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On the Liftings to the Sheaf of Triad Homotopy Groups

Ayhan Şerbetçí

Presented by P. Kenderov

Let $(X;A,B)$ be a triad such that $A \cap B$ is a nonempty connected, locally path connected, and semilocally simple connected subspace of X. Let $S_n(X;A,B)$, $n \geq 3$, be the sheaf of triad homotopy groups of the triad $(X;A,B)$. For $n > 3$, $S_n(X;A,B)$ is also an abelian and regular covering space of $A \cap B$. In this paper, we give the relations among the fibers, sections and cover transformations of $S_n(X;A,B)$. We solve the lifting problem for paths and arbitrary continuous maps. Finally, we determine the features of the group of liftings to $S_n(X;A,B)$.

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1. Introduction

Let $(X;A,B)$ be a triad (i.e., X is a topological space and A, B are two subspaces of X such that $A \cap B \neq \emptyset$) such that $A \cap B$ is a connected, locally path connected, and semilocally simple connected subspace of X. Let $\pi_n(X;A,B,x)$ be the n-th, $n \geq 3$, triad homotopy group of the triad $(X;A,B)$ with base point in $A \cap B$. For $n > 3$, $\pi_n(X;A,B,x)$ is an abelian group. Let $S_n(X;A,B)$ be the disjoint union of the triad homotopy groups obtained for each $x \in A \cap B$, i.e., $S_n(X;A,B) = \bigvee_{x \in A \cap B} \pi_n(X;A,B,x)$. Then $S_n(X;A,B)$ is a set over $A \cap B$ and the mapping $\varphi : S_n(X;A,B) \rightarrow A \cap B$ defined by $\varphi(\sigma) = \varphi([\alpha]_x) = x$, for any $\sigma = [\alpha]_x \in \pi_n(X;A,B,x) \subset S_n(X;A,B)$, is a natural projection. Let x_0 be an arbitrary fixed point of $A \cap B$ and $W = W(x_0)$ be a path connected open neighbourhood of x_0 in $A \cap B$ such that for any two points x and y in W, every pair of paths in W joining x to y are homotopic in $A \cap B$ with endpoints held fixed. There exists such an open neighbourhood W for every points in $A \cap B$

since $A \cap B$ is locally path connected and semilocally simple connected. For any $x \in W$, if γ is a path from x_0 to x for any $x \in W$, then γ induces isomorphism

$$(\gamma^*)_n : \pi_n(X; A, B, x_0) \to \pi_n(X; A, B, x)$$

for all n, defined by $(\gamma^*)_n([\alpha_0]_{x_0}) = [\alpha]_x$ for any fixed $[\alpha_0]_{x_0} \in \pi_n(X; A, B, x_0)$. Therefore we can define a mapping $s : W(x_0) \to S_n(X; A, B)$ with $s(x) = (\gamma^*)_n([\alpha_0]_{x_0}) = [\alpha]_x \in \pi_n(X; A, B, x)$ for every $x \in W(x_0)$. Clearly, s is well defined: If γ_1 and γ_2 are two paths in W from x_0 to x, then they are homotopic in $A \cap B$ with endpoints held fixed. Hence $(\gamma_1^*)_n = (\gamma_2^*)_n$. Furthermore,

$$s(x_0) = 1^*([\alpha_0])_{x_0} \in \pi_n(X; A, B, x_0) \quad \text{and} \quad \varphi \circ s = 1_W.$$

