambda < 0$) of the operator L_0. Using the Fourier transform, one obtains

$$\varphi(r) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{z \exp(i(z, r))}{4z^4 - z^2 + k^2} dz d\vartheta,$$

where $z = |z|$. Using the well known representation of the Bessel function J_0 [3, 4], one get

$$\varphi(r) = \frac{1}{\sqrt{|r - r_0|}} \int_0^\infty \frac{\sqrt{z} J_0(z|r - r_0|)}{4z^4 - z^2 + k^2} \sqrt{z|r - r_0|}\, dz.$$

Taking into account that

$$(z^4 - 4^{-1}z^2 + 4^{-1}k^2)^{-1} = (z^4 + 4^{-1}k^2)^{-1} + 4^{-1}z^2(z^4 + 4^{-1}k^2)^{-1}(z^4 - 4^{-1}z^2 + 4^{-1}k^2)^{-1},$$

and using the table of the Bessel transformation, one comes to the following expression for φ :

$$\varphi = (32ki^{-1}H_0^{(1)}(\sqrt{ik}|r-r_0|) + H_0^{(2)}(\sqrt{-ik}|r-r_0|))$$

$$+(|r-r_0|)^{-1}(J_0(4z^{5/2}(4z^4+\lambda)^{-1}(4z^4-z^2+\lambda)^{-1}))(|r-r_0|). \quad (2)$$

■

Lemma 1 allows to describe the behaviour of the second term of the expression (2) near the point r_0. To determine the deficiency elements of the operator $\overline{L_0}$, it is necessary to list such derivatives $\varphi_{x_1x_1}^{(j_1,j_2)}$, $r=(x_1,x_2)$, that belong to the space $L_2(\mathbb{R}^2)$. The information about the behaviour of the function φ allows one to come to the conclusion that this fact takes place if and only if $j_1+j_2\le 2$. Hence, the deficiency indices are (6,6).

R e m a r k. The situation is analogous to one for the operator $\triangle^2$ ([8, 11]).

Using the von Neumann theorem one comes to the following lemma.

Lemma 3. *The domain of the adjoint operator L_0^* consists of the set of functions*

$$u(r) = \sum_{i,j=1}^{2} c_{i,j}\varphi_{x_i,x_j}(r) + \sum_{i=1}^{2} c_i\varphi_{x-i}(r) + c_0\varphi(r)$$

$$+\varepsilon(r)(a_0 - \sum_{i=1}^{2} a_ix_i + \sum_{i,j=1}^{2} a_{ij}g_{ij}x_ix_j) + u_0(r). \quad (3)$$

Here $u_0\in D(L_0), g_{ij}=1, i\neq j, g_{i,j}=2^{-1}, i,j=1,2, \varepsilon(r)$ - smooth cutting function: $\varepsilon(r)=1$, $|r|\le 1$; $\varepsilon(r)=0$, $|r|\ge 2$, φ is the fundamental solution. The function $(L_0^*u)(r)$ at the point r $(r\neq 0)$ is computed as:

$$(L_0^*)(r) = lu(r).$$

The operator $\overline{L_0}$ has self-adjoint extensions, that may be classified by the description of their domains. These domains are linear subsets of the domain adjoint operator for the elements of which the following relation is valid:

$$I(u,v) = (L_0^*u,v) - (u,L_0^*v) = 0.$$

The form $I(u,v)$ is usually called the "boundary form". Using the representation (3) and taking into account the asymptotics of the fundamental solution φ and its derivatives near the point r_0, one comes to the the following lemma.

Lemma 4.

$$I(u,v) = a_0^u c_0^v - c_0^u a_0^v + \sum_{i=1,2} (a_i^u c_i^v - c_i^u a_i^v) + \sum_{i,j=1,2} (a_{ij}^u c_{ij}^v - c_{ij}^u - a_{ij}^v).$$

It is necessary to select such linear subset of $D(L_0^*)$, that the form I vanishes on the elements of this subset, to construct a domain of self-adjoint extension of the operator L_0. It is an ordinary problem of the linear algebra in a space C^6 (the space of the "boundary vectors" U_0, U_1; $U_1 = (c_0^u, c_1^u, c_2^u, c_{11}^u, c_{12}^u, c_{22}^u)$, $U_0 = (a_0^u, a_1^u, a_2^u, a_{11}^u, a_{12}^u, a_{22}^u)$). As a result we obtain the following theorem.

Theorem. *The operator L_s (extension) is self-adjoint if and only if $L_0 \subset L_s \subset L_0^*$. Here $D(L_s)$ is such a linear subset of $D(L_0^*)$ (having no extensions) that one of the following conditions is valid for the boundary vectors of any function from the set $D(L_s)$:*

1) $U = (U_0, U_1)$, $U_0 = AU_1$, $A : C^6 \to C^6$, $A = A^*$,
2) $U_1 = AU_0$, $a : C^6 \to C^6$, $A = A^*$,
3) $U_0 = \alpha + \gamma$, $U_1 = \beta + A\gamma$,

where α, β are vectors from arbitrary orthogonal fixed subspaces N^+ and N^-, $N^- \in C^6, \gamma \in N$, $N = C^6 \ominus N^+ \ominus N^-$, $A : N \to N$, the operator A is self-adjoint and reversible.

It is this self-adjoint operator that gives us the correct mathematical description of the zero-range interaction for the flame front equation.

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