We prescribe that all the sets $s(W) = \{[\alpha]_x \in S_n(X; A, B) : x \in W\}$ be open sets. Then the set $\{s(W) : W \subset A \cap B\}$ is a base for the topology on $S_n(X; A, B)$. In fact, if W_1, W_2 are any two path connected subspaces of $A \cap B$ and $\sigma \in s_1(W_1) \cap s_2(W_2)$, then s_1 and s_2 agree at $\varphi(\sigma) = \varphi([\alpha]_x) = x$, $(x \in W_1 \cap W_2)$ and by the definition of the mappings s_1 and s_2, $s_1(x) = s_2(x)$ for every $x \in W_1 \cap W_2$, i.e., $s_1(W_1 \cap W_2) = s_2(W_1 \cap W_2)$. Therefore σ has a basic neighborhood $s_1(W_1 \cap W_2)$ inside $s_1(W_1) = s_2(W_2)$. In this topology the mappings φ and s are continuous. Moreover, φ is a local homeomorphism since on $s(W)$ it has a continuous inverse s. Therefore $(S_n(X; A, B), \varphi)$ is a sheaf of groups. It is a sheaf of abelian groups for $n > 3$. The sheaf $(S_n(X; A, B), \varphi)$ is called the sheaf of relative homotopy groups of the triad $(X; A, B)$ [5]. The set $\varphi^{-1}(x) = \pi_n(X; A, B, x)$ is called the stalk of the sheaf and denoted by $S_n(X; A, B))_x$ for every $x \in A \cap B$. The continuous mapping $s : W \to S_n(X; A, B)$ such that $\varphi \circ s = 1_W$ is called a section of $S_n(X; A, B)$ over the path connected open set $W \subset A \cap B$. We denote by $\Gamma(W, S_n(X; A, B))$ the collection of all sections of $S_n(X; A, B)$ over W. A section $s \in \Gamma(A \cap B, S_n(X; A,))$ is called a global section. If $s_1, s_2 \in \Gamma(W, S_n(X; A, B))$ are obtained by means of the elements $[\alpha_1]_x$, $[\alpha_2]_2 \in \pi_n(X; A, B, x)$, then we define

$$(s_1 + s_2)(x) = s_1(x) + s_2(x) = [\alpha_1]_x + [\alpha_2]_x = [\alpha_1 + \alpha_2]_x.$$

It is easy to see that $\Gamma(W, S_n(X; A, B))$ is a group with this point wise addition operation.

Definition 1. Let the sheaves $(S_n(X; A, B), \varphi)$ and $(S_n(Y; C, D), \psi)$ be given. It is said that there is a homomorphism between these sheaves and it is denoted by $F = (f, f^*)$, if:

(i) $f : (X; A, B) \to (Y; C, D)$ is a continuous mapping such that $f(A) \subset C$, $f(B) \subset D$, and $f(A \cap B)) \subset C \cap D$,

(ii) $f^* : S_n(X; A, B) \to S_n(Y; C, D)$ is a continuous mapping,
(iii) f^* preserves the stalks with respect to f, i.e., $f \circ \varphi = \psi \circ f^*$,
(iv) For every $x \in A$, the restricted mapping

$$f^* \mid (S_n(X; A, B))_x : (S_n(X; A, B))_x \to (S_n(Y; C, D))_{f(x)}$$

is a homeomorphism.

$F = (f, f^*)$ is called an isomorphism, if the mappings f^* and f are homeomorphisms. Then the sheaves $S_n(X; A, B)$ and $S_n(Y; C, D)$ are said to be isomorphic.

Lemma. *Let the sheaves $S_n(X; A, B)$ and $S_n(Y; C, D)$ be given. A continuous mapping $f : (X; A, B) \to (Y; C, D)$ such that $f(A) \subset C$, $f(B) \subset D$, and $f(A \cap B) \subset C \cap D$, induces a homeomorphism of sheaves $f^* : S_n(X; A, B) \to S_n(Y; C, D)$ ([5]).*

2. The sheaf of triad homotopy groups $S_n(X; A, B)$ as a covering space of $A \cap B$

Theorem 1. *Let $(X; A, B)$ be a triad such that $A \cap B$ is a connected, locally path connected, and semilocally simple connected subspace of X. Let $S_n(X; A, B)$ be corresponding sheaf to the triad $(X; A, B)$ and $W = W(x_0) \subset A \cap B$ be an open neighborhood of $x_0 \in A \cap B$. Then the group of sections $\Gamma(W, S_n(X; A, B))$ over W is isomorphic to the stalk $(S_n(X; A, B))_{x_0}$.*

Proof. Let us define a mapping $\phi : \Gamma(W, S_n(X; A, B)) \to (S_n(X; A, B))_{x_0}$ by $\phi(s) = s(x_0)$ for all $s \in \Gamma(W, S_n(X; A, B))$. Then ϕ is a homomorphism. In fact, for all s_1, $s_2 \in \Gamma(W, S_n(X; A, B))$,

$$\phi(s_1 + s_2) = (s_1 + s_2)(x_0) = s_1(x_0) + s_2(x_0) = \phi(s_1) + \phi(s_2).$$

The mapping ϕ is one-to-one. In fact, if $\phi(s_1) = \phi(s_2)$, then $s_1(x_0) = s_2(x_0)$. By definition of a section, $s_1(x) = s_2(x)$ for all $x \in W$ and so $s_1 = s_2$. ϕ is onto, since for all $\sigma_{x_0} \in (S_n(X; A, B))_{x_0}$, there exists a unique $s \in \Gamma(W, S_n(X; A, B))$ such that $\phi(s) = s(x_0) = x_0$.

Thus ϕ is an isomorphism. ∎

We can state, as a result of Theorem 1, that the stalk $(S_n(X; A, B))_{x_0}$ completely determines the group of sections over W. In particular, if we take $W = A \cap B$, then the stalk $(S_n(X; A, B))_{x_0}$ completely determines the group of global sections over $A \cap B$.

Now let $(X; A, B)$ be a triad such that $A \cap B$ be a connected, locally path connected, and semilocally simple connected subspace of X. Let $W = W(x_0)$ be an open set in $A \cap B$. Then there exists a unique section $s \in \Gamma(W, S_n(X; A, B))$ such that $s(x_0) = \sigma_{x_0}$ for any $\sigma_{x_0} \in (S_n(X; A, B))_{x_0}$. Since $\varphi|_{s_i(W)} : s_i(W) \to W$ is a homeomorphism and $s_i = (\varphi|_{s_i(W)})^{-1}$, $\varphi^{-1}(W) = \bigvee_{i \in I} s_i(W)$ for all $s_i \in \Gamma(W, S_n(X; A, B))$.

Therefore the open set $W = W(x_0)$ is evenly covered by φ, and so φ covering mapping and $(S_n(X; A, B), \varphi)$ is a covering space of $A \cap B$. For $n > 3$, $(S_n(X; A, B), \varphi)$ is called an abelian covering space since the stalk (or fiber) $(S_n(X; A, B))_{x_0}$ is an abelian group for every $x_0 \in A \cap B$.

Definition 2. A mapping $t : (S_n(X; A, B), \varphi) \to (S_n(X; A, B)_\varphi)$ is called cover transformation of $S_n(X; A, B)$ if t is a stalk preserving homeomorphism. That is, t is a sheaf isomorphism and $\varphi \circ t = \varphi$. We denote by T the collection of all cover transformations of $S_n(X; A, B)$. It is easy to show that T is a group under the point wise addition operation.

Theorem 2. *The group T of cover transformations of $S_n(X; A, B)$ is isomorphic to the group $\Gamma(A \cap B, S_n(X; A, B))$ of global sections of $A \cap B$.*

Proof. Let σ_x be an arbitrary element in $S_n(X; A, B)$ and let $t \in T$. Then there exists a unique element $\delta_x \in (S_n(X; A, B))_x$ such that $t(\sigma_x) = \delta_x$. By Theorem 1, there exists a unique section $s \in \Gamma(A \cap B, S_n(X; A, B))$ such that $s(x) = \sigma_x$ for $x \in A \cap B$. Then we write $t(\sigma_x) = t(s(x)) = (t \circ s)(x) = \delta_x$. It is clear that $t \circ s \in \Gamma(A \cap B, S_n(X; A, B))$. Thus we can define a mapping $\phi : T \to \Gamma(A \cap B, S_n(X; A, B))$ by $\phi(t) = t \circ s$, ϕ is a homomorphism. In fact, for all $t_1, t_2 \in T$, $x \in A \cap B$, $s \in \Gamma(A \cap B, S_n(X; A, B))$ and $s(x) = \sigma_x \in (S_n(X; A, B))_x$,

$$\begin{aligned}\phi(t_1 + t_2)(\sigma_x) &= [(t_1 + t_2) \circ s](x) = (t_1 + t_2)(s(x)) \\ &= t_1(s(x)) + t_2(s(x)) = (t_1 \circ s)(x) + (t_2 \circ s)(x) \\ &= \phi(t_1)(\sigma(x)) + \phi(t_2)(\sigma(x)).\end{aligned}$$

The mapping ϕ is one-to-one. If $\phi(t_1) = \phi(t_2)$ for $t_1, t_2 \in T$, then for all $x \in A \cap B$, $(t_1 \circ s)(x) = (t_2 \circ s)(x) \Rightarrow t_1(s(x)) = t_2(s(x)) \Rightarrow t_1 = t_2$. Since t is a homeomorphism, it follows that ϕ is onto. Therefore ϕ is an isomorphism. ■

Corollary 1. *Let $(S_n(X; A, B)), \varphi)$ be the covering space of $A \cap B$. The cover transformation group T of $(S_n(X; A, B)), \varphi)$ operates transitively on $\varphi^{-1} = (S_n(X; A, B))_x$ for every $x \in A \cap B$.*

According to Corollary 1, $(S_n(X; A, B), \varphi)$ is a regular covering space of $A \cap B$, since the cover transformation group T operates transitively on fibers of $(S_n(X; A, B), \varphi)$.

3. Lifting of paths and arbitrary continuous mappings to $(S_n(X;A,B),\varphi)$

Now let $x_0 \in A\cap B$ be any point and γ be a path in $A\cap B$ with initial point x_0. Then the mapping $s\circ\gamma : I \to S_n(X;A,B)$ is a path in $S_n(X;A,B)$ and $\varphi\circ(s\circ\gamma)=\gamma$, where $s\in\Gamma(A\cap B, S_n(X;A\cap B))$. The path $s\circ\gamma=\gamma^*$ is called a lifting of γ from the initial point $(s\circ\gamma)(x_0)=\sigma_0$ over x_0 in $S_n(X;A,B)$. γ^* is unique since the mapping $\varphi|s(A\cap B) : s(A\cap B)\to A\cap B$ is a homeomorphism. If $F : I\times I \to A\cap B$ is a continuous mapping and $\sigma_{x_0}\in(S_n(X;A,B))_{x_0}$ such that $\varphi(\sigma_{x_0})=x_0=F(0,0)$, then $s\circ F : I\times I\to S_n(X;A,B)$ is a continuous mapping such that $\varphi\circ(s\circ F)=F$ and $(s\circ F)(0,0)=\sigma_{x_0}$, where $s\in\Gamma(A\cap B, S_n(X;A,B))$ and $s(x_0)=\sigma_{x_0}$. Thus $s\circ F$ is a lifting of F to $S_n(X;A,B)$.

Then we have the following theorem.

Theorem 3. (Homotopy path lifting theorem) *Let $(X;A,B)$ be a triad such that $A\cap B$ is a connected, locally path connected, and semilocally simple connected subspace of X. Let $S_n(X;A,B)$ be a corresponding sheaf (or covering space) of the triad $(X;A,B)$. Then:*

(i) Let $x_0\in A\cap B$ be any point and γ be a path in $A\cap B$ with initial point x_0. Then γ has a unique lifting γ^* in $S_n(X;A,B)$ with initial point σ_{x_0}.

(ii) Let $F : I\times I\to A\cap B$ be a continuous mapping and let $\sigma_{x_0}\in S_n(X;A,B)$ such that $\varphi(\sigma_{x_0} = F(0,0)$. Then there exists a unique mapping $F^* : I\times I\to S_n(X;A,B)$ such that $\varphi\circ F^*=F$ and $F^*(0,0)=\sigma_{x_0}$.

Theorem 4. (Monodramy) *Let $(S_n(X;A,B),\varphi)$ be the sheaf of n-th triad homotopy groups (or covering space of $A\cap B$) of the triad $(X;A,B)$. Let γ^* and δ^* be two paths in $S_n(X;A,B)$ with common initial point σ_{x_1} and common terminal point σ_{x_2}. Then γ^* and δ^* are homotopic paths in $S_n(X;A,B)$ if and only if $\varphi\circ\gamma^*$ and $\varphi\circ\delta^*$ are homotopic paths in $A\cap B$.*

Proof. If γ^* is homotopic to δ^* by a homotopy F, then the homotopy $\varphi\circ F$ demonstrates the homotopy of $\varphi\circ\gamma^*$ and $\varphi\circ\delta^*$. Conversely, let x_1 and x_2 denote the common initial point and common terminal point of $\varphi\circ\gamma^*$ and $\varphi\circ\delta^*$, respectively. Let $G : I\times I\to A\cap B$ be a homotopy between the paths $\varphi\circ\gamma^*$ and $\varphi\circ\delta^*$. On the other hand, if $\sigma_{x_1}\in S_n(X;A,B)$, then there exists a unique section $s\in\Gamma(A\cap B, S_n(X;A,B))$ such that $s(x_1)=\sigma_{x_1}$. Thus,

$$s\circ(\varphi\circ\gamma^*)=\gamma^* \quad\text{and}\quad s\circ(\varphi\circ\delta^*)=\delta^*.$$

Furthermore, the mapping $s\circ G=G^*$ is a homotopy between the paths γ^* and δ^*. This complete the proof. ■

Now let γ be a path in $A \cap B$ with initial point x_0. As a result of Theorems 3 and 4, we can state that any lifting γ^* of γ in $S_n(X; A, B)$ with initial point $\sigma_{x_0} \in \varphi^{-1}(x_0)$ is a closed path if and only if the path γ is closed in $A \cap B$. Hence, $S_n(X; A, B)$ is a regular covering space of $A \cap B$.

In Theorem 3 we gave the "lifting" of paths in $A \cap B$ to the covering space $S_n(X; A, B)$. We study the analogous problem for continuos mappings $f : (Y, y_0) \to (A \cap B, x_0)$ such that $f(y_0) = x_0$, where Y is connected and locally path connected topological space.

Theorem 5. (Homotopy lifting theorem) *Let $(S_n(X; A, B), \varphi)$ be the sheaf of triad homotopy groups (or covering space of $A \cap B$). of the triad $(X; A, B)$ and $(Y, y_0$ be a connected, locally path connected pointed topological space. Let $\sigma_{x_0} \in S_n(X; A, B)$ and $x_0 = \varphi(\sigma_{x_0})$. Then:*

(i) If $f : (Y, y_0) \to (A \cap B, x_0)$ is a continuous mapping such that $f(y_0) = x_0$, then f has a unique lifting

$$\tilde{f} : (Y, y_0) \to (S_n(X; A, B), \sigma_{x_0})$$

such that $\varphi \circ \tilde{f} = f$.

(ii) If $\tilde{f} : (Y, y_0) \to (S_n(X; A, B), \sigma_{x_0})$ is a continuous mapping and $F : Y \times I \to A \cap B$ is a continuous mapping such that $F(y, 0) = \varphi(\tilde{f}(y))$ for every $y \in Y$, then there exists a unique continuous mapping $\tilde{F} : Y \times I \to S_n(X; A, B)$ such that $\tilde{F}(y, 0) = \tilde{f}(y)$ and $\varphi \circ \tilde{F} = F$ for every $y \in Y$.

Proof. (i) Let $f : (Y, y_0) \to (A \cap B, x_0)$ be a continuous mapping and $\sigma_{x_0} \in \varphi^{-1}(x_0)$. Then there exists a unique section $s \in \Gamma(A \cap B, S_n(X; A, B))$ such that $s(x_0) = \sigma_{x_0}$. Thus,

$$s \circ f : (Y, y_0) \to (S_n(X; A, B), \sigma_{x_0})$$

is a continuous mapping and $\varphi \circ (s \circ f) = f$. Hence $s \circ f$ is a lifting of f to $S_n(X; A, B)$. Let us denote $s \circ f$ by $\tilde{f}$. Since s is unique and $\varphi | s(A \cap B) : s(A \cap B) \to A \cap B$ is a homeomorphism, $\tilde{f}$ is unique.

(ii) Let $\tilde{f} : (Y, y_0 \to (S_n(X; A, B), \sigma_{x_0})$ be a continuous mapping and $F : Y \times I \to A \cap B$ be a continuous mapping such that $F(y, 0) = \varphi(\tilde{f}(y))$ for every $y \in Y$. Hence $\tilde{f}(y_0) = \sigma_{x_0}$ and there exists a unique section $s \in \Gamma(A \cap B, S_n(X; A, B))$ such that $s(x_0) = \sigma_{x_0}$. Therefore, $s \circ F : Y \times I \to S_n(X; A, B)$ is a continuous mapping such that

$$\begin{aligned} (s \circ F)(y, 0) &= s(F(y, 0)) = s(\varphi(\tilde{f}(y)) \\ &= (s \circ \varphi)(\tilde{f}(y)) = \tilde{f}(y). \end{aligned}$$

and so, $\varphi \circ (s \circ F) = F$. If we denote $s \circ F$ by $\tilde{F}$, then $\tilde{F}$ is the desired lifting. Since s is unique and $\varphi | s(A \cap B) : s(A \cap B) \to A \cap B$ is a homeomorphism, $\tilde{F}$ is unique. ■

Corollary 2. *Let $S_n(X; A, B)$ be the sheaf of triad homotopy groups of the triad $(X; A, B)$ and (Y, y_0) be a connected, locally path connected pointed topological space. Let $\sigma_{x_0} \in S_n(X; A, B)$ and $x_0 = \varphi(\sigma_{x_0})$. If $\tilde{f}, \tilde{g} : (Y, y_0) \to (S_n(X; A, B)), \sigma_{x_0})$ be any two continuous mappings such that $\varphi \circ \tilde{f} = \varphi \circ \tilde{g}$, then $\tilde{f} = \tilde{g}$.*

Now we denote by $L(Y, f, S_n(X; A, B))$ the set of all liftings of f to $S_n(X; A, B))$. $L(Y, f, S_n(X; A, B))$ is a group under the point wise addition of functions. In fact, for any $\tilde{f}_1$, $\tilde{f}_2 \in L(Y, f, S_n(X; A, B))$ and $y \in Y$, we have for s_1, $s_2 \in \Gamma(A \cap B, S_n(X; A, B))$,

$$\begin{aligned}(\tilde{f}_1 + \tilde{f}_2)(y) &= \tilde{f}_1(y) + \tilde{f}_2(y) = (s_1 \circ f)(y) + (s_2 \circ f)(y) \\ &= s_1(f(y)) + s_2(f(y)) = (s_1 + s_2)(f(y)) \\ &= \tilde{f}_3(y),\end{aligned}$$

where $s_1 + s_2 \in \Gamma(A \cap B, S_n(X; A, B))$ and so $\tilde{f} \in L(Y, f, S_n(X; A, B))$. Therefore the operation is closed. On the other hand, if $O \in \Gamma(A \cap B, S_n(X; A, B))$ is zero section, then $\tilde{O} = O \circ f \in L(Y, f, S_n(X; A, B))$ is the identity lifting. The inverse of any lifting $\tilde{f}$ is $(\tilde{f})^{-1} = f^{-1} \circ s^{-1}$. Since $\Gamma(A \cap B, S_n(X; A, B))$ is commutative, $L(Y, f, S_n(X; A, B))$ is commutative. Hence, the group $\Gamma(A \cap B, S_n(X; A, B))$ of global sections completely determines the totality of all liftings of f to $S_n(X; A, B))$.

We then have the following theorem.

Theorem 6. *$L(Y, f, S_n(X; A, B))$ is isomorphic to $\Gamma(A \cap B, S_n(X; A, B))$ Then we have for every $x \in A \cap B$, $L(Y, f, S_n(X; A, B) \cong \Gamma(A \cap B, S_n(X; A, B)) \cong T \cong (S_n(X; A, B))_x$.*

Now let (Y, y_0) be a connected, locally path connected pointed topological space and $(X_1; A, B)$, $(X_2; C, D)$ be two triad such that $A \cap B$, $C \cap D$ are connected, locally path connected, and semilocally simple connected subspaces of X_1 and X_2, respectively. Let

$$F : (Y, y_0) \to (A \cap B, x_0), \quad g : (X_1; A, B, x_0) \to (X_2; C, D, z_0)$$

be two continuous mappings such that $f(y_0) = x_0$, $g(A) \subset C$, $g(B) \subset D$, $g(A \cap B) \subset C \cap D$, and $g(x_0) = z_0$. By Theorem 5, f has a unique lifting

$$\tilde{f} : (Y, y_0) \to (S_n(X; A, B), \sigma_{x_0}$$

such that $\tilde{f} = s \circ f$ and $\varphi \circ \tilde{f} = f$, where $\sigma_{x_0} = s(x_0)$ for a unique $s \in \Gamma(A \cap B, S_n(X; A, B))$. On the other hand, there exists a sheaf morphism

$$g^* : S_n(X_1; A, B) \to S_n(X_2; C, D)$$

such that $g^*(\sigma_{x_0}) = \sigma_{z_0}$, where $\sigma_{z_0} = t(z_0)$ for a unique $t \in \Gamma(C \cap D, S_n(X_2, C, D))$. Hence the mapping $t \circ (g \circ f)$ is the lifting of $h = g \circ f$ to $S_n(X_2, C, D))$ and $t \circ (g \circ f) = g^* \circ f$. Hence we obtain a one-to-one correspondence between the groups $L(Y, f, S_n(X, A, B))$ and $L(Y, h, S_n(X_2, C, D))$ defined by

$$\phi(\tilde{f}) = g^* \circ \tilde{f}.$$

ϕ is homomorphism. In fact, for any f_1, $f_2 \in L(Y, f, S_n(X_1, A, B))$ and $y \in Y$,

$$\begin{aligned}\phi(\tilde{f}_1 + \tilde{f}_2)(y) &= g^* \circ (\tilde{f}_1 + \tilde{f}_2)(y) = g^*(\tilde{f}_1(y) + \tilde{f}_2(y)) \\ &= g^*(\tilde{f}_1(y)) + g^*(\tilde{f}_2(y)) = (g^* \circ \tilde{f}_1)(y) + (g^* \circ \tilde{f}_2)(y) \\ &= \phi(f_1)(y) + \phi(f_2)(y).\end{aligned}$$

We then have the following theorem.

Theorem 7. *Let $g : (X_1; A, B, x_0) \to (X_2; C, D, z_0)$ be a continuous mapping such that $g(A) \subset C$, $g(B) \subset D$, $g(A \cap B) \subset C \cap D$ and $g(x_0) = z_0$. Then there exists a homeomorphism between the groups of liftings $L(Y, f, S_n(X_1; A, B))$ and $L(Y, h, S_n(X_n; C, D))$. Furthermore, if g is a homeomorphism, then the groups of liftings are isomorphic.*

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Department of Mathematics
Faculty of Sciences, University of Ankara
06100 Tandoğan, Ankara, TURKEY

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